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A WEAK CONVERGENCE THEOREM FOR A KRASNOSELSKIJ TYPE FIXED POINT ITERATIVE METHOD IN HILBERT SPACES USING AN ADMISSIBLE PERTURBATION

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Abstract. The aim of this paper is to prove a weak convergence theorem for a general Krasnoselskij type fixed point iterative method defined by means of the new concept of *admissible perturbation* of a firmly nonexpansive mapping and a nonspreading mapping in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A mapping F is said to be firmly nonexpansive if $\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$, for all $x, y \in C$; see, for instance, [7, 8, 9, 21, 23]. On the other hand, a mapping $Q : C \to C$ is said to be quasi-nonexpansive if $F(Q)$ is nonempty and

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$$\|Qx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(Q)$, where $F(Q)$ is the set of fixed points of Q. If $T : C \to C$ is nonexpansive and the set $F(T)$ of fixed points of T is nonempty, then T is quasi-nonexpansive.

In 2007, Aoyama et al. [1] proved the following strong convergence theorem of Halpern's iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \tag{1}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, α_n is a sequence in $[0, 1]$ and T_n is a sequence of nonexpansive mappings of C into itself which satisfies the AKTT-condition, that is,

$$\sum_{n=1}^{\infty} sup \{\|T_{n+1}z - T_n z\| : z \in C\} < \infty. \tag{2}$$

They proved that the sequence $\{x_n\}$ defined by (1) converges strongly to a common fixed point of $\{T_n\}$.

In 2008, Kohsaka and Takahashi [11] introduced the notion of nonspreading mapping, which is defined as a nonlinear mapping from a nonempty closed convex subset of C of a smooth, strictly convex and reflexive Banach space E into C, satisfying some contractive condition. In the special case where E is a Hilbert space, it is said that a mapping $S : C \to C$ is nonspreading if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2, \tag{3}$$

for all $x, y \in C$.

Example 1. [17] *Define* $T : [0, 2] \to [0, 2]$ *by*

$$Tx = \begin{cases} 0, & \text{if } x \in [0, 2), \\ 1, & \text{if } x = 2. \end{cases}$$

We have $F(T) = \{0\}$, T *is neither nonexpansive nor continuous. However,* T *is nonspreading.*

Definition 1. [22] *Let* X *be a nonempty set. A mapping* $G : X \times X \to X$ *is called admissible if it satisfies the following two conditions:*

$(A1)$ $G(x, x) = x,$ *for all* $x \in X$;

$(A2)$ $G(x, y) = x$ *implies* $y = x$.

Definition 2. [22] *Let X be a nonempty set. If $f : X \to X$ is a given mapping and $G : X \times X \to X$ is an admissible mapping, then the operator $f_G : X \to X$, defined by*

$$f_G(x) = G(x, f(x))\,; x \in X; \tag{4}$$

is called the admissible perturbation of f with respect to G.

Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\lambda \in (0,1)$, $f : X \to X$ and $G : X \times X \to X$ be defined by

$$G(x, y) := (1-\lambda)\,x + \lambda y,\ x, y \in X.$$

The admissible perturbation f_G is called in this case the Krasnoselskij perturbation of f. Let $f : X \to X$ be a nonlinear operator and $G : X \times X \to X$ be an admissible mapping. The the iterative algorithm generating the sequence $\{x_n\}$ given by $x_0 \in X$ and

$$x_{n+1} = G(x_n, f(x_n))\ \ n \geq 0 \tag{5}$$

is called the *Krasnoselskij algorithm* corresponding to G or the *GK-algorithm.*

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)\,y\|^2 = \alpha\|x\|^2 + (1-\alpha)\,\|y\|^2 - \alpha\,(1-\alpha)\,\|x-y\|^2$$

A space X is said to satisfy Opial's condition [20] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,\ \forall y \in X,\ x \neq y$$

Lemma 1. [10] *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \to C$ a nonspreading mapping with $F(S) \neq \phi$. Then S is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n S x_n \to 0$ imply $u \in F(S)$.*

