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SOLITARY SUBGROUPS AND SOLITARY QUOTIENTS OF ZM-GROUPS

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Abstract. The main goal of this note is to describe the lattices of solitary subgroups and of solitary quotients of a ZM-group.

1. Introduction

Given a group G, we will denote by $L(G)$ the subgroup lattice of G and by $N(G)$ the normal subgroup lattice of G. A subgroup $H \in L(G)$ is called a *solitary subgroup* of G if G does not contain another isomorphic copy of H, that is

$$\forall\, K \in L(G),\, K \cong H \Longrightarrow K = H.$$

By Theorem 25 of [5], the set $\mathrm{Sol}(G)$ of all solitary subgroups of G forms a lattice with respect to set inclusion, which is called *the lattice of solitary subgroups of* G. A "dual" for $\mathrm{Sol}(G)$ is the set $\mathrm{QSol}(G)$ of all normal subgroups of G that determine solitary quotients, i.e.

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$$\mathrm{QSol}(G) = \{H \in N(G) \mid \forall\, K \in N(G),\, G/K \cong G/H \Longrightarrow K = H\}\,.$$

This is also a lattice under inclusion (see Proposition 2.1 of [7]), which is called *the lattice of solitary quotients of* G.

The lattices $\mathrm{Sol}(G)$ and $\mathrm{QSol}(G)$ have been described in [5,7] only for few classes of finite groups G. This is the reason for which in the current note we will describe them for yet another important class of groups, namely for Zassenhaus metacyclic groups (ZM-groups, in short).

We recall that a ZM-group is a finite nonabelian group with all Sylow subgroups cyclic. By [4], such a group is of type

$$\mathrm{ZM}(m,n,r) = \langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r\rangle,$$

where the triple (m, n, r) satisfies the conditions

$$\gcd(m,n) = \gcd(m, r-1) = 1 \quad \text{and} \quad r^n \equiv 1 \ (\mathrm{mod}\, m).$$

It is clear that $|\mathrm{ZM}(m,n,r)| = mn$, $\mathrm{ZM}(m,n,r)' = \langle a\rangle$ (consequently, we have $|\mathrm{ZM}(m,n,r)'| = m$) and $\mathrm{ZM}(m,n,r)/\mathrm{ZM}(m,n,r)'$ is cyclic of order n. One of the most important (lattice theoretical) property of the ZM-groups is that these groups are exactly the finite groups whose poset of conjugacy classes of subgroups forms a distributive lattice (see Theorem A of [2]). We infer that they are DLN-groups, that is groups with distributive lattice of normal subgroups.

The subgroups of $\mathrm{ZM}(m,n,r)$ have been completely described in [3]. Set

$$L = \left\{(m_1, n_1, s) \in \mathbb{N}^3 \ \mid\ m_1|m,\ n_1|n,\ s < m_1,\ m_1|s\frac{r^n - 1}{r^{n_1} - 1}\right\}.$$

Then there exists a bijection between L and the subgroup lattice $L(\mathrm{ZM}(m,n,r))$ of $\mathrm{ZM}(m,n,r)$, namely the function that maps a triple $(m_1, n_1, s) \in L$ into the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)} = \bigcup_{k=1}^{\frac{n}{n_1}} \alpha(n_1, s)^k \langle a^{m_1}\rangle = \langle a^{m_1}, \alpha(n_1, s)\rangle,$$

where $\alpha(x, y) = b^x a^y$, for all $0 \le x < n$ and $0 \le y < m$. Remark also that $|H_{(m_1,n_1,s)}| = \frac{mn}{m_1 n_1}$, for any s satisfying $(m_1, n_1, s) \in L$.

The normal subgroup lattice of $\mathrm{ZM}(m,n,r)$ is given by Theorem 1 of [8]. It consists of all subgroups

$$H_{(m_1,n_1,s)} \in L(\mathrm{ZM}(m,n,r)) \text{ with } (m_1,n_1,s) \in L',$$

where

$$L' = \left\{(m_1,n_1,s) \in \mathbb{N}^3 \;\middle|\; m_1|\gcd(m, r^{n_1}-1),\ n_1|n,\ s=0\right\} \subseteq L.$$

Most of our notation is standard and will not be repeated here. Basic definitions and results on group theory can be found in [4]. For subgroup lattice theory we refer the reader to [6].

2. Main results

First of all, we indicate the structure of the lattice of solitary subgroups of a ZM-group.

Theorem 1. *We have*

$$\mathrm{Sol}(\mathrm{ZM}(m,n,r)) = \left\{H_{(m_1,n_1,0)} \mid m_1|m,\ n_1|n,\ \gcd(m_1, \frac{r^n-1}{r^{n_1}-1}) = 1\right\}.$$

Proof. We observe that two subgroups of $\mathrm{ZM}(m,n,r)$ are isomorphic if and only if they are conjugate, and consequently if and only if they have the same order by the result on the last page of [2].

