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SUZUKI TYPE RESULT IN PARTIAL $G-$ METRIC SPACES USING RATIONAL CONTRACTIVE CONDITION

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Abstract

In this paper, we obtain a Suzuki type common fixed point theorem in partial G - metric spaces, for a pair of weakly compatible maps in the sense of Jungck, by using a rational contractive condition. We have given an example which supports our main result.

1. Introduction and Preliminaries

Matthews [11] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification.

Romaguera [10] proved that a partial metric space is 0-complete if and only if every Caristi type mapping on X has a fixed point. The result of Romaguera extended Kirk's characterization of metric completeness [17] to a kind of complete partial metric spaces.

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In 2005, Mustafa and Sims introduced a new class of generalized metric spaces (see [19], [20]) which are called G-metric spaces as generalization of metric space (X, d). Many fixed point results on such spaces for mappings satisfying different contractive conditons appeared in ([4], [5], [7], [9], [14] – [16], [18], [20] – [23]).

In [6], Salimi and Vetro introduced the concept of a partial G-metric space and proved some properties of partial $G-$ metric spaces.They also proved a fixed and common fixed point theorems of Suzuki type. Now we present the following necessary known definitions.

Definition 1.1. *([20]) Let X be a nonempty set and let $G : X \times X \times X \to \mathcal{R}^+$ be a function satisfying the following properties :*
(G_1)*: $G(x, y, z) = 0$ if $x = y = z$,*
(G_2)*: $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,*
(G_3)*: $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,*
(G_4)*: $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$,symmetry in all three variables,*
(G_5)*: $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.*
Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. *([11]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathcal{R}^+$ such that for all $x, y, z \in X$,*
(p_1) $p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
The pair (X, p) is called a partial metric space (PMS).

Definition 1.3. *([6]) Let X be a nonempty set. A function $P : X \times X \times X \to [0, +\infty)$ is called a partial G-metric if the following conditions are satisfied:*

(P_1) : $P(x, x, x) + P(y, y, y) + P(z, z, z) \leq 3P(x, y, z)$ *for all $x, y, z, \in X$,*
(P_2) : $\frac{1}{3}P(x, x, x) + \frac{2}{3}P(y, y, y) < P(x, y, y)$ *for all $x, y, \in X$ with $x \neq y$,*
(P_3) : $P(x, x, y) - \frac{1}{3}P(x, x, x) \leq P(x, y, z) - \frac{1}{3}P(z, z, z)$ *for all points $x, y, z \in X$ with $y \neq z$,*
(P_4) : $P(x, y, z) = P(x, z, y) = P(y, z, x) = \cdots$ *(symmetry in three variables),*
(P_5) : $P(x, y, z) \leq P(x, a, a) + P(a, y, z) - P(a, a, a)$ *for any $x, y, z, a \in X$.*

Then the pair (X, P) is called a partial G-metric space (in brief PGMS)

Remark 1.4. *It is shown in ([6]) that to every partial G-metric space we can associate a G-metric defined by* $G_P(x,y,z) = 3P(x,y,z) - P(x,x,x) - P(y,y,y) - P(z,z,z)$.

Example 1.5. *([6]) Let* $X = [0, +\infty)$ *and define* $P(x,y,z) = \frac{1}{3}(\max\{x,y\} + \max\{y,z\} + \max\{x,z\})$,*for all points* $x, y, z \in X$. *Then* (X, P) *is a PGMS.*

The following proposition gives some properties of a partial G-metric.

Proposition 1.6. *([6]) Let* (X, P) *be a PGMS. Then for all* $x, y, z, a \in X$, *the following properties hold:*

(i) *If* $P(x,y,z) = P(x,x,x) = P(y,y,y) = P(z,z,z)$, *then* $x = y = z$;

(ii) *If* $P(x,y,z) = 0$ *then* $x = y = z$;

(iii) *If* $x \neq y$, *then* $P(x,y,y) > 0$;

(iv) $P(x,y,z) \leq P(x,x,y) + P(x,x,z) - P(x,x,x)$;

(v) $P(x,y,y) \leq 2P(x,x,y) - P(x,x,x)$;

(vi) $P(x,y,z) \leq P(x,a,a) + P(y,a,a) + P(z,a,a) - 2P(a,a,a)$;

(vii) $P(x,y,z) \leq P(x,a,z) + P(a,y,z) - \frac{2}{3}P(a,a,a) - \frac{1}{3}P(z,z,z)$ *if* $y \neq z$;

(viii) $P(x,y,y) \leq P(x,y,a) + P(a,y,y) - \frac{2}{3}P(a,a,a) - \frac{1}{3}P(y,y,y)$ *if* $x \neq y$.

