"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 25(2015), No. 2, 63-70

ON UNIVALENCE OF SOME INTEGRAL OPERATORS WHICH PRESERVE THE CLASS $\mathcal{S}$

VIRGIL PESCAR AND LAURA STANCIU

Abstract In this paper, we obtain new sufficient conditions for two integral operators to be univalent in the open unit disk $\mathcal{U}$, using a new result on univalence of analytical functions. These integral operators were considered in a recent work [8].

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the following normalization condition $f(0) = f'(0) - 1 = 0$, $\mathbb{C}$ being the set of complex numbers.

Keywords and phrases: Analytic functions, integral operators, univalence conditions.
(2010) Mathematics Subject Classification: 30C45, 30C75.

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathcal{U}$.

In our present investigation, we study the univalence conditions for the following integral operators:

$$F_1(f,g)(z) = \left(\alpha \int_0^z \left(f(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{\frac{1}{\alpha}}, \quad f,g \in \mathcal{A}; \quad \alpha \in \mathbb{C} \tag{1}$$

and

(2)

$$H_1(f,g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}e^{g(t)}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}, \quad f,g \in \mathcal{A}; \quad \alpha,\beta \in \mathbb{C}-\{0\}$$

In the proof of our main results (Theorem 2.1 and Theorem 2.2), we need the following univalence criterion, which is asserted by Theorem 1.1 bellow; it was proven by Pescar [6].

Theorem 1.1. [6] *Let α be a complex number,* $\mathrm{Re}\alpha > 0$ *and c be a complex number, $|c| \leq 1$, $c \neq -1$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + ...$ If*

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}} = z + ...$$

is regular and univalent in $\mathcal{U}$.

In order to derive our main results, we have to recall the following theorem:

Theorem 1.2. [3] *If $f \in \mathcal{S}$ then*

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{1+|z|}{1-|z|}, \quad z \in \mathcal{U}. \tag{3}$$

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [4]).

Theorem 1.3. *(***General Schwarz Lemma***) (see* [4]*) Let the function $f(z)$ be regular in the disk*

$$\mathcal{U}_R = \{z : z \in \mathbb{C} and\, |z| < R\}, where R > 0.$$

with

$$|f(z)| < M, \qquad z \in \mathbb{C},$$

for a fixed number $M > 0$. If the function $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then

$$|f(z)| \leqq \frac{M}{R^m} |z|^m, \qquad z \in \mathcal{U}_R. \tag{4}$$

The equality in (4) *holds true only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is a real constant.

2. Main Results

Theorem 2.1. *Let $f, g \in \mathcal{S}$, where g satisfies the inequality $|g(z)| < M$, M is a real positive number and α, c are complex numbers with $|c| \leq 1$, $c \neq -1$.*

If

$$|\alpha - 1| \leq \frac{(1 - |c|) \operatorname{Re}\alpha}{4(M+1)}, \quad \operatorname{Re}\alpha \in (0,1) \tag{5}$$

or

$$|\alpha - 1| \leq \frac{1 - |c|}{4(M+1)}, \quad \operatorname{Re}\alpha \in [1, \infty), \tag{6}$$

then the function $F_1(f,g)(z)$ defined by (1) is in the class $\mathcal{S}$.

Proof. We begin by observing that the function $F_1(f,g)(z)$ in (1) can be rewritten as follows:

$$F_1(f,g)(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(\frac{f(t)}{t} e^{g(t)} \right)^{\alpha-1} dt \right)^{\frac{1}{\alpha}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)} \right)^{\alpha-1} dt.$$

The function h is regular in $\mathcal{U}$ and satisfies the following normalization condition $h(0) = h'(0) - 1 = 0$. Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$h'(z) = \left(\frac{f(z)}{z} e^{g(z)} \right)^{\alpha-1} \tag{7}$$

and

(8)
$$h''(z) = (\alpha - 1)\left(\frac{f(z)}{z}e^{g(z)}\right)^{\alpha-2}\left(\frac{zf'(z) - f(z)}{z^2}e^{g(z)} + \frac{f(z)}{z}g'(z)e^{g(z)}\right).$$

We easily find from (7) and (8) that

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1)\left[\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z)\right],$$

which readily shows that

$$\begin{aligned}
&|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
&= |c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|(\alpha - 1)\left(\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z)\right)\right| \\
&\le |c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|\alpha - 1|\left(\left|\frac{zf'(z)}{f(z)}\right| + 1 + \left|\frac{zg'(z)}{g(z)}\right||g(z)|\right).
\end{aligned} \tag{9}$$

