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Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 25(2015), No. 2, 5-14

SURFACES GENERATED BY BLENDING INTERPOLATION ON A TRIANGLE

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Abstract.
We use some interpolation operators of Lagrange, Hermite and Birkhoff type in order to generate surfaces which satisfy some given conditions.

1. Introduction

There have been constructed interpolation operators of Lagrange, Hermite and Birkhoff type on a triangle with all straight sides, starting with the paper [5] of R.E. Barnhil, G. Birkhoff and W.J. Gordon, and in many others papers (see, e.g., [4], [6], [7], [10], [11]). Further there were considered interpolation operators on triangles with curved sides (one, two or all curved sides), many of them in connection with their applications in computer aided geometric design and in finite element analysis (see, e.g, [1], [2], [8], [9], [18], [19], [20]).

Keywords and phrases: interpolation operators, surfaces generation.
(2010)Mathematics Subject Classification: 41A05, 41A36, 65D05

In [18] the authors have considered a standard triangle, $\tilde{T}_h$, having the vertices $V_1 = (h, 0), V_2 = (0, h)$ and $V_3 = (0, 0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e. $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$ and F a real-valued function defined on $\tilde{T}_h$. (See Figure 1).

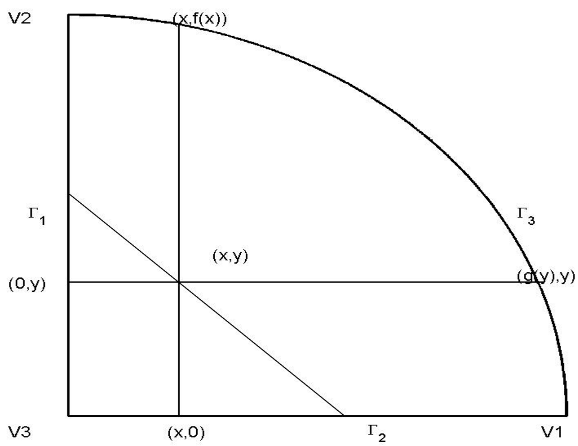

Figure 1.

They constructed certain Lagrange, Hermite and Birkhoff type operators, which interpolate a given function and some of its derivatives on the border of this triangle with one curved side, as well as some of their product and Boolean sum operators.

In [1] we have introduced an Lagrange operator which interpolates the function F on cathetus, on the curved side, but also on an interior line of the triangle $\tilde{T}_h$. We considered the case when the interior line is a median. Then in [2] we have introduced Hermite and Birkhoff type operators which interpolate a given function and some of its derivatives on median of the same triangle (see Figure 2).

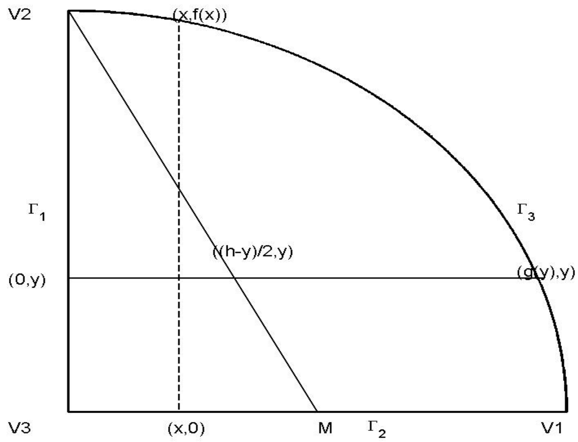

Figure 2.

The aim of this paper is to use some interpolation operators defined by us in [1], [2] and by the authors of [18], for construction of surfaces which satisfy some given conditions such as, for example, the roof of the halls (see, e.g., [3], [13]-[16]).

2. Surfaces generated by Lagrange, Hermite and Birkhoff type operators

Suppose that F is a real-valued function defined on $\tilde{T}_h$, and it has all partial derivatives needed.

