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A CLASS OF COMPLETE METRIZABLE Q-ALGEBRAS

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Abstract. The fundamental topological algebras, which extend both locally convex and locally bounded concepts, have been introduced before. Here we introduce a class of fundamental Q-algebras.

1. Introduction

In this note, we assume that all algebras are complex unital completely metrizable topological algebras. We recall that a topological algebra A is said Q-algebra if $InvA$ is open. In 1965, Allan [2] provided the definition of the radius of boundedness β to develop the spectral theory for locally convex topological algebras. In 1979, T. Husain [10] introduced the concepts of strongly sequential and infrasequential topological algebras and in 1990 [4], the first author introduced the concept of fundamental topological algebras to generalize the famous Cohen factorization theorem. In ([2], [3], [6], [11], [13]) β is extended and studied for general topological algebras and compared with spectral radius r. In particular, Kinani, Oubbi and Oudadess [11] showed in 1997 that every strongly sequential locally convex algebra for which $r \leq \beta$, is a Q-algebra.

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Also Anjidani [3] showed in 2014 that for every fundamental topological algebra, $r \leq \beta$. In this note, at first we combine two recent results and observe that every strongly sequential fundamental topological algebra is a Q-algebra and by using Theorem 2.1 of [3] we introduce another class of commutative fundamental Q-algebras. Then we give an example to show the existence of such algebras. Finally we list some properties of two subclasses of fundamental topological algebras.

2. Definitions and related results

We begin with the needed definitions and related preliminary results.

Definition 2.1. ([13]) Let x be an element of a topological algebra (A, τ). We will say that x is bounded if there exists some $r > 0$ such that the sequence $(\frac{x^n}{r^n})_n$ converges to zero. The radius of boundedness of x with respect to (A, τ) is denoted by $\beta(x)$ and defined by

$$\beta(x) = \inf\{r > 0 \ : \ (\frac{x^n}{r^n}) \to 0\} \tag{2.1}$$

with the convention : $\inf \emptyset = +\infty$. We also say A is a topological algebra with bounded elements if all elements of A are bounded.

Theorem 2.2. ([13]) *Suppose A is a topological algebra and $x \in A$. Then*
(i) $\beta(x) \geq 0$ and
$\beta(\lambda x) = |\lambda|\beta(x)$ *for every* $\lambda \in C$ *, with the convention* $0 \cdot \infty = \infty$.
(ii) $\beta(0) = r(0) = 0$.
(iii) $\beta(x) < +\infty$ if and only if x is bounded.
(iv) If $r > \beta(x)$, then the sequence $(\frac{x^n}{r^n})_n$ converges to 0.
(v) If A is commutative, then $\beta(xy) \leq \beta(x)\beta(y)$, for all $x, y \in A$.

Definition 2.3. ([10]) Let A be a topological algebra.
(i) A is said to be strongly sequential if there exists a neighbourhood U of 0 such that for all $x \in U$ we have $x^n \to 0$ as $n \to \infty$.
(ii) A is said to be infrasequential if for each bounded set $B \subseteq A$ there exists $\lambda > 0$ such that for all $x \in B$ we have $(\lambda x)^n \to 0$ as $n \to \infty$.

Proposition 2.4. ([10]) *With reference to the above definitions, (i) $\Rightarrow$ (ii), but the reverse implication needs not hold.*

Definition 2.5. ([4]) A topological algebra A is said to be fundamental if there exists $b > 1$ such that for every sequence (a_n) of A the convergence of $b^n(a_{n+1}-a_n)$ to zero in A implies that (a_n) is a Cauchy sequence.

Definition 2.6. Let A be a topological algebra and $x \in A$. The spectrum of x is the set $Sp(x)$ of complex numbers defined as follows:

$$Sp(x) = \{\lambda \in \mathbb{C} \ : \ \lambda - x \text{ is not invertible}\},$$

and its spectral radius is defined to be:

$$r(x) = \sup \{ \ |\lambda| \ : \lambda \in Sp(x)\}$$

Theorem 2.7. ([3], [5]) *Let A be a complete metrizable fundamental topological algebra and $a \in A$. Then*

$$\beta(a) < 1 \text{ implies } 1 - a \in InvA.$$

Corollary 2.8. ([3]) *Let A be a complete metrizable fundamental topological algebra and $a \in A$. Then*

$$r(a) \leq \beta(a).$$

3. Fundamental Q-algebras

Kinani, Oubbi and Oudadess ([11], Proposition III.1) proved that in locally convex algebras, β is continuous at zero if and only if the algebra is strongly sequential.