Lemma 2. [10] *Let H be a Hilbert space, C a nonempty closed convex subset of a real Hilbert space H and let S be a nonspreading mapping of C into itself and let $A = I - S$. Then*

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle + \frac{1}{2}\left(\|Ax\|^2 + \|Ax\|^2\right).$$

In [19] the following theorem is proved:

Theorem 3. [19] *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let $\{T_n\}$ be the sequences of firmly nonexpansive mappings of C into itself such that $F(S) \bigcap (\bigcap_{n=1}^{\infty} F(T_n))$ is nonempty. Let $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$.*

Suppose that $\{T_n\}$ satisfy the AKTT-condition and T be the mappings of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(Tn)$. Let x_n be a sequence defined by $x_0 = x \in C$ and $x_{n+1} = \alpha_n x_n + (1-\alpha_n) ST_n x_n, n \geq 0$.

Then $\{x_n\}$ converges weakly to $\hat{z} \in F(S) \bigcap (\bigcap_{n=1}^{\infty} F(Tn))$.

3. The result

The result of this paper is a modification of the above theorem. The changes are the following: first we replace the sequence $\{\alpha_n\} \subset [a,b]$ with the constant $\alpha \in [a,b]$ where a, $b \in (0,1)$, secondly we replace the sequence $\{x_n\}$ in the conclusion of the theorem with a much more general sequence $\{x_n\}$ defined by $x_0 \in X$ and the recurrence $x_{n+1} = G(x_n, f(x_n)), n \geq 0$, where G is an admissible perturbation operator.

Theorem 4. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let $\{T_n\}$ be a sequence of firmly nonexpansive mappings of C into itself such that $F(S) \bigcap (\bigcap_{n=1}^{\infty} F(T_n))$ is nonempty. Let $\alpha \in [a,b]$ for some $a, b \in (0,1)$.*

Suppose that $\{T_n\}$ satisfy the AKTT-condition and that T is a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Let $G : C \times C \to C$ be an addmisible perturbation operator which is affine Lipschitzian with constant $\alpha \in [0,1]$. Let $\{x_n\}$ be a sequence defined by $x_0 = x \in C$ and

$$x_{n+1} = G(x_n, ST_n x_n); \tag{6}$$

Then $\{x_n\}$ converges weakly to $\hat{z} \in F(S) \bigcap (\bigcap_{n=1}^{\infty} F(T_n))$.

Proof. Take a point $v \in F(S) \bigcap (\bigcap_{n=1}^{\infty} F(T_n))$ and put $y_n = T_n x_n$. We shall show that the sequence $\{x_n\}$ is bounded. First, we note that

$$\begin{aligned} \|Sy_n - v\| &\le \|y_n - v\| \\ &= \|T_n x_n - v\| \\ &\le \|x_n - v\|. \end{aligned}$$

We obtain:

$\|x_{n+1} - v\|^2 =$
$= \|G(x_n, Sy_n) - G(v,v)\|^2$
$= \|\alpha(x_n - v) + (1-\alpha)(Sy_n - v)\|^2$
$= \alpha\|x_n - v\|^2 + (1-\alpha)\|Sy_n - x_n\|^2 - \alpha(1-\alpha)\|Sy_n - v\|^2$
$\le \alpha\|x_n - v\|^2 + (1-\alpha)\|y_n - x_n\|^2 - \alpha(1-\alpha)\|Sy_n - v\|^2$
$= \|x_n - v\|^2 - \alpha(1-\alpha)\|Sy_n - x_n\|^2$
$\le \|x_n - v\|^2$

