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LOCAL FUNCTION Γ^* IN IDEAL TOPOLOGICAL SPACES

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Abstract. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset A of X, a local function $\Gamma^*(A)(\mathcal{I}, \tau)$ is defined as follows: $\Gamma^*(A)(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every regular open set U containing $x\}$. This coincides with the δ-local functions due to Hatir et al. [2]. By using $\Gamma^*(A)(\mathcal{I}, \tau)$, an operator $\Psi_{\Gamma^*} : \mathcal{P}(X) \to \tau^\delta$ is defined as the dual of the δ-local function and its relations with δ-codense ideals are investigated.

1. Introduction and Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), we denote by $Cl(A)$ and $Int(A)$ the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$.
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

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An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [3]). We simply write A^* instead of $A^*(\mathcal{I}, \tau)$ in case there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generated by the base $\beta(\mathcal{I}, \tau) = \{U - J : U \in \tau$ and $J \in \mathcal{I}\}$. It is known in Example 3.6 of [3] that $\beta(\mathcal{I}, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by τ^*. Recall that A is said to be $*$-dense in itself (resp. τ^*-closed, $*$-perfect) if $A \subseteq A^*$ (resp. $A^* \subseteq A$, $A = A^*$). For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X, τ^*), respectively.

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). By $RO(X)$ we denote the family of all regular open sets of (X, τ). A subset A is said to be δ-open [13] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a δ-open set is said to be δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(U)) \cap A \neq \emptyset$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$. The δ-interior of A is the union of all regular open sets of X contained in A and it is denoted by $Int_\delta(A)$. A is said to be δ-open if $Int_\delta(A) = A$. The collection of all δ-open sets of (X, τ) is denoted by $\delta O(X)$ and forms a topology τ^δ.

In this paper, we define and investigate an operator $\Psi_{\Gamma^*} : \mathcal{P}(X) \to \tau^\delta$ as follows: $\Psi_{\Gamma^*}(A) = X - \Gamma^*(X - A)$ for every subset A of X. Many relationships between the operator Ψ_{Γ^*} and a δ-codense ideal are obtained.

2. δ-local functions in ideal topological spaces

Definition 2.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset A of X, we define the following set: $\Gamma^*(A)(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in RO(x)\}$, where $RO(x) = \{U \in RO(X, \tau) : x \in U\}$.

Remark 2.2. In [2], $A^{\delta *}(\mathcal{I}, \tau)$ is defined as follows: $A^{\delta *}(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \tau^\delta(x)\}$, where $\tau^\delta(x) = \{V : x \in V \in \tau^\delta\}$ and it is called the δ-local function of A with respect to $\mathcal{I}$ and τ. We show that $\Gamma^*(A)(\mathcal{I}, \tau) = A^{\delta *}(\mathcal{I}, \tau)$ for every subset A of X.

Proof. Since $RO(X) \subseteq \tau^\delta$, it is obvious that $A^{\delta*}(\mathcal{I},\tau) \subseteq \Gamma^*(A)(\mathcal{I},\tau)$.

Conversely, suppose that $x \notin A^{\delta*}(\mathcal{I},\tau)$. Then, there exist $V \in \tau^\delta(x)$ such that $A \cap V \in \mathcal{I}$. Since there exists $U \in RO(X)$ such that $U \subseteq V$, $A \cap U \subseteq A \cap V$. Therefore, $A \cap U \in \mathcal{I}$ and hence $x \notin \Gamma^*(A)(\mathcal{I},\tau)$. Hence $\Gamma^*(A)(\mathcal{I},\tau) \subseteq A^{\delta*}(\mathcal{I},\tau)$. ∎

In this paper, we use the notion $\Gamma^*(A)(\mathcal{I},\tau)$ (briefly $\Gamma^*(A)$).

Lemma 2.3. [2] *Let (X,τ) be a topological space, $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X, and let A and B be subsets of X. Then the following properties hold:*

(1) *If $A \subseteq B$, then $\Gamma^*(A) \subseteq \Gamma^*(B)$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $\Gamma^*(A)(\mathcal{I}) \supseteq \Gamma^*(A)(\mathcal{J})$.*
(3) *$\Gamma^*(A) = Cl_\delta(\Gamma^*(A)) \subseteq Cl_\delta(A)$ and $\Gamma^*(A)$ is δ-closed.*
(4) *If $A \subseteq \Gamma^*(A)$, then $\Gamma^*(A) = Cl_\delta(A) = Cl_\delta(\Gamma^*(A))$.*
(5) *If $A \in \mathcal{I}$, then $\Gamma^*(A) = \emptyset$.*

Theorem 2.4. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A, B any subsets of X. Then the following properties hold:*

(1) *$\Gamma^*(\emptyset) = \emptyset$.*
(2) *$\Gamma^*(\Gamma^*(A)) \subseteq \Gamma^*(A)$.*
(3) *$\Gamma^*(A) \cup \Gamma^*(B) = \Gamma^*(A \cup B)$.*

By Theorem 3 of [2], we obtain that $Cl_{\Gamma^*}(A) = A \cup \Gamma^*(A)$ is a Kuratowski closure operator. We will denote by τ_{Γ^*} the topology generated by Cl_{Γ^*}, that is, $\tau_{\Gamma^*} = \{U \subseteq X : Cl_{\Gamma^*}(X-U) = X-U\}$.