Definition 1.7. *([6]) Let* (X, P) *be a PGMS. Then*

(1) *A sequence* $\{x_n\}$ *is called* $P - G$*-convergent to* $x \in X$, *if and only if*

$$P(x,x,x) = \lim_{n\to+\infty} P(x,x,x_n) = \lim_{n\to+\infty} P(x,x_n,x_n).$$

(2) *A sequence* $\{x_n\}$ *is called* $0 - P - G$*-Cauchy if and only if*

$$\lim_{m,n\to+\infty} P(x_n,x_m,x_m) = 0.$$

(3) *A partial G-metric spaces* (X, P) *is said to be* $0 - P - G$*-complete if and only if every* $0 - P - G$*-Cauchy sequence is* $P - G$*-convergent to some* $x \in X$ *such that* $P(x,x,x) = 0$.

Example 1.8. *([6]) Let* $X = Q \cap [0, +\infty)$ *where* Q *denotes the set rational numbers, and define* $P(x,y,z) = \frac{1}{3}(\max\{x,y\} + \max\{y,z\} + \max\{x,z\})$,*for all* $x, y, z \in X$. *Then* (X, P) *is a* $0 - P - G$*-complete partial G-metric space.*

Lemma 1.9. *([6]) Let (X, P) be a partial G-metric space and $\{x_n\}$ be a sequence in X. Assume that $\{x_n\}$ $P-G$-converges to $x \in X$ and $P(x, x, x) = 0$. Then $\lim\limits_{n\to+\infty} P(x_n, y, y) = P(x, y, y)$ for all $y \in X$. Moreover,*
$\lim\limits_{m,n\to+\infty} P(x_n, x_m, x) = 0.$

Similarly we can have the following Lemma.

Lemma 1.10. *Let (X, P) be a partial G-metric space and $\{x_n\}$ be a sequence in X. Assume that $\{x_n\}$ $P-G$-converges to $x \in X$ and $P(x, x, x) = 0$. Then $\lim\limits_{n\to+\infty} P(x_n, x_n, y) = P(x, x, y)$ for all $y \in X$.*

Definition 1.11. *([2])Let X be a non-empty set and $f, S : X \to X$.The pair (f, S) is said to be weakly compatible if $fSu = Sfu$ whenever $fu = Su$ for $u \in X$.*

Theorem 1.12. *([6], Th. 3.6) Let (X, P) be a partial G-metric space and $f, T : X \to X$ be satisfying*
(1.12.1) *$T(X) \subseteq f(X)$ and $\{f, T\}$ is a weakly compatible pair*
(1.12.2) *$f(X)$ is $0 - P - G$-complete subspace of X*
(1.12.3) *assume that there exists $r \in [0, 1)$ such that*
$\psi(r)\ P(fx, Tx, Tx) \le P(fx, fy, fy)$
implies that
$P(Tx, Ty, Ty) \le rP(fx, fy, fy)$ *for any $x, y \in X$, where $\psi : [0, 1) \to (\frac{1}{6}, \frac{\sqrt{3}}{6}]$ a function given by*

$$\psi(r) = \begin{cases} \frac{\sqrt{3}}{6}\ if\ 0 \le r < \frac{\sqrt{3}-1}{2} \\ \frac{1}{2(1+2r)}\ if\ \frac{\sqrt{3}-1}{2} \le r < 1 \end{cases}$$

Then T and f have unique common fixed point in X.

In the spirit of the above theorem, we prove a common fixed point result for a pair of weakly compatible self maps of a partial G-metric space, but using a rational contractive condition.