Hence $\{\|x_n - v\|\}$ is a decreasing sequence and therefore $lim_{n\to\infty}\|x_n - v\|$ exists and is finite. This implies that $\{x_n\}$, $\{y_n\}$ and $\{Sy_n\}$ are bounded. Since $\{T_n\}$ is firmly nonexpansive, it follows that for all $v \in F(S) \bigcap (\bigcap_{n=1}^{\infty} F(T_n))$ we have

$$\begin{aligned} \|T_n x_n - v\|^2 &= \|T_n x_n - T_n v\|^2 \\ &\le \langle T_n x_n - v, x_n - v\rangle \\ &= \tfrac{1}{2}\left(\|T_n x_n - v\|^2 + \|x_n - v\|^2 - \|x_n - T_n x_n\|^2\right) \end{aligned}$$

and hence

$$\|T_n x_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - T_n x_n\|^2$$

Thus, we get:

$$\begin{aligned} \|x_{n+1} - v\|^2 &= \|\alpha(x_n - v) + (1-\alpha)(Sy_n - T_n x_n)\|^2 \\ &\le \alpha\|x_n - v\|^2 + (1-\alpha)\|Sy_n - v\|^2 \\ &\le \alpha\|x_n - v\|^2 + (1-\alpha)\|y_n - v\|^2 \\ &= \alpha\|x_n - v\|^2 + (1-\alpha)\|T_n x_n - v\|^2 \\ &\le \alpha\|x_n - v\|^2 + (1-\alpha)(\|x_n - v\|^2 - \|x_n - T_n x_n\|^2). \end{aligned}$$

We obtain

$(1-\alpha)\|x_n - T_n x_n\|^2 \le \alpha\|x_n - v\|^2 + \|(1-\alpha)\|x_n - v\|^2 - \|x_{n+1} - v\|^2$
$= \|x_n - v\|^2 - \|x_{n+1} - v\|^2.$

Since $0 < a \le \alpha \le b < 1$ and $\lim_{n\to\infty}\|x_n - v\|^2 = \lim_{n\to\infty}\|x_{n+1} - v\|^2$ this implies

$$\|x_n - T_n x_n\| = \|x_n - y_n\| \to 0 \text{ as } n \to \infty.$$

Put $A_n = I - ST_n$. From $A_n v = 0$ it follows using Lemma 2 that

$$\begin{aligned}
&\|x_{n+1}-v\|^2=\|\alpha\left(x_n-v\right)-\left(1-\alpha\right)\left(Sy_n-v\right)\|^2\\
&=\|\alpha x_n+\left(1-\alpha\right)Sy_n-v\|^2\\
&=\|x_n-v-\left(1-\alpha\right)\left(x_n-ST_nx_n\right)\|^2\\
&=\|x_n-\left(1-\alpha\right)A_nx_n\|^2\\
&=\|x_n-v\|^2-2\left(1-\alpha\right)\langle x_n-v,A_nx_n-A_nv\rangle+\left(1-\alpha\right)^2\|A_nx_n\|^2\\
&\le\|x_n-v\|^2-2\left(1-\alpha\right)\left\{\|A_nx_n-A_nv\|^2-\tfrac{1}{2}\left(\|A_nx_n\|^2+\|A_nv\|^2\right)\right\}\\
&+\left(1-\alpha\right)^2\|A_nx_n\|^2\\
&=\|x_n-v\|^2-\left(1-\alpha\right)\|A_nx_n\|^2+\left(1-\alpha\right)\|A_nx_n\|^2\\
&=\|x_n-v\|^2-\alpha\left(1-\alpha\right)\|A_nx_n\|^2
\end{aligned}$$

and hence

$$\alpha\left(1-\alpha\right)\|A_nx_n\|^2\le\|x_n-v\|^2-\|x_{n+1}-v\|^2$$

Since $\alpha\left(1-\alpha\right)>0$ we have

$$\lim_{n\to\infty}\|A_nx_n\|=\lim_{n\to\infty}\|x_n-ST_nx_n\|=0.$$

So, we have

$$\begin{aligned}
\|y_n-Sy_n\| &=\|T_nx_n-ST_nx_n\|\\
&=\|T_nx_n-x_n+x_n-ST_nx_n\|\\
&\le\|T_nx_n-x_n\|+\|x_n-ST_nx_n\|\to 0 \text{ as } n\to\infty
\end{aligned}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to some $\hat{z}\in C$, i.e. $y_{n_i}\rightharpoonup\hat{z}$ as $i\to\infty$. By Lemma 1, we have $\hat{z}\in F(S)$. From $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $y_{n_i}\rightharpoonup\hat{z}$ as $i\to\infty$ we get $x_{n_i}\rightharpoonup\hat{z}$ as $i\to\infty$.