Corollary 2.5. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A, B be subsets of X with $B \in \mathcal{I}$. Then $\Gamma^*(A \cup B) = \Gamma^*(A) = \Gamma^*(A-B)$.*

Theorem 2.6. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then $\beta(\tau,\mathcal{I}) = \{V - I : V \in RO(X,\tau), I \in \mathcal{I}\}$ is a basis for τ_{Γ^*}.*

3. Ψ_{Γ^*}-OPERATOR IN IDEAL TOPOLOGICAL SPACES

Definition 3.1. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. An operator $\Psi_{\Gamma^*} : \mathcal{P}(X) \to \tau^\delta$ is defined as follows: for every $A \in X$, $\Psi_{\Gamma^*}(A) = \{x \in X : \text{there exists } U \in RO(x) \text{ such that } U - A \in \mathcal{I}\}$.

It is easily shown that $\Psi_{\Gamma^*}(A) = X - \Gamma^*(X-A)$.

Several basic facts concerning the behavior of the operator Ψ_{Γ^*} are included in the following theorem.

Theorem 3.2. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then the following properties hold:*

(1) *If* $A \subseteq X$*, then* $\Psi_{\Gamma^*}(A)$ *is* δ*-open.*
(2) *If* $A \subseteq B$*, then* $\Psi_{\Gamma^*}(A) \subseteq \Psi_{\Gamma^*}(B)$*.*
(3) *If* $A, B \subseteq X$*, then* $\Psi_{\Gamma^*}(A \cap B) = \Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(B)$*.*
(4) *If* $U \in \tau_{\Gamma^*}$*, then* $U \subseteq \Psi_{\Gamma^*}(U)$*.*
(5) *If* $A \subseteq X$*, then* $\Psi_{\Gamma^*}(A) \subseteq \Psi_{\Gamma^*}(\Psi_{\Gamma^*}(A))$*.*
(6) *Let* $A \subseteq X$*, then* $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(\Psi_{\Gamma^*}(A))$ *if and only if* $\Gamma^*(X - A) = \Gamma^*(\Gamma^*(X - A))$*.*
(7) *If* $A \in \mathcal{I}$*, then* $\Psi_{\Gamma^*}(A) = X - \Gamma^*(X)$*.*
(8) *If* $A \subseteq X$*, then* $A \cap \Psi_{\Gamma^*}(A) = Int_{\Gamma^*}(A)$*.*
(9) *If* $A \subseteq X$*,* $I \in \mathcal{I}$*, then* $\Psi_{\Gamma^*}(A - I) = \Psi_{\Gamma^*}(A)$*.*
(10) *If* $A \subseteq X$*,* $I \in \mathcal{I}$*, then* $\Psi_{\Gamma^*}(A \cup I) = \Psi_{\Gamma^*}(A)$*.*
(11) *If* $(A - B) \cup (B - A) \in \mathcal{I}$*, then* $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(B)$*.*

Proof. (1) This follows from Theorem 2.3 (3).
(2) This follows from Theorem 2.3 (1).
(3) It follows from (2) that $\Psi_{\Gamma^*}(A \cap B) \subseteq \Psi_{\Gamma^*}(A)$ and $\Psi_{\Gamma^*}(A \cap B) \subseteq \Psi_{\Gamma^*}(B)$. Hence $\Psi_{\Gamma^*}(A \cap B) \subseteq \Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(B)$. Now let $x \in \Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(B)$. There exist $U, V \in RO(x)$ such that $U - A \in \mathcal{I}$ and $V - B \in \mathcal{I}$. Let $G = U \cap V \in RO(x)$ and we have $G - A \in \mathcal{I}$ and $G - B \in \mathcal{I}$ by heredity. Thus $G - (A \cap B) = (G - A) \cup (G) - B) \in \mathcal{I}$ by additivity, and hence $x \in \Psi_{\Gamma^*}(A \cap B)$. We have shown $\Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(B) \subseteq \Psi_{\Gamma^*}(A \cap B)$ and the proof is complete.
(4) If $U \in \tau_{\Gamma^*}$, then $X - U$ is τ_{Γ^*}-closed which implies $\Gamma^*(X - U) \subseteq X - U$ and hence $U \subseteq X - \Gamma^*(X - U) = \Psi_{\Gamma^*}(U)$.
(5) This follows from (1) and (4).
(6) This follows from the facts:

(1) $\Psi_{\Gamma^*}(A) = X - \Gamma^*(X - A)$.
(2) $\Psi_{\Gamma^*}(\Psi_{\Gamma^*}(A)) = X - \Gamma^*[X - (X - \Gamma^*(X - A))] = X - \Gamma^*(\Gamma^*(X - A))$.