2. Main result

Theorem 2.1. *Let (X, P) be a partial G-metric space. Suppose that (f, T) is a weakly compatible pair of self maps of X satisfying the following*
(2.1.1) *there exists a constant $\theta \in [0, 1)$ such that*

$\eta(\theta)\ P(fx,Tx,Tx) \leq P(fx,fy,fy)$ implies that

$$P(Tx,Ty,Ty) \leq \theta \max \left\{ \begin{array}{c} P(fx,fy,fy), \\ \frac{1}{9}\frac{P(fx,Tx,Tx)P(fy,Ty,Ty)}{1+P(fx,fy,fy)}, \\ \frac{1}{9}\frac{P(fx,Ty,Ty)P(fy,Tx,Tx)}{1+P(fx,fy,fy)} \end{array} \right\}$$

for all $x, y \in X$, where $\eta(\theta) : [0,1) \to (\frac{1}{5}, \frac{1}{3}]$ is defined by $\eta(\theta) = \frac{1}{3+2\theta}$.
(2.1.2) *$T(X) \subseteq f(X)$, $f(X)$ is $0-P-G$-complete subspace of X.*
Then f and T have unique common fixed point in X.

Proof Let $x_0 \in X$ be an arbitrary point. From (2.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = Tx_n = fx_{n+1}, n = 0,1,2,...$$

Case 1. Assume that $y_n \neq y_{n+1}$ for all n.
Then $P(y_n, y_{n+1}, y_{n+1}) > 0$ for all n from Proposition 1.6(iii).
Since $\eta(\theta) \leq \frac{1}{3} < 1$, we have

$$\eta(\theta)P(fx_{n-1}, Tx_{n-1}, Tx_{n-1}) < P(fx_{n-1}, fx_n, fx_n).$$

From (2.1.1), we have

$$\begin{aligned} &P(fx_n, fx_{n+1}, fx_{n+1}) \\ &= P(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \theta \max \left\{ \begin{array}{c} P(fx_{n-1}, fx_n, fx_n), \\ \frac{1}{9}\frac{P(fx_{n-1},fx_n,fx_n)P(fx_n,fx_{n+1},fx_{n+1})}{1+P(fx_{n-1},fx_n,fx_n)}, \\ \\ \frac{1}{9}\frac{P(fx_{n-1},fx_{n+1},fx_{n+1})P(fx_n,fx_n,fx_n)}{1+P(fx_{n-1},fx_n,fx_n)} \\ \end{array} \right\}. \end{aligned}$$

We have the estimates

$$\begin{aligned} &\frac{1}{9}\frac{P(fx_{n-1},fx_{n+1},fx_{n+1})P(fx_n,fx_n,fx_n)}{1+P(fx_{n-1},fx_n,fx_n)} \\ &\leq \frac{1}{9}\frac{[P(fx_{n-1},fx_n,fx_n)+P(fx_n,fx_{n+1},fx_{n+1})]3P(fx_{n-1},fx_n,fx_n)}{1+P(fx_{n-1},fx_n,fx_n)} \\ &\leq \frac{2}{3}\max\{P(fx_{n-1}, fx_n, fx_n), P(fx_n, fx_{n+1}, fx_{n+1})\} \\ &< \max\{P(fx_{n-1}, fx_n, fx_n), P(fx_n, fx_{n+1}, fx_{n+1})\}. \end{aligned}$$

In the second line we used (P_5) and (P_1).

We also get
$\frac{1}{9}\frac{P(fx_{n-1},fx_n,fx_n)P(fx_n,fx_{n+1},fx_{n+1})}{1+P(fx_{n-1},fx_n,fx_n)} < \frac{1}{9}P(fx_n, fx_{n+1}, fx_{n+1}) \leq$
$\leq \frac{1}{9}\max\{P(fx_{n-1}, fx_n, fx_n), P(fx_n, fx_{n+1}, fx_{n+1})\}.$

Then
$P(fx_n, fx_{n+1}, fx_{n+1}) < \theta \max\{P(fx_n, fx_{n+1}, fx_{n+1}), P(fx_{n-1}, fx_n, fx_n)\}$.
Since $\theta \in [0,1)$ and $P(fx_n, fx_{n+1}, fx_{n+1}) > 0$, the above inequality implies

$$P(fx_n, fx_{n+1}, fx_{n+1}) \leq \theta P(fx_{n-1}, fx_n, fx_n),$$

for $n = 1, 2, 3, \cdots$, hence

$$P(fx_n, fx_{n+1}, fx_{n+1}) \leq \theta^n P(fx_0, fx_1, fx_1),$$

for $n = 1, 2, 3, \cdots$. Since $\theta \in [0,1)$, it follows that

$$P(fx_n, fx_{n+1}, fx_{n+1}) \to 0 \text{ as } n \to \infty \tag{1}$$

For $m > n$ and using (P_5) and induction over $(m-n)$ we have

$$\begin{aligned} P(fx_n, fx_m, fx_m) &\leq P(fx_n, fx_{n+1}, fx_{n+1}) + \\ &P(fx_{n+1}, fx_{n+2}, fx_{n+2}) + \cdots + P(fx_{m-1}, fx_m, fx_m) \\ &\leq (\theta^n + \theta^{n+1} + \cdots + \theta^{m-1}) P(fx_0, fx_1, fx_1) \\ &\leq \frac{\theta^n}{1-\theta} P(fx_0, fx_1, fx_1) \end{aligned}$$

