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ON $\delta-$SMALL SOFT SUBMODULES

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Abstract. In this paper, we will define $\delta-$small soft submodules and investigate the properties of them by using soft module theory. Moreover, we will introduce the notion of $\delta-$radical of a soft module and will prove some basic properties of $\delta-$radicals of a soft module.

1. Introduction and Preliminaries

Several problems with applications to economics, engineering, environment science, social sciences, medical sciences and other fields cannot be solved by classical mathematical methods since they have inherent difficulties due to the inadequacy of the parametrization tools. Exact solutions for the mathematical models are needed in classical mathematics. If the model is so complicated, then it is not easy to find an exact solution. To handle these kind of situations, many tools have been suggested. Probability theory, fuzzy sets, rough sets and other mathematical tools have their inherent difficulties.

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Consequently, Russian researcher Molodtsov proposed a new completely approach, which is so-called soft set theory. A soft set can be considered as an approximate description of an object precisely consisting of two parts, such as predicate and approximate value set. Now, work on the soft set theory is progressing rapidly.

In [10], Maji et al. described the application of soft set theory to a decision making problem using rough sets and they also published a detailed theoretical study on soft sets in [11].

In [7], a new definition of soft set parametrization reduction is presented. Rosenfeld proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets in [14]. Aktaş and Çağman in [2] defined soft groups and compared soft sets with fuzzy sets and rough sets. After giving the definition of fuzzy soft group by some authors in [6,9], soft semirings and soft rings are introduced in [8] and [1], respectively. In addition, soft isomorphism theorems of soft groups were studied in [4].

Qiu-Mei Sun et al. defined soft modules and investigated their basic properties in [15]. Following this, Atagun and Sezgin studied soft substructures of rings and modules in [5].

Sums and direct sums of soft submodules of a soft module are introduced by Türkmen and Pancar using soft set theory in [16]. Also small soft submodules of a soft module and basic properties of small submodules are studied in there.

Throughout the paper, R will be an associative ring with identity and all modules will be unital left $R-$modules unless otherwise specified. Let M be an $R-$module. Recall that a submodule N of M shown by $N \leq M$ is called *small* and denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. A submodule L of M is said to be *essential* in M and denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \leq M$. A module M is said to be *singular* if $M \cong \frac{N}{L}$ for some module N and a submodule $L \leq N$ with $L \trianglelefteq N$. Also we will denote Jacobson radical of M with $Rad(M)$.

By Zhou [19], a submodule L of M is called $\delta-$*small* in M and denoted by $L \ll_\delta M$ if for any submodule N of M with $\frac{M}{N}$ singular and $M = N + L$ implies that $M = N$. The sum of $\delta-$small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is $\delta-$small in M, so $Rad(M) \subseteq \delta(M)$ and $Rad(M) = \delta(M)$ if M is singular. Also any non-singular semisimple submodule of M is $\delta-$small in M and $\delta-$small submodules of a singular module are small submodules.

The main purpose of this paper is to define the notion of $\delta-$small soft submodules and to investigate their properties by using soft module theory.

1.1. $\delta-$**Small Submodules.** The following definition and two lemmas with their proofs can be found in [19].

Definition 1.1. *For a ring R and a left $R-$module M, a submodule N of M is said to be $\delta-$small in M and denoted by $N \ll_\delta M$, provided $M \neq N + X$ for any proper submodule X of M with $\frac{M}{X}$ singular.*

Lemma 1.1. *For any module M, the following are satisfied:*
(1) For submodules N, K, L of M with $K \leq N$, we have
(i) $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $\frac{N}{K} \ll_\delta \frac{M}{K}$,
(ii) $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.
(2) If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \leq N$, then $K \ll_\delta N$.
(3) Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.

Lemma 1.2. *Let M and N be two $R-$modules.*
(1) $\delta(M) = \sum \{L \leq M : L$ is a $\delta-$ small submodule of $M\}$.
(2) If $f : M \to N$ is a homomorphism, then $f(\delta(M)) \leq \delta(N)$.
(3) If $M = \bigoplus\limits_{i \in I} M_i$, then $\delta(M) = \bigoplus\limits_{i \in I} \delta(M_i)$.

1.2. **Soft Sets.** Firstly, let us recall some basic concepts of soft set theory. Throughout this section U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A \subset E$. Molodtsow defined the soft set in the following manner in [12].

Definition 1.2. [12] *A pair (F, A) is called a soft set over U if and only if F is a mapping given by $F : A \to P(U)$.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For any $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$elements of the soft set (F, A), or as the set of $\varepsilon-$approximate elements of the soft set.