(7) By Corollary 2.5 we obtain that $\Gamma^*(X - A) = \Gamma^*(X)$ if $A \in \mathcal{I}$.
(8) If $x \in A \cap \Psi_{\Gamma^*}(A)$, then $x \in A$ and there exists $U_x \in RO(x)$ such that $U_x - A \in \mathcal{I}$. Then by Theorem 2.6, $U_x - (U_x - A)$ is a τ_{Γ^*}-open neighborhood of x and $x \in Int_{\Gamma^*}(A)$. On the other hand, if $x \in Int_{\Gamma^*}(A)$, there exists a basic τ_{Γ^*}-open neighborhood $V_x - I$ of x, where $V_x \in RO(X, \tau)$ and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq A$ which implies $V_x - A \subseteq I$ and hence $V_x - A \in \mathcal{I}$. Hence $x \in A \cap \Psi_{\Gamma^*}(A)$.
(9) This follows from Corollary 2.5 and $\Psi_{\Gamma^*}(A - I) = X - \Gamma^*[X - (A - I)] = X - \Gamma^*[(X - A) \cup I] = X - \Gamma^*(X - A) = \Psi_{\Gamma^*}(A)$.
(10) This follows from Corollary 2.5 and $\Psi_{\Gamma^*}(A \cup I) = X - \Gamma^*[X - (A \cup I)] = X - \Gamma^*[(X - A) - I] = X - \Gamma^*(X - A) = \Psi_{\Gamma^*}(A)$.
(11) Assume $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$.

Observe that $I, J \in \mathcal{I}$ by heredity. Also observe that $B = (A - I) \cup J$. Thus $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(A - I) = \Psi[(A - I) \cup J] = \Psi_{\Gamma^*}(B)$ by (9) and (10). ∎

Corollary 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $U \subseteq \Psi_{\Gamma^*}(U)$ for every δ-open set $U \subseteq X$.*

Proof. We know that $\Psi_{\Gamma^*}(U) = X - \Gamma^*(X - U)$. Now $\Gamma^*(X - U) \subseteq Cl_\delta(X - U) = X - U$, since $X - U$ is δ-closed. Therefore, $U = X - (X - U) \subseteq X - \Gamma^*(X - U) = \Psi_{\Gamma^*}(U)$. ∎

Now we give an example of a set A which is not δ-open but satisfies $A \subseteq \Psi_{\Gamma^*}(A)$.

Example 3.4. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, and $\mathcal{I} = \{\emptyset, , \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{a\}$. Then $\Psi_{\Gamma^*}(\{a\}) = X - \Gamma^*(X - \{a\}) = X - \Gamma^*(\{b, c, d\}) = X - \{b, d\} = \{a, c\}$. Therefore, $A \subseteq \Psi_{\Gamma^*}(A)$, but A is not open and not δ-open.*

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

(1) $\Psi_{\Gamma^*}(A) = \cup\{U \in RO(X) : U - A \in \mathcal{I}\}$.
(2) $\Psi_{\Gamma^*}(A) \supseteq \cup\{U \in RO(X) : (U - A) \cup (A - U) \in \mathcal{I}\}$.

Proof. (1) This follows immediately from the definition of the Ψ_{Γ^*}-operator.
(2) Since $\mathcal{I}$ is heredity, it is obvious that $\cup\{U \in RO(X) : (U - A) \cup (A - U) \in \mathcal{I}\} \subseteq \cup\{U \in RO(X) : U - A \in \mathcal{I}\} = \Psi_{\Gamma^*}(A)$ for every $A \subseteq X$. ∎

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\sigma = \{A \subseteq X : A \subseteq \Psi_{\Gamma^*}(A)\}$. Then σ is a topology for X and $\sigma = \tau_{\Gamma^*}$.*

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \Psi_{\Gamma^*}(A)\}$. First, we show that σ is a topology. Observe that $\emptyset \subseteq \Psi_{\Gamma^*}(\emptyset)$ and $X \subseteq \Psi_{\Gamma^*}(X) = X$, and thus $\emptyset$ and $X \in \sigma$. Now if $A, B \in \sigma$, then by Theorem 3.2(3) $A \cap B \subseteq \Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(B) = \Psi_{\Gamma^*}(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Psi_{\Gamma^*}(A_\alpha) \subseteq \Psi_{\Gamma^*}(\cup A_\alpha)$ for every $\alpha \in \Delta$ and hence $\cup A_\alpha \subseteq \Psi_{\Gamma^*}(\cup A_\alpha)$. This shows that σ is a topology. Now if $U \in \tau_{\Gamma^*}$ then by Theorem 3.2(4) $U \subseteq \Psi_{\Gamma^*}(U)$ and we have shown $\tau_{\Gamma^*} \subseteq \sigma$. Now let $A \in \sigma$, then we have $A \subseteq \Psi_{\Gamma^*}(A)$, that is, $A \subseteq X - \Gamma^*(X - A)$ and $\Gamma^*(X - A) \subseteq X - A$. This shows that $X - A$ is τ_{Γ^*}-closed and hence $A \in \tau_{\Gamma^*}$. Thus $\sigma \subseteq \tau_{\Gamma^*}$ and hence $\sigma = \tau_{\Gamma^*}$. ∎