Since $\theta \in [0,1)$, it follows that

$$\lim_{m,n\to\infty} P(fx_n, fx_m, fx_m) = 0 \tag{2}$$

This implies that $\{fx_n\}$ is a $0-P-G-$Cauchy sequence in X.
Suppose $f(X)$ is $0-P-G$ complete. Then it follows that $\{fx_n\}$ is $P-G-$ convergent to a point $x = ft$ for some $t \in X$ such that

$$P(x, x, x) = 0 \tag{3}$$

That is

$$\lim_{n\to\infty} P(fx_n, ft, ft) = \lim_{n\to\infty} P(fx_n, fx_n, ft) = P(ft, ft, ft) = 0. \tag{4}$$

From Lemma (1.9), we have $\lim_{n\to\infty} P(fx_n, fx_{n+1}, ft) = 0$.
Since $fx_n \neq fx_{n+1}$ for all n and $fx_n \to ft$ it follows that $fx_n \neq ft$ for infinitely many n.
Claim: $P(ft, Tz, Tz) \leq \theta P(ft, fz, fz)$ for $z \in X$ with $ft \neq fz$.
Let $z \in X$ with $ft \neq fz$.
Then from Proposition 1.6(iii), we have $P(fz, ft, ft) > 0$.
Since $fx_n \to ft$ as $n \to \infty$, there exists a positive integer N such that for all $n \geq N$, we have

$$\begin{cases} P(fx_n, ft, ft) \leq \frac{1}{3} P(fz, ft, ft,) \\ P(fx_n, fx_n, ft) \leq \frac{1}{3} P(fz, ft, ft,) \\ P(fx_n, fx_{n+1}, ft) \leq \frac{1}{3} P(fz, ft, ft,) \end{cases} \tag{5}$$

From Proposition 1.6$(viii)$, we have
$P(fx_n, fx_{n+1}, fx_{n+1}) \leq P(fx_n, fx_{n+1}, ft) + P(ft, fx_{n+1}, fx_{n+1})$
and from (P_5), we have
$P(fz, ft, ft) - P(fx_n, ft, ft) \leq P(fz, fx_n, fx_n)$.
Hence for all $n \geq N$, we have

$$\begin{aligned}
3\eta(\theta)P(fx_n, Tx_n, Tx_n) &\leq P(fx_n, Tx_n, Tx_n) \\
&= P(fx_n, fx_{n+1}, fx_{n+1}) \\
&= P(fx_{n+1}, fx_n, fx_{n+1}) \\
&\leq P(fx_{n+1}, ft, fx_{n+1}) + P(ft, fx_n, fx_{n+1}), \\
&\leq \tfrac{2}{3}P(fz, ft, ft) \\
&= P(fz, ft, ft) - \tfrac{1}{3}P(fz, ft, ft) \\
&\leq P(fz, ft, ft) - P(fx_n, ft, ft) \\
&\leq P(fz, fx_n, fx_n) \\
&\leq 2P(fz, fz, fx_n) \\
&< 3P(fx_n, fz, fz).
\end{aligned}$$

We used Proposition 1.6(vii) in the fourth line and Proposition 1.6(v) the ninth line, as well as (5) in the fifth line and (P_5) in the eigth line.

Thus $\eta(\theta)P(fx_n, Tx_n, Tx_n) \leq P(fx_n, fz, fz)$.
Now from (2.1.1), we have

$$P(Tx_n, Tz, Tz) \leq \theta \max \left\{ \begin{array}{c} P(fx_n, fz, fz), \\ \frac{1}{9}\frac{P(fx_n,Tx_n,Tx_n)P(fz,Tz,Tz)}{1+P(fx_n,fz,fz)}, \\ \frac{1}{9}\frac{P(fx_n,Tz,Tz)P(fz,Tx_n,Tx_n)}{1+P(fx_n,fz,fz)} \end{array} \right\}.$$

Letting $n \to \infty$, we have from $(1), (3)$ and Lemmas $(1.9), (1.10)$ that

$$\begin{aligned}
P(ft, Tz, Tz) &\leq \theta \max \left\{ \begin{array}{c} P(ft, fz, fz), 0 \\ \frac{1}{9}\frac{P(ft,Tz,Tz)P(fz,ft,ft)}{1+P(ft,fz,fz)} \end{array} \right\} \\
&\leq \theta \max \left\{ P(ft, fz, fz), \tfrac{2}{9}P(ft, Tz, Tz) \right\},
\end{aligned} \tag{6}$$

from Proposition $(1.6)(v)$.