From the hypothesis of Theorem 2.1, we have $|g(z)| < M$, using Theorem 1.3, we get $|g(z)| \le M\,|z|$ and because $f, g \in \mathcal{S}$ by Theorem 1.2 and (9), we obtain

$$\begin{aligned}
&|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\le |c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|\alpha - 1|\left(\frac{2}{1 - |z|} + \frac{1 + |z|}{1 - |z|}M\,|z|\right) \\
&\le |c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}2\,|\alpha - 1|\left(\frac{1 + M\,|z|}{1 - |z|}\right).
\end{aligned} \tag{10}$$

Suppose that $\mathrm{Re}\alpha \in (0, 1)$. Define a function $\Phi : (0, 1) \to \mathbb{R}$ by

$$\Phi(x) = 1 - a^{2x}, \quad 0 < a < 1.$$

Then Φ is an increasing function and consequently for $|z| = a$, $z \in \mathcal{U}$, we obtain

$$1 - |z|^{2\mathrm{Re}\alpha} < 1 - |z|^2, \quad z \in \mathcal{U}. \tag{11}$$

We thus find (10) and (11) that

$$\begin{aligned}
&|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\leq |c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2}}{\mathrm{Re}\alpha} 2\,|\alpha-1|\left(\frac{1+M\,|z|}{1-|z|}\right) \\
&\leq |c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1+|z|}{\mathrm{Re}\alpha} 2\,|\alpha-1|\,(1+M\,|z|) \\
&\leq |c| + \frac{4\,|\alpha-1|}{\mathrm{Re}\alpha}\,(M+1)\,.
\end{aligned}$$

Using the hypothesis (5) for $\mathrm{Re}\alpha \in (0,1)$, we readily get

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad z \in \mathcal{U}. \tag{12}$$

Suppose that $\mathrm{Re}\alpha \in [1,\infty)$. We define a function $\Phi : [1,\infty) \to \mathbb{R}$ by

$$\Phi(x) = \frac{1-a^{2x}}{x}, \quad 0 < a < 1.$$

We observe that the function Ψ is decreasing and consequently for $|z| = a$, $z \in \mathcal{U}$, we have

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \leq 1-|z|^{2}, \quad z \in \mathcal{U}. \tag{13}$$

It follows from (10) and (13) that

$$\begin{aligned}
&|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\leq |c|\,|z|^{2\mathrm{Re}\alpha} + \left(1-|z|^{2}\right) 2\,|\alpha-1|\left(\frac{1+M\,|z|}{1-|z|}\right) \\
&\leq |c| + 4\,|\alpha-1|\,(M+1)\,.
\end{aligned}$$

Using the hypothesis (6), when $\mathrm{Re}\alpha \in [1,\infty)$, we easily get

$$|c|\,|z|^{2\mathrm{Re}\alpha} + \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad z \in \mathcal{U}. \tag{14}$$

From (12) and (14), applying Theorem 1.1, we conclude that the integral operator $F_1(f,g)(z)$ defined by (1) is in the class $\mathcal{S}$. □

Theorem 2.2. *Let $f, g \in \mathcal{S}$, where g satisfies the inequality $|g(z)| < M$, M is a real positive number and α, c are complex numbers with*

$|c| \leq 1$, $c \neq -1$.

If

$$\frac{1}{|\alpha|} \leq \frac{(1-|c|)\,\mathrm{Re}\beta}{4\,(M+1)}, \quad \mathrm{Re}\beta \in (0,1) \tag{15}$$

or

$$\frac{1}{|\alpha|} \leq \frac{1-|c|}{4\,(M+1)}, \quad \mathrm{Re}\beta \in [1,\infty), \tag{16}$$

then the function $H_1(f,g)(z)$ defined by (2) is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t}e^{g(t)}\right)^{\frac{1}{\alpha}} dt. \tag{17}$$

The function h is regular in $\mathcal{U}$. From (17), we have

$$h'(z) = \left(\frac{f(z)}{z}e^{g(z)}\right)^{\frac{1}{\alpha}}$$

and

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\left(\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z)\right),$$

which readily shows that

$$|c|\,|z|^{2\mathrm{Re}\beta} + \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right|$$
$$\leq |c|\,|z|^{2\mathrm{Re}\beta} + \frac{1-|z|^{2\mathrm{Re}\beta}}{|\alpha|\,\mathrm{Re}\beta}\left(\left|\frac{zf'(z)}{f(z)}\right| + 1 + \left|\frac{zg'(z)}{g(z)}\right|\,|g(z)|\right). \tag{18}$$