4. Ψ_{Γ^*}-OPERATOR AND δ-COMPATIBLE TOPOLOGY

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say the τ is δ-compatible with the ideal $\mathcal{I}$, denoted $\tau \sim^*_* \mathcal{I}$, if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $U \in RO(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau \sim^*_* \mathcal{I}$ if and only if $\Psi_{\Gamma^*}(A) - A \in \mathcal{I}$ for every $A \subseteq X$.*

Proof. Necessity. Assume $\tau \sim^*_* \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \Psi_{\Gamma^*}(A) - A \in \mathcal{I}$ if and only if $x \notin A$ and $x \notin \Gamma^*(X - A)$ if and only if $x \notin A$ and there exists $U_x \in RO(x)$ such that $U_x - A \in \mathcal{I}$ if and only if there exists $U_x \in RO(x)$ such that $x \in U_x - A \in \mathcal{I}$. Now, for each $x \in \Psi_{\Gamma^*}(A) - A$ there exists $U_x \in RO(x)$ such that $U_x \cap (\Psi_{\Gamma^*}(A) - A) \in \mathcal{I}$ by heredity and hence $\Psi_{\Gamma^*}(A) - A \in \mathcal{I}$ by assumption that $\tau \sim^*_* \mathcal{I}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in RO(x)$ such that $U_x \cap A \in \mathcal{I}$. Observe that $\Psi_{\Gamma^*}(X - A) - (X - A) = \{x :$ there exists $U_x \in RO(x)$ such that $x \in U_x \cap A \in \mathcal{I}\}$. Thus we have $A \subseteq \Psi_{\Gamma^*}(X - A) - (X - A) \in \mathcal{I}$ and hence $A \in \mathcal{I}$ by heredity of $\mathcal{I}$. ∎

Proposition 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^*_* \mathcal{I}$, $A \subseteq X$. If N is a nonempty regular open subset of $\Gamma^*(A) \cap \Psi_{\Gamma^*}(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq \Gamma^*(A) \cap \Psi_{\Gamma^*}(A)$, then $N - A \subseteq \Psi_{\Gamma^*}(A) - A \in \mathcal{I}$ by Theorem 4.2 and hence $N - A \in \mathcal{I}$ by heredity. Since $N \in RO(X) - \{\emptyset\}$ and $N \subseteq \Gamma^*(A)$, we have $N \cap A \notin \mathcal{I}$ by the definition of $\Gamma^*(A)$. ∎

As a consequence of the above theorem, we have the following.

Corollary 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^*_* \mathcal{I}$. Then $\Psi_{\Gamma^*}(\Psi_{\Gamma^*}(A)) = \Psi_{\Gamma^*}(A)$ for every $A \subseteq X$.*

Proof. $\Psi_{\Gamma^*}(A) \subseteq \Psi_{\Gamma^*}(\Psi_{\Gamma^*}(A))$ follows from Theorem 3.2 (5). Since $\tau \sim^*_* \mathcal{I}$, it follows from Theorem 4.2 that $\Psi_{\Gamma^*}(A) \subseteq A \cup I$ for some $I \in \mathcal{I}$ and hence $\Psi_{\Gamma^*}(\Psi_{\Gamma^*}(A)) = \Psi_{\Gamma^*}(A)$ by Theorem 3.2 (10). ∎

Theorem 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^*_* \mathcal{I}$. Then $\Psi_{\Gamma^*}(A) = \cup\{\Psi_{\Gamma^*}(U) : U \in RO, \Psi_{\Gamma^*}(U) - A \in \mathcal{I}\}$.*

Proof. Let $\emptyset(A) = \cup\{\Psi_{\Gamma^*}(U) : U \in RO, \Psi_{\Gamma^*}(U) - A \in \mathcal{I}\}$. Let $x \in \emptyset(A)$. Then there exists $U \in RO(X)$ such that $\Psi_{\Gamma^*}(U) - A \in \mathcal{I}$ and

$x \in \Psi_{\Gamma^*}(U)$. Therefore, there exists $V \in RO(x)$ such that $V - U \in \mathcal{I}$. By Corollary 3.3, $U \subset \Psi_{\Gamma^*}(U)$ and $U - A \subseteq \Psi_{\Gamma^*}(U) - A$ and hence $U - A \in \mathcal{I}$. Therefore, $V - A \subseteq (V - U) \cup (U - A) \in \mathcal{I}$ and hence $V - A \in \mathcal{I}$. Since $V \in RO(x)$ and $V - A \in \mathcal{I}$, $x \in \Psi_{\Gamma^*}(U)$. Hence, $\emptyset(A) \subseteq \Psi_{\Gamma^*}(A)$. Now let $x \in \Psi_{\Gamma^*}(A)$. Then there exists $U \in RO(x)$ such that $U - A \in \mathcal{I}$. By Corollary 3.3, $U \subseteq \Psi_{\Gamma^*}(U)$ and $\Psi_{\Gamma^*}(U) - A \subseteq [\Psi_{\Gamma^*}(U) - U] \cup [U - A]$. By Theorem 4.2, $\Psi_{\Gamma^*}(U) - U \in \mathcal{I}$ and hence $\Psi_{\Gamma^*}(U) - A \in \mathcal{I}$. Hence $x \in \emptyset(A)$ and $\emptyset(A) \supseteq \Psi_{\Gamma^*}(A)$. Consequently, we obtain $\emptyset(A) = \Psi_{\Gamma^*}(A)$. ∎