Definition 1.3. [11] *For any two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) and write $(F, A) \widetilde{\subset} (G, B)$, if*

(i) $A \subset B$,

(ii) for any $\varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

(F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A).

Definition 1.4. [11] *Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).*

Definition 1.5. [11] *The union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) where $C = A \cup B$ and*

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This relationship is denoted by $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

Definition 1.6. [11] *The intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cap B$ and $H(\varepsilon) = F(\varepsilon)$ or $G(\varepsilon)$ for each $\varepsilon \in C$ (as both are same set). This relationship is denoted by $(F, A) \widetilde{\cap} (G, B) = (H, C)$.*

Pei and Miao gave an alternative definition for the intersection of soft sets as following in [13].

Definition 1.7. *Let (F, A) and (G, B) be two soft sets over a common universe U. The intersection of (F, A) and (G, B) is denoted by $(F, A) \cap (G, B)$, and is defined as $(F, A) \cap (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.*

Since the notation of soft set intersection of Pei and Miao is similar to that of set theory, this may mislead the readers, Sezgin and Atagun denoted the intersection of (F, A) and (G, B) by $(F, A) \Cap (G, B)$ as Ali et al. used in [3]. In addition to the above definition, Ali et al. introduced a new definition for intersection which is called restricted intersection as follows:

Definition 1.8. *Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The restricted intersection of (F, A) and (G, B) is denoted by $(F, A) \Cap (G, B)$, and is defined as $(F, A) \Cap (G, B) = (H, C)$, where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.*

1.3. **Soft Modules.** Let A be a nonempty set and (F, A) be a soft set over the module M.

Definition 1.9. [15] *Let (F, A) and (G, B) be two soft modules over M. Then (G, B) is a soft submodule of (F, A) if* *(i)* $B \subseteq A$,

(ii) $G(x) \leq F(x)$, *for all* $x \in B$. *This is denoted by* $(G, B) \widetilde{\leq} (F, A)$.

Proposition 1.1. [15] *Let* (F, A) *and* (G, B) *be two soft modules over* M. *Then* (G, B) *is soft submodule of* (F, A) *if* $G(x) \leq F(x)$ *for all* $x \in A$.

Definition 1.10. [15] *Let* (F, A), (G, B) *be two soft modules over* M, N *respectively and* $f : M \to N$, $g : A \to B$ *be two functions. Then we say that* (f, g) *is a soft homomorphism if the following conditions are satisfied:*

(1) $f : M \to N$ *is a homomorphism of modules,*

(2) $g : A \to B$ *is mapping,*

(3) $f(F(x)) = G(g(x))$ *for all* $x \in A$.

Under the assumptions of Definition 1.10, we say (F, A) is soft homomorphic to (G, B) and denote it by $(F, A) \simeq (G, B)$.

Definition 1.11. [16] *Let* (F, A) *be a soft module over* M *and* $\{(F_i, A_i)\}_{i \in I}$ *be any collection of soft submodules satisfying* $(F_i, A_i) \widetilde{\leq} (F, A)$ *where* I *is a nonempty set. The sum of the soft submodules* (F_i, A_i) *of* (F, A) *is defined as* $\widetilde{\sum\limits_{i \in I}} (F_i, A_i) = \left(H, \bigcup\limits_{i \in I} A_i\right)$ *for all* $a \in \bigcup\limits_{i \in I} A_i$ *such that* $H(a) = \sum\limits_{i \in I(a)} F_i(a)$ *where* $I(a)$ *is the set of all elements* $i \in I$ *such that* $a \in A_i$.

Definition 1.12. [16] *Let* (F, A) *be a soft module over* M *and* $(G, B) \widetilde{\leq} (F, A)$. *If there exists a soft submodule* (T, C) *of* (F, A) *such that* $(G, B) \widetilde{+} (T, C) = (F, A)$ *and* $(G, B) \Cap (T, C)$ *is trivial, then* (G, B) *is said to be a direct summand of* (F, A) *and denoted by* $(G, B) \widetilde{\oplus} (T, C) = (F, A)$.

Theorem 1.1. [16] *Let* (G, A), (T, A) *be two soft submodules of* (F, A) *over* M. *Then* $(G, A) \widetilde{\oplus} (T, A) = (F, A)$ *if and only if* $G(a) \oplus T(a) = F(a)$ *for every element* $a \in A$.

Definition 1.13. [16] *Let* (F, A) *be a soft module over* M *and* $(G, B) \widetilde{\leq} (F, A)$. *Then* (G, B) *is called small in* (F, A) *and denoted by* $(G, B) \widetilde{\ll} (F, A)$, *if* $G(b)$ *is a small submodule of* $F(b)$ *for every element* $b \in B$.