If $\max\left\{P(ft, fz, fz), \frac{2}{9}P(ft, Tz, Tz)\right\} = \frac{2}{9}P(ft, Tz, Tz)$, then from (6)

$$P(ft, Tz, Tz) \leq \frac{2}{9}P(ft, Tz, Tz)$$

which yields that $ft = Tz$.
Also

$$P(ft, fz, fz) \leq \frac{2}{9}P(ft, Tz, Tz) = 0$$

which in turn yields that $ft = fz$
This is a contradiction with

$P(fz, ft, ft) > 0$. Then $\max\left\{P(ft, fz, fz), \frac{2}{9}P(ft, Tz, Tz)\right\} = P(ft, fz, fz)$ and (6) implies

$$P(ft, Tz, Tz) \leq \theta P(ft, fz, fz), \tag{7}$$

hence the claim.
Now we will show that $Tt = x$.

$$\begin{aligned} P(fz, Tz, Tz) &\leq P(fz, ft, ft) + P(ft, Tz, Tz) \\ &\leq P(fz, ft, ft) + \theta P(ft, fz, fz) \\ &\leq (3 + 2\theta)P(fz, ft, ft) \end{aligned}$$

We used (P_5) in the first line, (7) in the second line and Proposition 1.6(v) in the third line.

Thus $\eta(\theta)P(fz, Tz, Tz) \leq P(fz, ft, ft)$ for $ft \neq fz$.
From (2.1.1), we have

$$P(Tz, Tt, Tt) \leq \theta \max\left\{\begin{array}{c} P(fz, ft, ft), \\ \frac{1}{9}\frac{P(fz,Tz,Tz)P(ft,Tt,Tt)}{1+P(fz,ft,ft)}, \\ \frac{1}{9}\frac{P(fz,Tt,Tt)P(ft,Tz,Tz)}{1+P(fz,ft,ft)} \end{array}\right\} \tag{8}$$

Since $ft \neq fx_n$ for infinitely many n,
putting $z = x_n$ in (8), we get

$$P(Tx_n, Tt, Tt) \leq \theta \max\left\{\begin{array}{c} P(fx_n, ft, ft), \\ \frac{1}{9}\frac{P(fx_n,Tx_n,Tx_n)P(ft,Tt,Tt)}{1+P(fx_n,ft,ft)}, \\ \frac{1}{9}\frac{P(fx_n,Tt,Tt)P(ft,Tx_n,Tx_n)}{1+P(fx_n,ft,ft)} \end{array}\right\}.$$

Letting $n \to \infty$, we get $P(x, Tt, Tt) \leq 0$ from (1) and (4).
Thus $Tt = x$, from Proposition 1.6(ii). Hence $ft = x = Tt$.
Since (f, T) is a weakly compatible pair, we have $fx = Tx$.
Suppose $fx = Tx \neq x$. Then $P(x, Tx, Tx) > 0$ and $P(x, fx, fx) > 0$.
Since

$$\begin{aligned} \eta(\theta)P(ft, Tt, Tt) &= \eta(\theta)P(x, x, x) \\ &= 0 \\ &< P(ft, fx, fx). \end{aligned}$$

From (2.1.1), we have

$$\begin{aligned} P(x,Tx,Tx) &= P(Tt,Tx,Tx) \\ &\le \theta \max\left\{ \begin{array}{c} P(ft,fx,fx), \\ \frac{1}{9}\frac{P(ft,Tt,Tt)P(fx,Tx,Tx)}{1+P(ft,fx,fx)}, \\ \frac{1}{9}\frac{P(ft,Tx,Tx)P(fx,Tt,Tt)}{1+P(ft,fx,fx)} \end{array} \right\} \\ &= \theta \max\{P(x,Tx,Tx), 0, \tfrac{2}{9}P(x,Tx,Tx)\}, \\ &\quad from\ Propostion 1.6(v) \\ &= \theta P(x,Tx,Tx) \\ &< P(x,Tx,Tx). \end{aligned}$$