In [7], Newcomb defines $A = B$ [mod $\mathcal{I}$] if $(A-B) \cup (B-A) \in \mathcal{I}$ and observes that $=$ [mod $\mathcal{I}$] is an equivalence relation. By Theorem 3.2 (11), we have that if $A = B$ [mod $\mathcal{I}$], then $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(B)$.

Definition 4.6. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a Baire set with respect to τ and $\mathcal{I}$, denoted $A \in \mathcal{B}_r(X, \tau, \mathcal{I})$, if there exists a regular open set U such that $A = U$ [mod $\mathcal{I}$].

Lemma 4.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^*_* \mathcal{I}$. If U, $V \in RO(X)$ and $\Psi_{\Gamma^*}(U) = \Psi_{\Gamma^*}(V)$, then $U = V$ [mod $\mathcal{I}$].*

Proof. Since $U \in RO(X)$, we have $U \subseteq \Psi_{\Gamma^*}(U)$ and hence $U - V \subseteq \Psi_{\Gamma^*}(U) - V = \Psi_{\Gamma^*}(V) - V \in \mathcal{I}$ by Theorem 4.2. Therefore, $U - V \in \mathcal{I}$. Similarly $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence $U = V$ [mod $\mathcal{I}$]. ∎

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim^*_* \mathcal{I}$. If A, $B \in \mathcal{B}_r(X, \tau, \mathcal{I})$, and $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(B)$, then $A = B$ [mod $\mathcal{I}$].*

Proof. Let $U, V \in RO(X)$ such that $A = U$ [mod $\mathcal{I}$] and $B = V$ [mod $\mathcal{I}$]. Now $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(U)$ and $\Psi_{\Gamma^*}(B) = \Psi_{\Gamma^*}(V)$ by Theorem 3.2(11). Since $\Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(B)$ implies that $\Psi_{\Gamma^*}(U) = \Psi_{\Gamma^*}(V)$, $U = V$ [mod $\mathcal{I}$] by Lemma 4.7. Hence $A = B$ [mod $\mathcal{I}$] by transitivity. ∎

5. δ-codense ideals

Definition 5.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then an ideal $\mathcal{I}$ is said to be δ-codense if $RO(X, \tau) \cap \mathcal{I} = \emptyset$.

Lemma 5.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset $A \subseteq X$, if $A \subseteq \Gamma^*(A)$, then $Cl_\delta(A) = \Gamma^*(A) = Cl_{\Gamma^*}(A) = Cl_{\Gamma^*}(\Gamma^*(A))$.*

Proof. This follows from Lemma 2.3(4) and Theorem 2.4(2). ∎

A subset S of a topological space (X, τ) is said to be δ-semiopen [11] if $S \subseteq Cl(Int\delta(S))$, equivalently if there exists a δ-open set U such that $U \subseteq S \subseteq Cl(U)$. We recall that $Cl(U) = Cl_\delta(U)$ for every open set U of (X, τ).

Theorem 5.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following properties are equivalent:*

(1) *$\mathcal{I}$ is δ-codense.*
(2) *$G \subseteq \Gamma^*(G)$ for every $G \in \delta O(X)$.*
(3) *$S \subseteq \Gamma^*(S)$ for every δ-semiopen set S.*
(4) *$Cl_\delta(G) = \Gamma^*(G)$ for every $G \in \delta O(X)$.*
(5) *$Cl_\delta(S) = \Gamma^*(S)$ for every δ-semiopen set S.*
(6) *$Int_\delta(A) \subseteq Int_\delta(\Gamma^*(A))$ for every subset A of X.*