2. Properties of $\delta-$Small Soft Submodules

Definition 2.1. *Let* (F, A) *be a soft module over* M *and* (G, B) *be a soft submodule of* (F, A). *We say that* (G, B) *is* $\delta-$*small in* (F, A) *and* *we denote this by* $(G,B)\widetilde{\ll}_\delta(F,A)$*, if* $G(b)$ *is a* $\delta-$*small submodule of* $F(b)$ *for every element* $b\in B$*.*

Definition 2.2. *Let* (F,A) *and* (H,A) *be two soft modules over* M *such that* $H(a)\leq M$ *for all* $a\in A$*. The sum of the soft modules* (F,A) *and* (H,A) *is defined as* $(F,A)+(H,A)=(K,A)$ *where* $F(a)+H(a)=K(a)$ *for all* $a\in A$*.*

Theorem 2.1. *Let* (F,A) *and* (H,A) *be two soft modules over* M *and* $H(a)\leq M$ *for all* $a\in A$*. Then the sum* $(F,A)+(H,A)$ *is a soft module over* M*.*

Proof. If (F,A) and (H,A) are soft modules over M then $F(a),H(a)\leq M$ for all $a\in A$. Since $F(a),H(a)$ are submodules $F(a)+H(a)$ is also a submodule of M for all $a\in A$. Finally the sum $(F,A)+(H,A)$ is a soft module over M. ∎

Let us define a mapping $\frac{F}{H}$ as $\left(\frac{F}{H}\right)(a)=\frac{F(a)}{H(a)}$ such that it goes from A to the set of quotient module of M. The module $\frac{F(a)}{H(a)}$ induces a submodule of M as $\{x+y:x\in F(a)\ ,y\in H(a)\}$. In place of $\frac{F(a)}{H(a)}$, if we consider the submodule $\{x+y:x\in F(a)\ ,y\in H(a)\}$, then $\left(\frac{F}{H},A\right)$ became a soft module over M.

Lemma 2.1. *Let* (F,A) *be a soft module over* M*. For soft submodules* (G,A)*,* (H,A) *and* (K,A) *of* (F,A) *with* $(H,A)\,\widetilde{\subseteq}\,(G,A)$*, we have* $(G,A)\widetilde{\ll}_\delta(F,A)$ *if and only if* $(H,A)\widetilde{\ll}_\delta(F,A)$ *and* $\left(\frac{G}{H},A\right)\widetilde{\ll}_\delta\left(\frac{F}{H},A\right)$*.*

Proof. $(\Rightarrow)$ Suppose that $(G,A)\widetilde{\ll}_\delta(F,A)$. Then for every $a\in A$, we get $G(a)\ll_\delta F(a)$. Since $(H,A)\,\widetilde{\subseteq}\,(G,A)$, we have $H(a)\subseteq G(a)\ll_\delta F(a)$ for every $a\in A$ and obtain $(H,A)\widetilde{\ll}_\delta(F,A)$. On the other hand, using Lemma 1.1 (1) (i) $\frac{G(a)}{H(a)}\ll_\delta\frac{F(a)}{H(a)}$ implies that $\left(\frac{G}{H},A\right)\widetilde{\ll}_\delta\left(\frac{F}{H},A\right)$.

$(\Leftarrow)$ Assume that $(H,A)\widetilde{\ll}_\delta(F,A)$ and $\left(\frac{G}{H},A\right)\widetilde{\ll}_\delta\left(\frac{F}{H},A\right)$. Then we can write $H(a)\ll_\delta F(a)$ and $\frac{G(a)}{H(a)}\ll_\delta\frac{F(a)}{H(a)}$ for every $a\in A$. By Lemma 1.1(1)(i), we have $G(a)\ll_\delta F(a)$ for every $a\in A$ and so $(G,A)\widetilde{\ll}_\delta(F,A)$. ∎

Lemma 2.2. *Let* (F,A) *be a soft module over* M*. Then for soft submodules* (G,B)*,* (H,C) *of* (F,A)*, we have* $(G,B)\,\widetilde{+}\,(H,C)\widetilde{\ll}_\delta(F,A)$ *if and only if* $(G,B)\widetilde{\ll}_\delta(F,A)$ *and* $(H,C)\widetilde{\ll}_\delta(F,A)$*.*

Proof. ($\Rightarrow$) Let $(G,B)\,\widetilde{+}\,(H,C)\,\widetilde{\ll}_\delta\,(F,A)$. Since $(G,B)\,\widetilde{+}\,(H,C) = (K,B\cup C)$, we have $K(a)\ll_\delta F(a)$ for every $a\in A$ where

$$K(a)=\begin{cases} G(a), & a\in B-C\\ H(a), & a\in C-B\\ G(a)+H(a), & a\in B\cap C\end{cases}.$$