Here we drop the local convexity and get the same result.

Proposition 3.1. *Let A be a topological algebra. Then β is continuous at zero if and only if A is strongly sequential.*

Proof. Suppose β is continuous at zero. Then for $\varepsilon = 1$ there exists a neighbourhood U of zero such that for every $x \in U$, $| \ \beta(x) - \beta(0) \ | < 1$. Since $\beta(0) = 0$ and β is non-negative $x \in U$ implies $x^n \to 0$. Thus A is strongly sequential. Now assume that the algebra A is strongly sequential. Suppose U is the neighbourhood of zero defined in 2.3 (i). Then for every $\varepsilon > 0$, $x \in \frac{\varepsilon}{2} U$ implies $\beta(x) < \varepsilon$; for if $x = \frac{\varepsilon}{2} u \in \frac{\varepsilon}{2} U$, then by Theorem 2.2 (i) and (2.1), we have $\beta(x) = \frac{\varepsilon}{2} \beta(u) \leq \frac{\varepsilon}{2} < \varepsilon$. ∎

Also Kinani, Oubbi and Oudadess ([11], Proposition III.2) proved that every locally convex algebra for which $r \leq \beta$, is a Q-algebra.

This proposition is also true for the fundamental strongly sequential algebras with the same proof of [11]; so we have:

Proposition 3.2. *Every strongly sequential fundamental topological algebra is a Q-algebra.*

Theorem 3.3. *Let A be a commutative fundamental topological algebra. Let d denote a translation-invariant metric defining the topology on A, and we write*

$$q(x) = d(x, 0) \ (x \in A). \tag{3.1}$$

We also suppose the existence of a strictly increasing function $\varphi \ : \ [0, \infty) \to [0, \infty)$ such that

$$\beta(x) \leq \varphi(q(x)). \tag{3.2}$$

Then A is a Q- algebra.

Proof. Suppose $a \in InvA$. We show that $N(a; r) \subseteq InvA$, where $r = \varphi^{-1}(\varphi(q(a^{-1})))^{-1}$ and by $N(a; r)$ we mean $\{x \in A \ : \ d(x, a) < r\}$. To see this, suppose $x \in N(a; r)$; then by (3.1) and (3.2) we have

$$\beta(x - a) \leq \varphi(q(x - a)) = \varphi(d(x - a, 0)). \tag{3.3}$$

Since d is translation-invariant, and φ is strictly increasing and

$$d(x, a) < \varphi^{-1}(\varphi(q(a^{-1})))^{-1},$$

we have

$$\varphi(d(x - a, 0)) = \varphi(d(x, a)) < \varphi(\varphi^{-1}(\varphi(q(a^{-1})))^{-1}) = (\varphi(q(a^{-1})))^{-1}. \tag{3.4}$$

Since A is commutative, by (3.3), (3.4) and Theorem 2.1 (v), we conclude:

$$\beta(1 - xa^{-1}) \leq \beta(a^{-1})\beta(a - x) \leq \varphi(q(a^{-1}))(\varphi(q(a^{-1})))^{-1} < 1.$$

Then by Theorem 2.7 we have,

$$xa^{-1} = 1 - (1 - xa^{-1}) \in InvA.$$

Since $InvA$ is a group; then $x \in InvA$ as desired. ∎

The following lemma is obvious. We apply it to prove the existence of a strongly sequential fundamental algebra which is neither locally bounded and nor locally convex.

Lemma 3.4. *In every locally bounded algebra with p-norm $\|.\|_p$ we have $\beta(x) \leq \|x\|_p^{\frac{1}{p}}$.*

Theorem 3.5. *There exists a commutative strongly sequential fundamental algebra which is neither locally bounded nor locally convex and also satisfies the conditions of Theorem 3.3.*