Proof. (1) $\Rightarrow$ (2): Let $G \in \delta O(X)$ and $x \in G$. There exists $G_0 \in RO(X)$ such that $x \in G_0 \subseteq G$. For any $V \in RO(x)$, $x \in G_0 \cap V \in RO(X)$. Since $\mathcal{I}$ is δ-codense, $G_0 \cap V \notin \mathcal{I}$ and $G \cap V \notin \mathcal{I}$ and hence $x \in \Gamma^*(G)$. Therefore, $G \subseteq \Gamma^*(G)$.
(2) $\Rightarrow$ (3): Let S be a δ-semiopen set, then there exits $V \in \delta O(X)$ such that $V \subseteq S \subseteq Cl_\delta(V)$. By Lemma 2.3(4), $S \subseteq Cl_\delta(V) = \Gamma^*(V)$ and hence $S \subseteq \Gamma^*(V) \subseteq \Gamma^*(S)$.
(3) $\Rightarrow$ (4): Let $G \in \delta O(X)$. Since every δ-open set is δ-semiopen, by (3) $G \subseteq \Gamma^*(G)$ and hence by Lemma 2.3(4) we have $Cl_\delta(G) = \Gamma^*(G)$.
(4) $\Rightarrow$ (5): Let S be any δ-semiopen set. Then there exists $G \in \delta O(X)$ such that $G \subseteq S \subseteq \Gamma^*(G)$. Therefore, by Lemma 2.3, $Cl_\delta(S) = Cl_\delta(G) = \Gamma^*(G) \subseteq \Gamma^*(S) \subseteq Cl_\delta(S)$ and hence $Cl_\delta(S) = \Gamma^*(S)$.
(5) $\Rightarrow$ (6): let A be any subset of X and $x \in Int_\delta(A)$. There exists $U \in \delta O(X)$ such that $x \in U \subseteq A$. By (5), $U \subseteq Cl_\delta(U) = \Gamma^*(U) \subseteq \Gamma^*(A)$ and $x \in Int_\delta(\Gamma^*(A))$. Therefore, $Int_\delta(A) \subseteq Int_\delta(\Gamma^*(A))$
(6) $\Rightarrow$ (1): Let $\emptyset \neq V \in RO(X)$. Then there exists $x \in V$ and $V = Int_\delta(V) \subseteq Int_\delta(\Gamma^*(V)) \subseteq \Gamma^*(V)$ and hence $V \notin \mathcal{I}$. Therefore, $RO(X) \cap \mathcal{I} = \emptyset$. ∎

Proposition 5.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*

(1) *If $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then there exists $A \in RO(X) - \{\emptyset\}$ such that $B = A$ [mod $\mathcal{I}$].*
(2) *Let $\mathcal{I}$ be δ-codense. Then $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ if and only if there exists $A \in RO(X) - \{\emptyset\}$ such that $B = A$ [mod $\mathcal{I}$].*

Proof. (1) Assume $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $B \in \mathcal{B}_r(X, \tau, \mathcal{I})$. Then there exists $A \in RO(X)$ such that $B = A$ [mod $\mathcal{I}$]; $(B-A) \cup (A-B) \in \mathcal{I}$. Hence $B - A \in \mathcal{I}$. Suppose that $A = \emptyset$. Then $B \in \mathcal{I}$ which is a

contradiction.
(2) Assume there exists $A \in RO(X) - \{\emptyset\}$ such that $B = A$ [mod $\mathcal{I}$]. Then $A = (B - J) \cup I$, where $J = B - A \in \mathcal{I}, I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$ by heredity and additivity, which contradicts that $RO(X) \cap \mathcal{I} = \emptyset$. ∎

Proposition 5.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. If $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $\Psi_{\Gamma^*}(B) \cap Int_\delta(\Gamma^*(B)) \neq \emptyset$.*

Proof. Assume $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then by Proposition 5.4(1), there exists $A \in RO(X) - \{\emptyset\}$ such that $B = A$ [mod $\mathcal{I}$]. This implies that $\emptyset \neq A \subseteq \Gamma^*(A) = \Gamma^*((B - J) \cup I) = \Gamma^*(B)$, where $J = B - A, I = A - B \in \mathcal{I}$ by Theorem 5.3 and Corollary 2.5. Also $\emptyset \neq A \subseteq \Psi_{\Gamma^*}(A) = \Psi_{\Gamma^*}(B)$ by Corollary 3.3 and Theorem 3.2 (11), so that $A \subseteq \Psi_{\Gamma^*}(B) \cap Int_\delta(\Gamma^*(B))$. ∎

Given an ideal topological space $(X, \tau, \mathcal{I})$, let $\mathcal{U}(X, \tau, \mathcal{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Proposition 5.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. The following properties are equivalent:*