We shall show that $\hat{z}\in F(T)$.

Since $\|Tx_n-x_n\|\le\|Tx_n-T_nx_n\|+\|T_nx_n-x_n\|$, from $\lim_{n\to\infty}\|T_nx_n-x_n\|=0$ and the AKTT-condition (that implies the uniform convergence of $(T_n)_{n\ge1}$ to T it follows that

$$\|Tx_n-x_n\|\to 0 \text{ as } n\to\infty.$$

Assume $\hat{z}\notin F(T)$. Since $x_{n_i}\rightharpoonup\hat{z}$ and $\hat{z}\neq T\hat{z}$, by the Opial's condition we have

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i}-\hat{z}\| &<\liminf_{i\to\infty}\|x_{n_i}-T\hat{z}\|\\
&\le\liminf_{i\to\infty}\{\|x_{n_i}-Tx_{n_i}\|+\|Tx_{n_i}-T\hat{z}\|\}.
\end{aligned}$$

But $\lim\limits_{i\to\infty}\|x_{n_i}-Tx_{n_i}\|=0$, therefore $\liminf\limits_{i\to\infty}\|x_{n_i}-\hat{z}\|<\liminf\limits_{i\to\infty}\|Tx_{n_i}-T\hat{z}\|\leq\liminf\limits_{n\to\infty}\|x_n-\hat{z}\|$, a contradiction.

So, we get $\hat{z}\in F(T)$. Moreover, $\hat{z}\in F(S)\bigcap(\bigcap_{n=1}^{\infty}F(T_n))$.

Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $\tilde{z}\in C$. We show that $\hat{z}=\tilde{z}$.

Suppose $\hat{z}\neq\tilde{z}$. Since $\lim_{n\to\infty}\|x_n-v\|$ exists for all $v\in F(S)\bigcap(\bigcap_{n=1}^{\infty}F(T_n))$, it follows by the Opial's condition that

$$\begin{aligned}\lim_{n\to\infty}\|x_n-\hat{z}\| &= \liminf_{i\to\infty}\|x_{n_i}-\hat{z}\|<\liminf_{i\to\infty}\|x_{n_i}-\tilde{z}\|\\ &= \lim_{n\to\infty}\|x_n-\tilde{z}\|=\liminf_{k\to\infty}\|x_{n_k}-\tilde{z}\|\\ &< \liminf_{k\to\infty}\|x_{n_k}-\hat{z}\|=\lim_{n\to\infty}\|x_n-\hat{z}\|,\end{aligned}$$

hence, $\lim\limits_{n\to\infty}\|x_n-\hat{z}\|<\lim\limits_{n\to\infty}\|x_n-\hat{z}\|$, a contradiction.

Thus, we have $\hat{z}=\tilde{z}$. This implies that $\{x_n\}$ converges weakly to $\hat{z}\in F(S)\bigcap(\bigcap_{n=1}^{\infty}F(T_n))$ and the proof is completed. ∎

4. Conclusions

In order to obtain theorem 4 we modified a theorem in [19] where we replaced the sequence $\{\alpha_n\}$ with the constant λ and therefore we were able to use the GK-algorithm as it was defined by Rus in [22].

In the particular case in which $G(x,y)=(1-\lambda)x+\lambda y$ in the main theorem we obtain a convergence theorem for the Krasnoselskij iteration.

A more general case for the Mann algorithm associated to G or the GM-algorithm

$$G_n(x,y)=(1-\lambda_n)x+\lambda_n y$$

will be studied in a forthcoming paper.

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