On the other hand, for every divisors m_1 of m and n_1 of n, the number of subgroups of order $\frac{mn}{m_1n_1}$ in $\mathrm{ZM}(m,n,r)$ is $\gcd(m_1, \frac{r^n-1}{r^{n_1}-1})$. Indeed, the subgroups of order $\frac{mn}{m_1n_1}$ in $\mathrm{ZM}(m,n,r)$ are $H_{(m_1,n_1,s)}$ with $0 \le s < m_1$ and $m_1|s\frac{r^n-1}{r^{n_1}-1}$. Let $d = \gcd(m_1, \frac{r^n-1}{r^{n_1}-1})$. Then $\frac{m_1}{d}|s$, and therefore $s \in \left\{0, \frac{m_1}{d}, 2\frac{m_1}{d}, ..., (d-1)\frac{m_1}{d}\right\}$. This completes the proof. $\square$

Next we present some consequences of Theorem 1.

Corollary 2. *The lattice of solitary subgroups of* $\mathrm{ZM}(m,n,r)$ *contains two sublattices* $L_1 = \left\{H_{(1,n_1,0)} \mid n_1|n\right\}$ *and* $L_2 = \left\{H_{(m_1,n,0)} \mid m_1|m\right\}$ *which are isomorphic to the lattices of divisors of* n *and* m*, respectively.*

From the above corollary we infer that the subgroups $H_{(1,n,0)}$ and $H_{(m,1,0)}$ of $\mathrm{ZM}(m,n,r)$ are always solitary. This shows that:

Corollary 3. $\mathrm{ZM}(m,n,r)$ *is not solitary free, that is it contains proper nontrivial solitary subgroups.*

By Definition 1.4 of [1] a group G is called *strongly* (*normal*) *solitary solvable* if it has a (normal) series $1 = G_0 \le G_1 \le ... \le G_n = G$

such that every G_i is solitary and every $\frac{G_{i+1}}{G_i}$ is abelian. Clearly, for $\mathrm{ZM}(m,n,r)$ such a series is

$$1 = H_{(m,n,0)} \leq H_{(1,n,0)} \leq H_{(1,1,0)} = \mathrm{ZM}(m,n,r)$$

and consequently:

Corollary 4. $\mathrm{ZM}(m,n,r)$ *is strongly normal solitary solvable.*

We also remark that Theorem 1 allows us to determine explicitly the solitary subgroups of several particular ZM-groups. We take first $n=2$, m odd, and $r=-1$. Then $\mathrm{ZM}(m,2,-1)$ is in fact the dihedral group D_{2m} and its solitary subgroups are the following.

Example 5. $\mathrm{Sol}(D_{2m}) = \{D_{2m}\} \bigcup \left\{H_{(m_1,2,0)} \mid m_1|m\right\}$.

Assume next that both m, n are prime and $n \mid m-1$. Then there is a unique nonabelian group $G_{m,n}$ of order mn and this is isomorphic to $\mathrm{ZM}(m,n,r)$. Its solitary subgroups follow immediately from Theorem 1.

Example 6. $\mathrm{Sol}(G_{m,n}) = \left\{H_{(1,1,0)}, H_{(1,n,0)}, H_{(m,n,0)}\right\}$.

Our second theorem gives the structure of the lattice of solitary quotients of a ZM-group.

Theorem 7. *We have*

$$\mathrm{QSol}(\mathrm{ZM}(m,n,r)) = N(\mathrm{ZM}(m,n,r)).$$

Proof. It is clear that all subgroups in $\mathrm{QSol}(\mathrm{ZM}(m,n,r))$ are normal. Conversely, let $H_{(m_1,n_1,0)} \in N(\mathrm{ZM}(m,n,r))$. If for $H_{(m_2,n_2,0)} \in N(\mathrm{ZM}(m,n,r))$ we have $\frac{\mathrm{ZM}(m,n,r))}{H_{(m_2,n_2,0)}} \cong \frac{\mathrm{ZM}(m,n,r))}{H_{(m_1,n_1,0)}}$, then $H_{(\frac{m}{m_2},\frac{n}{n_2},0)} \cong H_{(\frac{m}{m_1},\frac{n}{n_1},0)}$. We infer that $m_2n_2 = m_1n_1$, and consequently $m_2 = m_1$ and $n_2 = n_1$ by the condition $\gcd(m,n) = 1$. Thus $H_{(m_2,n_2,0)} = H_{(m_1,n_1,0)}$, completing the proof. □

Since the group $\mathrm{ZM}(m,n,r)$ is not simple, by Theorem 7 we infer that it has at least one proper nontrivial solitary quotient. Consequently:

Corollary 8. $\mathrm{ZM}(m,n,r)$ *is not quotient solitary free.*

Finally, we indicate an open problem concerning the above study.

Open problem. Describe the lattices of solitary subgroups and of solitary quotients for other important classes of finite groups.

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