(1) $A \in \mathcal{U}(X, \tau, \mathcal{I})$;
(2) $\Psi_{\Gamma^*}(A) \cap Int_\delta(\Gamma^*(A)) \neq \emptyset$;
(3) $\Psi_{\Gamma^*}(A) \cap \Gamma^*(A) \neq \emptyset$;
(4) $\Psi_{\Gamma^*}(A) \neq \emptyset$;
(5) $Int_{\Gamma^*}(A) \neq \emptyset$;
(6) *There exists $N \in RO(X) - \{\emptyset\}$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): Let $A \in \mathcal{U}(X, \tau, \mathcal{I})$, then there exists $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Then $Int_\delta(\Gamma^*(B)) \subseteq Int_\delta(\Gamma^*(A))$ and $\Psi_{\Gamma^*}(B) \subseteq \Psi_{\Gamma^*}(A)$ and hence $Int_\delta(\Gamma^*(B)) \cap \Psi_{\Gamma^*}(B) \subseteq Int_\delta(\Gamma^*(A)) \cap \Psi_{\Gamma^*}(A)$. By Proposition 5.5, we have $\Psi_{\Gamma^*}(A) \cap Int_\delta(\Gamma^*(A)) \neq \emptyset$.
(2) $\Rightarrow$ (3): The proof is obvious.
(3) $\Rightarrow$ (4): The proof is obvious.
(4) $\Rightarrow$ (5): If $\Psi_{\Gamma^*}(A) \neq \emptyset$, then there exists $U \in RO(X) - \{\emptyset\}$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 3.2, $\emptyset \neq (U \cap A) \subseteq \Psi_{\Gamma^*}(U) \cap A = \Psi_{\Gamma^*}((U - A) \cup (U \cap A)) \cap A = \Psi_{\Gamma^*}(U \cap A) \cap A \subseteq \Psi_{\Gamma^*}(A) \cap A = Int_{\Gamma^*}(A)$. Hence $Int_{\Gamma^*}(A) \neq \emptyset$.
(5) $\Rightarrow$ (6): If $Int_{\Gamma^*}(A) \neq \emptyset$, then by Theorem 2.6 there exists $N \in RO(X) - \{\emptyset\}$ and $I \in \mathcal{I}$ such that $\emptyset \neq N - I \subseteq A$. We

have $N - A \in \mathcal{I}$, $N = (N - A) \cup (N \cap A)$ and $N \notin \mathcal{I}$. This implies that $N \cap A \notin \mathcal{I}$.
(6) $\Rightarrow$ (1): Let $B = N \cap A \notin \mathcal{I}$ with $N \in RO(X) - \{\emptyset\}$ and $N - A \in \mathcal{I}$. Then $B \in \mathcal{B}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$. ∎

Theorem 5.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. Then $\Psi_{\Gamma^*}(A) \subseteq \Gamma^*(A)$ for every subset A of X.*

Proof. Suppose $x \in \Psi_{\Gamma^*}(A)$ and $x \notin \Gamma^*(A)$. Then there exists $U_x \in RO(x)$ such that $U_x \cap A \in \mathcal{I}$. Since $x \in \Psi_{\Gamma^*}(A)$, by Theorem 3.5 $x \in \cup\{U \in RO(X) : U - A \in \mathcal{I}\}$ and there exists $V \in RO(X)$ such that $x \in V$ and $V - A \in \mathcal{I}$. Now we have $U_x \cap V \in RO(x)$, $(U_x \cap V) \cap A \in \mathcal{I}$ and $(U_x \cap V) - A \in \mathcal{I}$ by heredity. Hence by finite additivity we have $((U_x \cap V) \cap A) \cup ((U_x \cap V) - A) = (U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in RO(x)$, this is contrary to $RO(X) \cap \mathcal{I} = \emptyset$. Therefore, $x \in \Gamma^*(A)$. This implies that $\Psi_{\Gamma^*}(A) \subseteq \Gamma^*(A)$. ∎

Corollary 5.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. Then $\Psi_{\Gamma^*}(A) \subseteq Cl_\delta(\Gamma^*(A))$ for every subset A of X.*

Theorem 5.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. Then $\Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(X - A) = \emptyset$ for every subset A of X.*

Proof. Assume that $z \in \Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(X - A)$ for some $z \in X$, then there exist regular open sets U, V containing z such that $U - A \in \mathcal{I}$ and $V \cap A \in \mathcal{I}$, respectively. Hence $(U \cap V) - A \in \mathcal{I}$ and $(U \cap V) \cap A \in \mathcal{I}$ so $U \cap V \in \mathcal{I}$ and $U \cap V \in RO(X)$. Since $\mathcal{I}$ is δ-codense, we have $U \cap V = \emptyset$. This is a contradiction. Hence $\Psi_{\Gamma^*}(A) \cap \Psi_{\Gamma^*}(X - A) = \emptyset$. ∎

Corollary 5.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. Then $\Gamma^*(A) \cup \Gamma^*(X - A) = X$ for every subset A of X.*

Theorem 5.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

(1) *$\mathcal{I}$ is δ-codense;*
(2) *$\Psi_{\Gamma^*}(\emptyset) = \emptyset$;*
(3) *If $A \subseteq X$ is regular closed, then $\Psi_{\Gamma^*}(A) - A = \emptyset$;*
(4) *If $I \in \mathcal{I}$, then $\Psi_{\Gamma^*}(I) = \emptyset$.*

Proof. (1) $\Rightarrow$ (2): Since $RO(X) \cap \mathcal{I} = \emptyset$, by Theorem 3.5 we have $\Psi_{\Gamma^*}(\emptyset) = \cup\{U \in RO(X) : U \in \mathcal{I}\} = \emptyset$.
(2) $\Rightarrow$ (3): Suppose $x \in \Psi_{\Gamma^*}(A) - A$, then there exists a $U_x \in RO(x)$ such that $x \in U_x - A \in \mathcal{I}$ and $U_x - A \in RO(X)$. But $U_x - A \in \{U \in RO(X) : U \in \mathcal{I}\} = \Psi_{\Gamma^*}(\emptyset)$ which implies that $\Psi_{\Gamma^*}(\emptyset) \neq \emptyset$. Hence