If we take an element from $B-C$ or from $C-B$ or from $B\cap C$ respectively, then we get $(G,B)\,\widetilde{+}\,(H,C) = K(a) = G(a) \ll_\delta F(a)$ or $(G,B)\,\widetilde{+}\,(H,C) = K(a) = H(a) \ll_\delta F(a)$ or $(G,B)\,\widetilde{+}\,(H,C) = K(a) = G(a)+H(a) \ll_\delta F(a)$ respectively. If we remember Lemma 1.1 (1) (ii), we obtain $G(a)\ll_\delta F(a)$ and $H(a)\ll_\delta F(a)$. Therefore $(G,B)\,\widetilde{\ll}_\delta\,(F,A)$ and $(H,C)\,\widetilde{\ll}_\delta\,(F,A)$ for every $b\in B$, $c\in C$.

($\Leftarrow$) Let $(G,B)\,\widetilde{\ll}_\delta\,(F,A)$ and $(H,C)\,\widetilde{\ll}_\delta\,(F,A)$. Then $G(b)\ll_\delta F(b)$ and $H(c)\ll_\delta F(c)$ for every $b\in B$, $c\in C$.
To prove $(G,B)\,\widetilde{+}\,(H,C) = (K,B\cup C)\,\widetilde{\ll}_\delta\,(F,A)$, we must show $K(x)\ll_\delta F(x)$ for every $x\in B\cup C$. If we take an element x from $B-C, C-B$ and $B\cap C$ in the order, then we get $K(x)=G(x)\ll_\delta F(x)$, $K(x)=H(x)\ll_\delta F(x)$ and $K(x)=G(x)+H(x)\ll_\delta F(x)$ by Lemma 1.1(1)(ii). ∎

Lemma 2.3. *Let (F,A) and (H,C) be soft modules over M and N respectively. If $(G,B)\,\widetilde{\ll}_\delta\,(F,A)$ and $\left(\widetilde{f,g}\right):(F,A)\to(H,C)$ is a soft homomorphism, then $\left(\widetilde{f,g}\right)(G,B)\,\widetilde{\ll}_\delta\,(H,C)$. In particular, if $(G,B)\,\widetilde{\ll}_\delta\,(F,A)\,\widetilde{\subseteq}\,(K,D)$, then we have $(G,B)\,\widetilde{\ll}_\delta\,(K,D)$.*

Proof. Let $(G,B)\,\widetilde{\ll}_\delta\,(F,A)$. Then we have $f\left(G\left(b\right)\right)\widetilde{\ll}_\delta f\left(F\left(b\right)\right) = H\left(g\left(b\right)\right)$ for every $b\in B$. This means that $\left(f\left(G\right),G\left(B\right)\right)\widetilde{\ll}_\delta\,(H,C)$. In particular, suppose that $(G,B)\,\widetilde{\ll}_\delta\,(F,A)\,\widetilde{\subseteq}\,(K,D)$. Then we can write $G\left(b\right)\ll_\delta F(b)$ for every $b\in B$. On the other hand, since $B\subseteq A\subseteq D$ and $F(a)\subseteq K(a)$, we obtain $G\left(b\right)\ll_\delta F(b)\subseteq K(b)$ and so $G\left(b\right)\ll_\delta K(b)$ by Lemma 1.1(2). So $(G,B)\,\widetilde{\ll}_\delta\,(K,D)$. ∎

Lemma 2.4. *Let $(G_1,A)\,\widetilde{\leq}\,(H_1,A)\,\widetilde{\leq}\,(F,A)$, $(G_2,A)\,\widetilde{\leq}\,(H_2,A)\,\widetilde{\leq}\,(F,A)$ and $(F,A)=(H_1,A)\,\widetilde{\oplus}\,(H_2,A)$. Then we get*
$(G_1,A)\,\widetilde{\oplus}\,(G_2,A)\,\widetilde{\ll}_\delta\,(H_1,A)\,\widetilde{\oplus}\,(H_2,A)$ if and only if $(G_1,A)\,\widetilde{\ll}_\delta\,(H_1,A)$ and $(G_2,A)\,\widetilde{\ll}_\delta\,(H_2,A)$.