Let A be a commutative locally bounded but non-locally convex algebra with metric d_1 such that $d_1(x,y) = \|x-y\|_p$, $(p \in [0,1))$ and, X be a complete metrizable locally convex and non-locally bounded topological vector space with translation invariant metric d_2. Denote by e the unit element of A. Suppose $(a,x) \to xa$ is a bilinear and continuous mapping of $A \times X$ into X satisfying $x(a_1a_2) = (xa_1)a_2$ and $xe = x$ for all $a_1, a_2 \in A$ and $x \in X$. Then X is a topological unit linked right A-module with module multiplication defined by $(a,x) \to xa$ and $Z = X \times A$ is a non-locally bounded, non-locally convex, fundamental topological vector space with pointwise operations and metric d such that $d((x_1,a_1),(x_2,a_2)) = d_1(a_1,a_2) + d_2(x_1,x_2)$. Define the multiplication on Z by $(x_1,a_1)(x_2,a_2) = (x_1a_2 + x_2a_1, a_1a_2)$ for all $x_1, x_2 \in X,\ a_1, a_2 \in A$. Now, Z is an algebra and since the module multiplication is continuous, Z is a topological algebra. It is clear that $(0,e)$ is the unit element of Z (see [3], [5]).
Now we show $\beta(z) \leq (q(z))^{\frac{1}{p}}$ which implies that Z is a strongly sequential fundamental topological algebra. Suppose $z = (x,a) \in Z$, then $(x,a)^n = (nxa^{n-1}, a^n)$ for all $n \in N$. Since A is a locally bounded topological algebra, if $a \in A$ and $\varepsilon > 0$, by Lemma 3.4 there exists $r = (\|a\|_p)^{\frac{1}{p}} + \frac{\varepsilon}{2} \in R^+$ such that $\frac{1}{r^n}a^{n-1} \to 0$ and so $\frac{1}{r^n}xa^{n-1} \to 0$ in X. Hence, $\frac{n}{(r+\frac{\varepsilon}{2})^n}xa^{n-1} \to 0$ in X and $\frac{1}{(r+\frac{\varepsilon}{2})^n}a^n \to 0$ in A. Therefore, $\frac{1}{(r+\frac{\varepsilon}{2})^n}(x,a)^n \to 0$; since ε is arbitrary, we have,

$$\beta(z) \leq (\|a\|_p)^{\frac{1}{p}} = (d_1(a,0))^{\frac{1}{p}} \leq (d((x,a),(0,0)))^{\frac{1}{p}} = (q(z))^{\frac{1}{p}}.$$

Finally, we note that if in Theorem 3.3 we suppose $\varphi(x) = x^{\frac{1}{p}}$ then Z satisfies the assumptions of this theorem.

The next example guarantees the consistency of the assumptions of Theorem 3.5.

Example 3.6. ([3], [5]) Let $0 < p < 1$ and $T = \{t_1, t_2, ...\}$ be a set of symbols and S be the commutative semigroup generated by T with operation defined by $t_it_j = t_{min(i,j)}$ if $i \neq j$ and $t_i^n t_i = t_i^{n+1}$. Then $S = \{t_i^j\ i,j \in N\}$. Let

$$A = \{\Sigma_{i,j=1}^{\infty}\ \alpha_{ij}\ t_i^j\ :\ \Sigma_{i,j=1}^{\infty}\ |\alpha_{ij}|^p (j+1)^p\ <\ \infty,\ \alpha_{ij} \in \mathbb{C}\}.$$

Then A is a locally bounded and non-locally convex F-algebra generated by S with p-norm defined by $\|\Sigma_{i,j=1}^{\infty}\ \alpha_{ij}t_i^j\|_p = \Sigma_{i,j=1}^{\infty}\ |\alpha_{ij}|^p\ (j+1)^p$. Let $\widetilde{A} = A \oplus C$ be the unitization of A. Now $\widetilde{A}$ is unital locally bounded and non-locally convex. Let also

$X = \{\Sigma_{i,j=1}^{\infty} \alpha_{ij}\, t_i^j \ : \ \alpha_{ij} \in \mathbb{C},\ \Sigma_{i,j=1}^{\infty} |\alpha_{ij}|^p\, p_m(t_i^j) < \infty, for\ all\ m \in N\}$,

where,

$$p_m(t_i^j) = \begin{cases} (j+1)^p, & \text{if } i \leq m \\ 1, & \text{if } i \ > \ m, \end{cases}$$

and let $p_m(\Sigma_{i,j=1}^{\infty}\ \alpha_{ij}\ t_i^j) = \Sigma_{i,j=1}^{\infty}\ |\alpha_{ij}|^p\ p_m(t_i^j)$. Then, every p_m is a seminorm on X, X is a locally convex and non-locally bounded algebra and A is a subalgebra of X and so X is a locally convex right $\widetilde{A}$-module with module multiplication given by

$$x(a, \lambda) = xa + \lambda x \text{ (for } x \in X,\ a \in A,\ \lambda \in \mathbb{C}).$$

Therefore, $Z = X \times \widetilde{A}$ with the algebra operations defined as in Theorem 3.5, is a strongly sequential fundamental topological algebra which is neither locally bounded and nor locally convex. Also Z^* separates the points on Z.