From the hypothesis of Theorem 2.2, we have $|g(z)| < M$, using Theorem 1.3, we get $|g(z)| \leq M\,|z|$ and because $f, g \in \mathcal{S}$ by Theorem 1.2 and (18), we have

$$|c|\,|z|^{2\mathrm{Re}\beta} + \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right|$$
$$\leq |c|\,|z|^{2\mathrm{Re}\beta} + \frac{1-|z|^{2\mathrm{Re}\beta}}{|\alpha|\,\mathrm{Re}\beta}2\left(\frac{1+M\,|z|}{1-|z|}\right). \tag{19}$$

Suppose that $\mathrm{Re}\beta \in (0,1)$. Define a function $\Phi : (0,1) \to \mathbb{R}$ by

$$\Phi(x) = 1 - a^{2x}, \quad 0 < a < 1.$$

Then Φ is an increasing function and consequently for $|z| = a$, $z \in \mathcal{U}$, we obtain

$$1 - |z|^{2\mathrm{Re}\beta} < 1 - |z|^2, \quad z \in \mathcal{U}. \tag{20}$$

We thus find (10) and (20) that

$$\begin{aligned} &|c|\,|z|^{2\mathrm{Re}\beta} + \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \\ &\leq |c|\,|z|^{2\mathrm{Re}\beta} + \frac{2\left(1 - |z|^2\right)}{|\alpha|\,\mathrm{Re}\beta}\left(\frac{1 + M\,|z|}{1 - |z|}\right) \\ &\leq |c| + \frac{4\,(M+1)}{|\alpha|\,\mathrm{Re}\beta}. \end{aligned}$$

Using the hypothesis (15), for $\mathrm{Re}\beta \in (0, 1)$, we readily get

$$|c|\,|z|^{2\mathrm{Re}\beta} + \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad z \in \mathcal{U}. \tag{21}$$

Suppose that $\mathrm{Re}\beta \in [1, \infty)$. We define a function $\Phi : [1, \infty) \to \mathbb{R}$ by

$$\Phi(x) = \frac{1 - a^{2x}}{x}, \quad 0 < a < 1.$$

We observe that the function Φ is decreasing and consequently for $|z| = a$, $z \in \mathcal{U}$, we have

$$\frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \leq 1 - |z|^2, \quad z \in \mathcal{U}. \tag{22}$$

It follows from (10) and (22) that

$$\begin{aligned} &|c|\,|z|^{2\mathrm{Re}\beta} + \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| \\ &\leq |c|\,|z|^{2\mathrm{Re}\beta} + 2\left(1 - |z|^2\right)\left(\frac{1 + M\,|z|}{|\alpha|\,(1 - |z|)}\right) \\ &\leq |c| + \frac{4\,(M+1)}{|\alpha|}. \end{aligned}$$

Using the hypothesis (16), when $\mathrm{Re}\beta \in [1, \infty)$, we easily get

$$|c|\,|z|^{2\mathrm{Re}\beta} + \frac{1 - |z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad z \in \mathcal{U}. \tag{23}$$

From (21) and (23), applying Theorem 1.1, we conclude that the integral operator $H_1(f, g)(z)$ defined by (2) is in the class $\mathcal{S}$. $\square$

References

1. D. Blezu, **On univalence criteria**, General Math. 14(2006), 77-84.
2. D. Blezu, R. N. Pascu, **Univalence criteria for integral operators**, Glas. Mat. **36** (2001), 241-245.
3. P.L. Duren, **Univalent Functions,** Grundlehren der Mathematischen Wissenshaften. Bd. 259, Springer-Verlag, New York, 1983.
4. Z. Nehari, Conformal Mapping, Dover, New York, NY, USA, 1975.
5. V. Pescar, **On the univalence of some integral operators,** Gen. Math. **14** (2006), no. 2, 77-84.
6. V. Pescar, **New generalizations of Ahlfors's, Becker's and Pascu's univalence criterions,** Acta Universitatis Apulensis, no. 34 (2013), pp. 173-178.
7. V. Pescar, **Univalence of certain integral operators**, Acta Univ. Apul. 12 (2006), 43- 48.
8. L. Stanciu, **The univalence conditions of some integral operators,** Abstract and Applied Analysis, ID 924645, 9 pages.

Virgil Pescar
"Transilvania" University of Braşov
Faculty of Mathematics and Computer Science
Department of Mathematics
Braşov, România
E-mail address: *virgilpescar@unitbv.ro*

Laura Stanciu
University of Piteşti
Department of Mathematics
Argeş, România.
E-mail address: *laura_stanciu_30@yahoo.com*