Let us consider the Lagrange interpolation operators L_1^y and L_2^x defined by

$$(L_1^y F)(x,y) = \frac{f(x)-y}{f(x)} F(x,0) + \frac{y}{f(x)} F(x, f(x)),$$

$$(L_2^x F)(x,y) = \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}$$

$$F(0,y) + \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]} F\left(\frac{h-y}{2}, y\right) + \frac{x(2x-h-y)}{g(y)[2g(y)-h+y]} F(g(y),y),$$

the Hermite interpolation operators H_3^y and H_2^x, corresponding to the double nodes, defined by

$(H_3^y F)(x,y) = \frac{[f(x-y)]^2[f(x)+2y]}{f^3(x)}F(x,0) + \frac{y[f(x)-y]^2}{f^2(x)}F^{(0,1)}(x,0)$
$+ \frac{y^2[3f(x)-2y]}{f^3(x)}F(x,f(x)) + \frac{y^2[y-f(x)]}{f^2(x)}F^{(0,1)}(x,f(x))$
and
$(H_2^x F)(x,y) = \frac{(2x-h+y)^2[x-g(y)]^2[g(y)(y-h)+2x(y-h-2g(y))]}{(y-h)^3g^3(y)}F(0,y)$
$+ \frac{x(2x-h+y)^2[x-g(y)]^2}{(h-y)^2g^2(y)}F^{(1,0)}(0,y)$
$+ \frac{16x^2[x-g(y)]^2[(h-y)(h-y-2g(y))-2(2x-h+y)^2(h-y-g(y))]}{(h-y)^3[h-y-2g(y)]^3}$
$F\left(\frac{h-y}{2},y\right) + \frac{8x^2[x-g(y)^2](2x-h+y)}{(h-y)^2[h-y-2g(y)]^2}F^{(1,0)}\left(\frac{h-y}{2},y\right)$
$+ \frac{x^2(2x-h+y)^2[g(y)(10g(y)-3h+3y)+2x(h-y-4g(y))]}{g^3(y)[2g(y)-h+y]^3}F(g(y),y)$
$+ \frac{x^2[x-g(y)](2x-h+y)^2}{g^2(y)[2g(y)-h+y]^2}F^{(1,0)}(g(y),y),$
and the Birkhoff interpolation operators B_1^y and B_1^x defined by
$(B_1^y F)(x,y) = F(x,0) + yF^{(0,1)}(x,f(x)),$
$(B_1^x F)(x,y) = F(0,y)+\frac{x[2g(y)-x]}{2g(y)-h+y}F^{(1,0)}\left(\frac{h-y}{2},y\right)+\frac{x(x-h+y)}{2g(y)-h+y}F^{(1,0)}(g(y),y).$

A. Using the natural condition that the roof stays on its support, i.e.,

$$F|_{\Gamma_3} = 0,$$
$$F|_{V_2M} = 0,$$

we get that
$(L_1^y F)(x,y) = \frac{f(x)-y}{f(x)}F(x,0),$
$(L_2^x F)(x,y) = \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y),$

$(H_3^y F)(x,y) = \frac{[f(x-y)]^2[f(x)+2y]}{f^3(x)}F(x,0) + \frac{y[f(x)-y]^2}{f^2(x)}F^{(0,1)}(x,0)$
$+ \frac{y^2[y-f(x)]}{f^2(x)}F^{(0,1)}(x,f(x)),$
$(H_2^x F)(x,y) = \frac{(2x-h+y)^2[x-g(y)]^2[g(y)(y-h)+2x(y-h-2g(y))]}{(y-h)^3g^3(y)}F(0,y)$
$+ \frac{x(2x-h+y)^2[x-g(y)]^2}{(h-y)^2g^2(y)}F^{(1,0)}(0,y)$
$+ \frac{8x^2[x-g(y)^2](2x-h+y)}{(h-y)^2[h-y-2g(y)]^2}F^{(1,0)}\left(\frac{h-y}{2},y\right)$
$+ \frac{x^2[x-g(y)](2x-h+y)^2}{g^2(y)[2g(y)-h+y]^2}F^{(1,0)}(g(y),y).$