We obtained $P(x,Tx,Tx) < P(x,Tx,Tx)$, which is a contradiction. Hence $Tx = x$, consequently $Tx = fx = x$.
Thus x is a common fixed point of T and f.
Suppose that y is common fixed point of T and f with $y \neq x$. Then $P(x,y,y) > 0$ from Proposition 1.6(iii).
Since $\eta(\theta)\ P(fx,Tx,Tx) = 0 < P(fx,fy,fy)$,it follows from (2.1.1) that

$$\begin{aligned} P(x,y,y) &= P(Tx,Ty,Ty) \\ &\le \theta\ \max\left\{ \begin{array}{c} P(fx,fy,fy), \\ \frac{1}{9}\frac{P(fx,Tx,Tx)P(fy,Ty,Ty)}{1+P(fx,fy,fy)}, \\ \frac{1}{9}\frac{P(fx,Ty,Ty)P(fy,Tx,Tx)}{1+P(fx,fy,fy)} \end{array} \right\} \\ &= \theta\ P(x,y,y), \end{aligned}$$

We used (3) and Proposition 1.6(v). We obtained $P(x,y,y) \le \theta\ P(x,y,y)$, which is a contradiction. Hence x is unique common fixed point of T and f.
Case 2. Assume that $y_n = y_{n+1}$ for some n. Then $fx_{n+1} = Tx_{n+1}$ i.e. $f\alpha = T\alpha$ where $\alpha = x_{n+1} \in X$. Let $f\alpha = T\alpha = \beta$.
Since (f,T) is a weakly compatible pair, we have

$$f\beta = T\beta. \tag{9}$$

Suppose that $T\beta \neq \beta$. Then $P(\beta,T\beta,T\beta) > 0$ and $P(\beta,f\beta,f\beta) > 0$. Since

$$\begin{aligned} \eta(\theta)P(f\alpha,T\alpha,T\alpha) &= \eta(\theta)P(\beta,\beta,\beta) \\ &\le \eta(\theta)3P(\beta,f\beta,f\beta),\ from\ (P_1) \\ &\le P(f\alpha,f\beta,f\beta), \end{aligned}$$

it follows from (2.1.1) that

$$\begin{aligned} P(\beta,T\beta,T\beta) &= P(T\alpha,T\beta,T\beta) \\ &\le \theta\max\left\{\begin{array}{c} P(f\alpha,f\beta,f\beta), \\ \frac{1}{9}\frac{P(f\alpha,T\alpha,T\alpha)P(f\beta,T\beta,T\beta)}{1+P(f\alpha,f\beta,f\beta)}, \\ \frac{1}{9}\frac{P(f\alpha,T\beta,T\beta)P(f\beta,T\alpha,T\alpha)}{1+P(f\alpha,f\beta,f\beta)} \end{array}\right\} \\ &\le \theta\max\left\{\begin{array}{c} P(f\alpha,f\beta,f\beta), \\ \frac{1}{9}\frac{3P(f\alpha,f\beta,f\beta)3P(f\alpha,f\beta,f\beta)}{1+P(f\alpha,f\beta,f\beta)}, \\ \frac{1}{9}\frac{P(f\alpha,f\beta,f\beta)2P(f\alpha,f\beta,f\beta)}{1+P(f\alpha,f\beta,f\beta)} \end{array}\right\}, \text{ from } (P_1) \\ &= \theta P(f\alpha,f\beta,f\beta) \\ &< P(\beta,T\beta,T\beta). \end{aligned}$$

which is a contradiction.
Hence $T\beta=\beta$.
Now from (9) $f\beta=\beta$. Thus β is common fixed point of T and f.
Suppose β^1 is another common fixed point of T and f.
Since

$$\begin{aligned} \eta(\theta)P(f\beta,T\beta,T\beta) &\le \eta(\theta)3P(f\beta,f\beta^1,f\beta^1),\ from\ (P_1) \\ &\le P(f\beta,f\beta^1,f\beta^1), \end{aligned}$$