Proof. By the assumption, for every $a\in A$ we have $G_1(a)\leq H_1\left(a\right)\leq F\left(a\right)$, $G_2(a)\leq H_2\left(a\right)\leq F\left(a\right)$ and $(F,a)=(H_1,a)\,\widetilde{\oplus}\,(H_2,a)$. Using Lemma 1.1(3), we obtain $G_1\left(a\right)\oplus G_2\left(a\right)\ll_\delta H_1\left(a\right)\oplus H_2\left(a\right)$ if and only if $G_1(a)\ll_\delta H_1\left(a\right)$ and $G_2(a)\ll_\delta H_2\left(a\right)$. ∎

Definition 2.3. *Let (F, A) be a soft module over M. $\delta-$radical of (F, A) is defined as (H, A), and denoted by $(H, A) = \delta_M(F, A)$ where $H(a) = \delta(F(a))$ for all $a \in A$.*

Proposition 2.1. *Let (F, A) be a soft module over M. Then $\delta_M(F, A)$ is a soft submodule of (F, A).*

Proof. It is obvious that $\delta_M(F, A)$ is a soft module over the module M. Nevertheless, we have $\delta_M(F, A) \widetilde{\leq} (F, A)$ for every $a \in A$ since $\delta(F(a)) \leq F(a)$. ∎

Proposition 2.2. *Let (F, A) be a soft module over M and $(G, B) \widetilde{\leq} (F, A)$. Then $\delta_M(G, B) \widetilde{\leq} \delta_M(F, A)$.*

Proof. Assume that $(G, B) \widetilde{\leq} (F, A)$ and $b \in B$. Then $G(b) \leq F(b)$. Hence, we have $\delta(G(b)) \leq \delta(F(b))$ for every $b \in B$ and
$\delta_M(G, B) \widetilde{\leq} \delta_M(F, A)$. ∎

Theorem 2.2. *Let (F, A) be a soft module over M. If $(F, A) = \widetilde{\bigoplus_{i \in I}} (F_i, A_i)$, then $\delta_M(F, A) = \widetilde{\bigoplus_{i \in I}} \delta_M(F_i, A_i)$.*

Proof. For an arbitrary $R-$module K, we know that if $K = \bigoplus_{i \in I} K_i$, then $\delta(K) = \bigoplus_{i \in I} \delta(K_i)$. So the proof is obvious by Lemma 1.2(3). ∎

Proposition 2.3. *Let (F, A) be a soft module over M. If M is a noetherian module, then $\delta_M(F, A) \widetilde{\ll}_\delta (F, A)$.*

Proof. Let M be a noetherian module. Since every submodule of M is finitely generated, then we have $\delta(F(a)) \ll_\delta F(a)$ for every $a \in A$. So according to Definition 2.3, we have $\delta_M(F, A) \widetilde{\ll}_\delta (F, A)$. ∎

Theorem 2.3. *Every small soft submodule is a $\delta-$small soft submodule.*

Proof. Since every small submodule of a module M is $\delta-$small in M, the result follows. ∎

Now, we give an example of $\delta-$small soft submodule that is not small soft submodule.

Example 1. [18] *Let $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ be $\mathbb{Z}_6$-module and (F, A) be a soft set over $\mathbb{Z}_6$, where $A = \mathbb{Z}_6$ and $F : A \to P(\mathbb{Z}_6)$ is defined by $F(x) = \{y \in \mathbb{Z}_6 | \ x\rho y \Leftrightarrow x + y \in \mathbb{Z}\}$ for all $x \in A$. Then $F(\overline{0}) = F(\overline{1}) = F(\overline{2}) = F(\overline{3}) = F(\overline{4}) = F(\overline{5}) = \mathbb{Z}_6$ are submodules of $\mathbb{Z}_6$.*

Hence (F, A) is a soft module over $\mathbb{Z}_6$.
Let (H, B) be a soft set over $\mathbb{Z}_6$, where $B = \{\overline{0}, \overline{2}, \overline{4}\}$ and $H : B \rightarrow P(\mathbb{Z}_6)$ is defined by $H(x) = \{y \in Z_6 | \ x\rho y \Leftrightarrow x + y \in \{\overline{0}, \overline{2}, \overline{4}\}\}$ for all $x \in B$. Then $H(\overline{0}) = \{\overline{0}, \overline{2}, \overline{4}\}$ is a $\delta-$small submodule of $F(\overline{0})$, $H(\overline{2}) = \{\overline{0}, \overline{2}, \overline{4}\}$ is a $\delta-$small submodule of $F(\overline{2})$ and $H(\overline{4}) = \{\overline{0}, \overline{2}, \overline{4}\}$ is a $\delta-$small submodule of $F(\overline{4})$. Hence (H, B) is a soft submodule of (F, A). Moreover, (H, B) is a $\delta-$small submodule of (F, A) over $\mathbb{Z}_6$ but (H, B) is not small submodule of (F, A) [17].

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