We consider the blending function generated by the Boolean sum of the operators L_1^y and L_2^x, i.e.,
$((L_1^y \oplus L_2^x)F)(x,y) = \frac{f(x)-y}{f(x)}F(x,0) + \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)$
$- \frac{f(x)-y}{f(x)}\frac{(2x-h)(x-h)}{h^2}F(0,0).$

In order to obtain a scalar approximation for F, in the second level we use the following approximations:

$F(0,y) := (H_3^y F)(0,y) = \frac{(h-y)^2(h+2y)}{h^3} F(0,0) + \frac{y(h-y)^2}{h^2} F^{(0,1)}(0,0)$
$+ \frac{y^2(y-h)}{h^2} F^{(0,1)}(0,h)$,
$F(x,0) := (H_2^x F)(x,0) = \frac{(2x-h)^2(x-h)^2(h+6x)}{h^5} F(0,0) +$
$\frac{x(2x-h)^2(x-h)^2}{h^4} F^{(1,0)}(0,0)$
$+ \frac{8x^2(x-h)^2(2x-h)}{h^4} F^{(1,0)}\left(\frac{h}{2},0\right) +$
$\frac{x^2(x-h)(2x-h)^2}{h^4} F^{(1,0)}(h,0)$.

We consider the interpolation operator

$$P_1 = L_1^y H_2^x + L_2^x H_3^y - L_1^y L_2^x,$$

with

$$(P_1F)(x,y) = \Bigg[\frac{(2x-h+y)(x-g(y))}{(h-y)g(y)} \frac{(h-y)^2(h+2y)}{h^3}$$
$$+ \frac{f(x)-y}{f(x)} \frac{(2x-h)^2(x-h)^2(h+6x)}{h^5} - \frac{f(x)-y}{f(x)} \frac{(2x-h)(x-h)}{h^2} \Bigg] F(0,0)$$
$$+ \frac{f(x)-y}{f(x)} \frac{x(2x-h)^2(x-h)^2}{h^4} F^{(1,0)}(0,0)$$
$$+ \frac{f(x)-y}{f(x)} \frac{8x^2(x-h)^2(2x-h)}{h^4} F^{(1,0)}\left(\frac{h}{2},0\right)$$
$$+ \frac{f(x)-y}{f(x)} \frac{x^2(x-h)(2x-h)^2}{h^4} F^{(1,0)}(h,0)$$
$$+ \frac{(2x-h+y)(x-g(y))}{(h-y)g(y)} \frac{y(h-y)^2}{h^2} F^{(0,1)}(0,0)$$
$$+ \frac{(2x-h+y)(x-g(y))}{(h-y)g(y)} \frac{y^2(y-h)}{h^2} F^{(0,1)}(0,h).$$

Example 1. *Consider the function* $f(x) = \sqrt{h^2 - x^2}$ *with* $h = 4$ *and* $F : \tilde{T}_h \to \mathbb{R}$. *In Figure 3 we plot the graph of the surface* P_1F *assigning to the data* $(F(0,0), F^{(1,0)}(0,0), F^{(1,0)}\left(\frac{h}{2},0\right), F^{(1,0)}(h,0), F^{(0,1)}(0,0), F^{(0,1)}(0,h))$ *the values* $(4,-1,-1,-1,-1,1)$.