$\Psi_{\Gamma^*}(A) - A = \emptyset$.
(3) $\Rightarrow$ (4): Let $I \in \mathcal{I}$ and since $\emptyset$ is regular closed, then $\Psi_{\Gamma^*}(I) = \Psi_{\Gamma^*}(I \cup \emptyset) = \Psi_{\Gamma^*}(\emptyset) = \emptyset$.
(4) $\Rightarrow$ (1): Suppose $A \in RO(X) \cap \mathcal{I}$, then $A \in \mathcal{I}$ and by (4) $\Psi_{\Gamma^*}(A) = \emptyset$. Since $A \in RO(X)$, by Corollary 3.3 we have $A \subseteq \Psi_{\Gamma^*}(A) = \emptyset$. Hence $RO(X) \cap \mathcal{I} = \emptyset$. ∎

Definition 5.12. A subset A in an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_R$-dense if $\Gamma^*(A) = X$.

The collection of all $\mathcal{I}_R$-dense sets in $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}_R D(X, \tau)$. The collection of all dense sets in (X, τ) is denoted by $D(X, \tau)$. Now we show that the collection of dense sets in a topological space (X, τ_{Γ^*}) and the collection of $\mathcal{I}_R$-dense sets in an ideal topological space $(X, \tau, \mathcal{I})$ are equal if $\mathcal{I}$ is δ-codense.

Theorem 5.13. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is δ-codense , then $\mathcal{I}_R D(X, \tau) = D(X, \tau_{\Gamma^*})$.*

Proof. Let $D \in \mathcal{I}_R D(X, \tau)$. Then $Cl_{\Gamma^*}(D) = D \cup \Gamma^*(D) = X$, i.e. $D \in D(X, \tau_{\Gamma^*})$. Therefore, $\mathcal{I}_R D(X, \tau) \subseteq D(X, \tau_{\Gamma^*})$.

Conversely, let $D \in D(X, \tau_{\Gamma^*})$. Then $Cl_{\Gamma^*}(D) = D \cup \Gamma^*(D) = X$. We prove that $\Gamma^*(D) = X$. Let $x \in X$ such that $x \notin \Gamma^*(D)$. Therefore there exists $\emptyset \neq U \in RO(X)$ such that $U \cap D \in \mathcal{I}$. Since $U \notin \mathcal{I}$, $U \cap (X - D) \notin \mathcal{I}$ and hence $U \cap (X - D) \neq \emptyset$. Let $x_0 \in U \cap (X - D)$. Then $x_0 \notin D$ and also $x_0 \notin \Gamma^*(D)$. Because $x_0 \in \Gamma^*(D)$ implies that $U \cap D \notin \mathcal{I}$ which is contrary to $U \cap D \in \mathcal{I}$. Thus $x_0 \notin D \cup \Gamma^*(D) = Cl_{\Gamma^*}(D) = X$. This is a contradiction. Therefore, we obtain $D \in \mathcal{I}_R D(X, \tau)$. Therefore, $D(X, \tau_{\Gamma^*}) \subseteq \mathcal{I}_R D(X, \tau)$. Hence $\mathcal{I}_R D(X, \tau) = D(X, \tau_{\Gamma^*})$. ∎

Theorem 5.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for $x \in X$, $X - \{x\}$ is $\mathcal{I}_R$-dense if and only if $\Psi_{\Gamma^*}(\{x\}) = \emptyset$.*

Proof. The proof follows from the definition of $\mathcal{I}_R$-dense sets, since $\Psi_{\Gamma^*}(\{x\}) = X - \Gamma^*(X - \{x\}) = \emptyset$ if and only if $X = \Gamma^*(X - \{x\})$. ∎

Proposition 5.15. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\mathcal{I}$ be δ-codense. Then $\Psi_{\Gamma^*}(A) \neq \emptyset$ if and only if A contains the nonempty τ_{Γ^*}-interior.*

Proof. Let $\Psi_{\Gamma^*}(A) \neq \emptyset$. By Theorem 3.5(1), $\Psi_{\Gamma^*}(A) = \cup\{U \in RO(X) : U - A \in \mathcal{I}\}$ and there exists a nonempty set $U \in RO(X)$ such that $U - A \in \mathcal{I}$. Let $U - A = P$, where $P \in \mathcal{I}$. Now $U - P \subseteq A$. By Theorem 2.6, $U - P \in \tau_{\Gamma^*}$ and A contains the nonempty τ_{Γ^*}-interior.

Conversely, suppose that A contains the nonempty τ_{Γ^*}-interior. Hence there exists $U \in RO(X)$ and $P \in \mathcal{I}$ such that $U - P \subseteq A$. So $U - A \subseteq P$. Let $H = U - A \subseteq P$, then $H \in \mathcal{I}$. Hence $\cup\{U \in RO(X) : U - A \in \mathcal{I}\} = \Psi_{\Gamma^*}(A) \neq \emptyset$. ∎

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