Remark 3.7. In ([11], Lemma II.1) it is proved that $1 + \{x : \beta(x) < 1\} \subseteq InvA$, for locally convex algebras; the same result holds by Theorem 2.7 for fundamental topological algebras.

4. Infrasequential and strongly sequential fundamental algebras

In this section, we list some properties of infrasequential and strongly sequential fundamental algebras.

Definition 4.1. Let A be a topological algebra and $E \subseteq A$. We say E is β-bounded if there exists $M > 0$ such that for every $x \in E$ we have $\beta(x) \leq M$.

Obviously, the definition of T. Husain for the infrasequential topological algebra [10] is equivalent to this fact that every bounded set, is β -bounded.

Proposition 4.2. *Let A be an infrasequential fundamental algebra. Then the following properties hold.*
(i) The spectrum of every element is compact.
(ii) The mapping $F(z) = (z - a)^{-1}$ is a holomorphic mapping of $\mathbb{C} \setminus Sp(a)$ into A, i.e. $\varphi \circ F$ is a holomorphic mapping of $\mathbb{C} \setminus Sp(a)$ into $\mathbb{C}$ for all $\varphi \in A^*$.
(iii) If A^* separates the points on A, then, the spectrum $Sp(a)$ of a is

nonempty; in particular if A is a division algebra, then A is isomorphic to $\mathbb{C}$.

Proof. Since every infrasequential fundamental algebra is a topological algebra with bounded elements; by [3], the results hold. ∎

Proposition 4.3. *Let A be a strongly sequential fundamental algebra. Then the following properties hold.*
(i) The spectrum of every element is compact. The mapping $F(z) = (z-a)^{-1}$ is a holomorphic mapping of $\mathbb{C} \setminus Sp(a)$ into A, i.e. $\varphi \circ F$ is a holomorphic mapping of $\mathbb{C} \setminus Sp(a)$ into $\mathbb{C}$ for all $\varphi \in A^*$. If A^* separates the points on A then, the spectrum $Sp(a)$ of a is nonempty; in particular if A is a division algebra, then A is isomorphic to $\mathbb{C}$.
(ii) Let $a_n \in InvA$ $(n = 1, 2, ...)$, $a = \lim_{n\to\infty} a_n$ and let $\{a_n^{-1} : n \in N\}$ be bounded. Then $a \in InvA$.
(iii) If, moreover, the boundedness of sets is equivalent with β-boundedness in A and $a \in \partial\,(InvA)$ then a is a joint topological divisor of zero (a is a joint topological divisor of zero if and only if there exists a sequence $(x_n)_n$ as $x_n \nrightarrow 0$, $ax_n \to 0$ and $x_n a \to 0$).
(iv) Every multiplicative linear functional on A is continuous.
(v) If φ is a linear functional on A such that $\varphi(1) = 1$ and $ker\varphi \subseteq Sing(A)$ then φ is continuous.

Proof. (i) Since every strongly sequential algebra is infrasequential algebra, by proposition 4.2 and also according to Proposition 3.2, every fundamental strongly sequential algebra is a Q-algebra (see e.g.[7], [9], [12]), and the result holds.
(ii) Since according to Proposition 3.2, every fundamental strongly sequential algebra is a Q-algebra; by Balachandran ([7], p.167) we get the result.
(iii) The proof is similar to ([8], Chapter 1, Section 2, Theorem 14).
(iv) The proof is similar to ([5], Theorem 4.5).
(v) The proof is similar to ([5], Theorem 4.6). ∎

Remark 4.4. According to [1] every locally pseudoconvex topological algebra is a fundamental topological algebra; and thus all the above statements which are true for fundamental topological algebras, also hold for locally pseudoconvex topological algebras and of course for locally convex and locally bounded algebras.

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