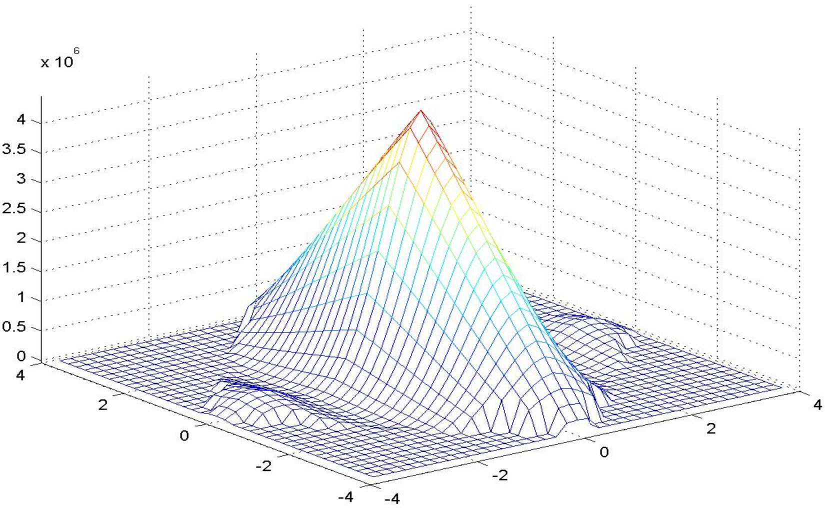

Figure 3.

B. Using the conditions

$$F|_{\Gamma_3} = F^{(0,1)}|_{\Gamma_3} = F^{(1,0)}|_{\Gamma_3} = 0, F|_{V_2M} = F^{(1,0)}|_{V_2M} = 0,$$

we get that

$(L_1^y F)(x,y) = \frac{f(x)-y}{f(x)} F(x,0),$

$(L_2^x F)(x,y) = \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)} F(0,y),$

$(H_3^y F)(x,y) = \frac{[f(x)-y]^2[f(x)+2y]}{f^3(x)} F(x,0) + \frac{y[f(x)-y]^2}{f^2(x)} F^{(0,1)}(x,0),$

$(H_2^x F)(x,y) = \frac{(2x-h+y)^2[x-g(y)]^2[g(y)(y-h)+2x(y-h-2g(y))]}{(y-h)^3 g^3(y)} F(0,y)$

$+ \frac{x(2x-h+y)^2[x-g(y)]^2}{(h-y)^2 g^2(y)} F^{(1,0)}(0,y).$

We obtain the interpolation operator

$$P_2 = L_1^y H_2^x + L_2^x H_3^y - L_1^y L_2^x,$$

with

$$(P_2F)(x,y) = \Bigg[\frac{(2x-h+y)(x-g(y))}{(h-y)g(y)} \frac{(h-y)^2(h+2y)}{h^3}$$

$$+ \frac{f(x)-y}{f(x)} \frac{(2x-h)^2(x-h)^2(h+6x)}{h^5} - \frac{f(x)-y}{f(x)} \frac{(2x-h)(x-h)}{h^2} \Bigg] F(0,0)$$

$+ \frac{f(x)-y}{f(x)} \frac{x(2x-h)^2(x-h)^2}{h^4} F^{(1,0)}(0,0)$
$+ \frac{(2x-h+y)(x-g(y))}{(h-y)g(y)} \frac{y(h-y)^2}{h^2} F^{(0,1)}(0,0).$

Example 2. *Consider the function* $f(x) = \sqrt{h^2 - x^2}, h = 4$ *and* $F : \tilde{T}_h \to \mathbb{R}$. *In Figure 4 we plot the graph of the surface* P_2F *assigning to the data*
$(F(0,0), F^{(1,0)}(0,0), F^{(0,1)}(0,0))$ *the values* $(4,0,0)$.

