"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 26(2016), No. 1, 25-38

R_{cl}-LOCALLY CONNECTED SPACES

J. K. KOHLI, D. SINGH AND B. K. TYAGI

Abstract: A new generalization of local connectedness called 'R_{cl}-local connectedness' is introduced. Basic properties of R_{cl}-locally connected spaces are studied and their place in the hierarchy of variants of local connectedness which already exist in the literature is discussed. R_{cl}-local connectedness is preserved in the passage to r_{cl}-open sets and is invariant under disjoint topological sums. A necessary and sufficient condition for a product of R_{cl}-local connected spaces be R_{cl}-locally connected proved. Preservation of R_{cl}-local connectedness under mappings is investigated. It is shown that R_{cl}-local connectedness is preserved under quotients and an r_{cl}-quotient of an R_{cl}-locally connected space is locally connected. The category of R_{cl}-locally connected spaces is properly contained in the category of sum connected spaces (Math. Nachrichten 82(1978), 121-129; Ann. Acad. Sci. Fenn. AI Math. 3 (1977), 185-205) and constitutes a mono-coreflective subcategory of TOP.

Keywords and phrases: R_{cl}-locally connected space, sum connected space, r_{cl}-open set, R_{cl}-space, mildly compact space, mono-coreflective subcategory.
(2010) Mathematics Subject Classification: 54D05;54A10, 54C05, 54C08, 54C10, 54C35.

1. Introduction:

Local connectedness is an important topological property. Several variants / generalizations of local connectedness occur in mathematical literature. For example see ([12] [24] [26] [29] [33]). The full subcategory of locally connected spaces is a mono-coreflective subcategory of TOP ($\equiv$ the category of topological spaces and continuous maps) [10].

However, the larger category of almost locally connected spaces [29] is not a coreflective subcategory of TOP. In ([24] [26]) the categories of quasi (pseudo) locally connected spaces and Z-locally connected spaces are introduced and are shown to be mono-coreflective subcategories of TOP. In this paper we introduce the category of 'R_{cl}-locally connected spaces' which also turns out to be a mono-coreflective subcategory of TOP.

The category of R_{cl}-locally connected spaces is properly contained in the category of sum connected spaces [15] which in its turn strictly contains each of the categories of almost (quasi, pseudo) locally connected spaces as well as the category of Z-locally connected spaces. Organization of the paper is as follows: Section 2 is devoted to preliminaries and basic definitions. In Section 3 we introduce the notion of an 'R_{cl}-locally connected space' and reflect upon its place in the hierarchy of variants of local connectedness that already exist in the literature. Basic properties of R_{cl}–locally connected spaces are studied in Section 4, wherein it is shown that (i) An R_{cl}-locally connected, R_{cl}-space is locally connected; (ii) A T_0 R_{cl}-locally connected, R_{cl}-space is discrete; (iii) A mildly compact R_{cl}-locally connected space has at most finitely many components; (iv) R_{cl}-local connectedness is r_{cl}-open hereditary and is preserved under disjoint topological sums; (v) A product theorem for R_{cl}-locally connected spaces is formulated and proved. Section 5 is devoted to investigate the interplay / preservation under mappings wherein it is shown that (i) R_{cl}-local connectedness is preserved under quotients; and (ii) r_{cl}-quotient of an R_{cl}-locally connected space is locally connected. In Section 6 we consider function spaces of R_{cl}–locally connected spaces. Section 7 is devoted to discuss the technique of change of topology and its correlation with R_{cl}–locally connected spaces.

2. Preliminaries, basic definitions and almost locally connected spaces.

Let X be a topological space. A subset A of X is said to be **regular open** if it is the interior of its closure, i.e., $A = \overset{\circ}{\overline{A}}$ The complement of a regular open set is referred to as a **regular closed** set. A union of regular open sets is called **δ-open** [42]. The complement of a δ-open set is referred to as a **δ-closed** set. A subset A of a space X is called **regular G_δ -set** [28] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^0$, where each F_n is a closed subset of X (here F_n^0 denotes the interior of F_n). The complement of a regular G_δ-set is called a **regular F_σ-set**. A subset A of a space X is called **$\mathbf{d}_\delta$ − open** [18] if for each $x \in A$, there exists a regular F_σ–set H such that $x \in H \subseteq A$; equivalently, A is expressible as a union of regular F_σ-sets. A point $x \in X$ is called a **θ-adherent point** [42] of $A \subset X$ if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denote the set of all θ-adherent points of A. The set A is called **θ-closed** [42] if $A = cl_\theta A$. The complement of a θ-closed set is referred to as a **θ-open set.** The correspondence $A \to A_\theta$ is not a Kuratowski closure closure operator since it is not idempotent. However, it is Čech closure operator in the sense of [7].