it follows from (2.1.1) that

$$\begin{aligned} P(\beta,\beta^1,\beta^1) &= P(T\beta,T\beta^1,T\beta^1) \\ &\le \theta\max\left\{\begin{array}{c} P(\beta,\beta^1,\beta^1), \\ \frac{1}{9}\frac{P(\beta,\beta,\beta)P(\beta^1,\beta^1,\beta^1)}{1+P(\beta,\beta^1,\beta^1)}, \\ \frac{1}{9}\frac{P(f\beta,T\beta^1,T\beta^1)P(f\beta^1,T\beta,T\beta)}{1+P(f\beta,f\beta^1,f\beta^1)} \end{array}\right\} \\ &\le \theta\max\left\{\begin{array}{c} P(\beta,\beta^1,\beta^1), \\ \frac{1}{9}\frac{3P(\beta,\beta^1,\beta^1)3P(\beta,\beta^1,\beta^1)}{1+P(\beta,\beta^1,\beta^1)}, \\ \frac{1}{9}\frac{3P(\beta,\beta^1,\beta^1)2P(\beta,\beta^1,\beta^1)}{1+P(\beta,\beta^1,\beta^1)} \end{array}\right\}, \\ &\quad from\ (P_1)\ and\ Proposition 1.6(v) \\ &= \theta P(\beta,\beta^1,\beta^1) \\ &< P(\beta,\beta^1,\beta^1). \end{aligned}$$

We obtained a contradiction.
Therefore β is the unique common fixed point of T and f.

The following example illustrates our Theorem 2.1

Example 2.2. *Let (X,P) be a partial G-metric space where $X=[0,1]$.*
$P(x,y,z)=max\{x,y\}+max\{y,z\}+max\{x,z\}$.

Let $f, T : X \to X$ *be defined by*
$Tx = \frac{x^2}{16}$, $fx = \frac{x}{2}$, $\forall\ x, y \in X$.
$\theta = \frac{1}{8} \in [0, 1)$ *and* $\eta(\theta) = \frac{1}{3+2\theta} = \frac{4}{13} \in (\frac{1}{5}, \frac{1}{3}]$.
Clearly $T(X) = [0, \frac{1}{16}] \subseteq [0, \frac{1}{2}] = f(X)$ *and* (T, f) *is a weakly compatible pair.*
Without loss of generality assume $x \leq y$
Now

$$\begin{aligned}\eta(\theta)P\left(fx, Tx, Tx\right) &\leq max\{fx, Tx\} + max\{Tx, Tx\} + max\{fx, Tx\}\\ &= max\{\tfrac{x}{2}, \tfrac{x^2}{16}\} + max\{\tfrac{x^2}{16}, \tfrac{x^2}{16}\} + max\{\tfrac{x}{2}, \tfrac{x^2}{16}\}\\ &= \tfrac{x}{2} + \tfrac{x^2}{16} + \tfrac{x}{2}\\ &\leq \tfrac{x}{2} + \tfrac{x}{2} + \tfrac{x}{2}\\ &\leq max\{\tfrac{x}{2}, \tfrac{y}{2}\} + max\{\tfrac{y}{2}, \tfrac{y}{2}\} + max\{\tfrac{x}{2}, \tfrac{y}{2}\}\\ &= P(fx, fy, fy).\end{aligned}$$

Now

$$\begin{aligned}P\left(Tx, Ty, Ty\right) &= max\{Tx, Ty\} + max\{Ty, Ty\} + max\{Tx, Ty\}\\ &= max\{\tfrac{x^2}{16}, \tfrac{y^2}{16}\} + max\{\tfrac{y^2}{16}, \tfrac{y^2}{16}\} + max\{\tfrac{x^2}{16}, \tfrac{y^2}{16}\}\\ &= \tfrac{1}{8}P(fx, fy, fy)\\ &\leq \tfrac{1}{8}\max\left\{\begin{array}{c} P(fx, fy, fy),\\ \frac{1}{9}\frac{P(fx,Tx,Tx)P(fy,Ty,Ty)}{1+P(fx,fy,fy)},\\ \frac{1}{9}\frac{P(fx,Ty,Ty)P(fy,Tx,Tx)}{1+P(fx,fy,fy}\end{array}\right\}.\end{aligned}$$

Hence all conditions of Theorem (2.1.1) *are satisfied.*
Clearly 0 *is the unique common fixed point of* f *and* T.

3. Conclusion

In this paper our main result for two maps satisfying Suzuki type contractive condition in partial G-metric space generalizes and improves some known results in existing literature in Partial G-metric spaces. We also provided an example to illustrate our main theorem.

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