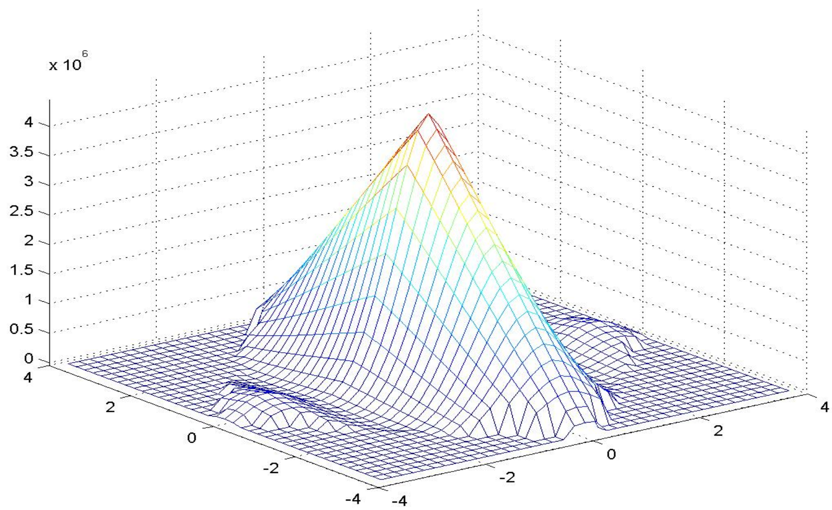

Figure 4.
C. We consider the conditions

$$F|_{\Gamma_3} = 0, F|_{V_2M} = 0,$$

and the interpolation operator
$P_3 = L_1^y B_1^x + L_2^x B_1^y - L_1^y L_2^x$,
with

$$(P_3F)(x,y) = \left[\frac{f(x)-y}{f(x)} + \frac{(2x-h+y)(x-g(y))}{(h-y)g(y)} - \frac{f(x)-y}{f(x)} \frac{(2x-h)(x-h)}{h^2} \right] F(0,0)$$

$$+\frac{f(x)-y}{f(x)}\frac{x(2h-x)}{h+y}F^{(1,0)}\left(\frac{h}{2},0\right)+\frac{f(x)-y}{f(x)}\frac{x(x-h)}{h+y}F^{(1,0)}(h,0)$$
$$+\frac{y(2x-h+y)(x-g(y))}{(h-y)g(y)}F^{(0,1)}(0,h).$$

Example 3. *Consider the function* $f(x)=\sqrt{h^2-x^2}$ *with* $h=4$ *and* $F:\tilde{T}_h\to\mathbb{R}$*. In Figure 5 we plot the graph of the surface* P_3F *assigning to the data* $(F(0,0),F^{(1,0)}\left(\frac{h}{2},0\right),\ F^{(1,0)}(h,0)),F^{(0,1)}(0,h))$ *the values* $(-1,-1,-1,1)$.

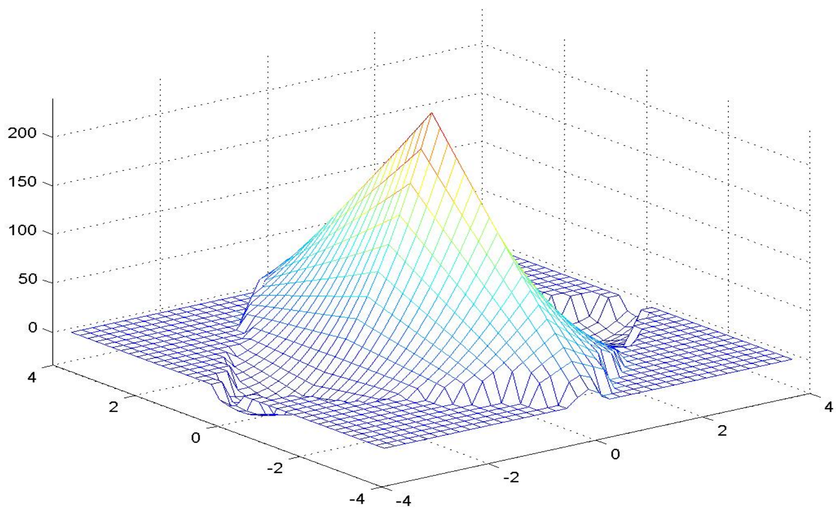

Figure 5.

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