2.1 Definitions. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*

(i) ***strongly continuous*** [27] *if $f(\overline{A}) \subset f(A)$ for every subset A of X.*

(ii) ***perfectly continuous*** ([23] [34]) *if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.*

(iii) ***cl-supercontinuous*** [36] *(=* ***clopen continuous*** [35]*) if for each $x \in X$ and each open set V containing $f(x)$, there is a clopen set U containing x such that $f(U) \subset V$.*

2.2 Definitions: *A topological space X is said to be*

(i) ***almost locally connected*** ([29] [33]) *if for each $x \in X$ and each regular open set U containing x there exists a connected open set V containing x such that $V \subset U$.*

(ii) ***quasi (pseudo) locally connected*** [24] *if for each $x \in X$ and each θ-open set (regular F_σ-set) U containing x there is an open connected set V containing x such that $V \subset U$.*

(iii) ***Z-locally connected*** [26] *if for each $x \in X$ and each cozero set U containing x there exists an open connected set V such that $x \in V \subset U$.*

(iv) ***sum connected*** [15] *if each $x \in X$ has a connected neighbourhood, or equivalently each component of X is open in X.*

Sum connected spaces have also been referred to as weakly locally connected spaces in the literature (see [6] [30] [32]).

For the categorical terms used but not defined in the paper we refer the reader to Herrlich and Strecker ([10] [11]).

3. R_{cl}-LOCALLY CONNECTED SPACES

Let X be a topological space. Any intersection of clopen subsets of X is called **cl-closed** [36]. The complement of a cl-closed set is referred to as **cl-open.** An open set U of X is said to be r_{cl}**-open** [41] if for each $x \in U$ there exists a cl-closed set C_x containing x such that $C_x \subset U$;equivalently U is expressible as a union of cl-closed sets. The complement of an r_{cl}-open set is called r_{cl}**-closed**. Not every open set in a space is r_{cl}-open. For example, no nonempty proper open subset of the real line $\mathbb{R}$ is r_{cl}-open. On the other hand, if X is a zero dimensional space, then every open subset of X is r_{cl}-open.

3.1 Definition ([20] [41])**:** *A topological space X is said to be an* R_{cl}***-space*** *if every open set in X is r_{cl}-open.*

Clearly every zero dimensional space as well as every ultra Hausdorff space is an R_{cl}-space. However, the space of strong ultrafilter topology [40, Example 113, p. 135] is a Hausdorff, extremally disconnected, R_{cl}-space but not a zero dimensional space. A nondegenerate indiscrete space is an R_{cl}-space which is not ultra Hausdorff.

3.2 Definition: A topological space X is said to be R_{cl}**-locally connected** at $x \in X$ if for each r_{cl}-open set U containing x there exists an open connected set V such that $x \in V \subset U$. The space X is said to be R_{cl}-locally connected if it is R_{cl}-locally connected at each $x \in X$.

Clearly every locally connected space is R_{cl}-locally connected. However, an R_{cl}-locally connected space need not be locally connected. On the other hand, every R_{cl}-space, which is R_{cl}-locally connected is locally connected (see Proposition 4.1).

The following diagram well illustrates the place of R_{cl}-local connectedness in the hierarchy of variants of local connectedness which already exist in the literature.

$$\begin{array}{ccccc}
\text{locally connected} & \rightarrow & \text{almost locally connected} & \rightarrow & \text{quasi locally connected} \\
\downarrow & & & & \downarrow \\
R_{cl}-\text{locally connected} & & & & \text{pseudo locally connected} \\
\downarrow & & & & \downarrow \\
\text{sum connected } (\equiv \text{weakly locally connected}) & & \leftarrow & & Z\text{- locally connected}
\end{array}$$

4. Basic properties of R_{cl}-locally connected spaces

4.1 Proposition. *An R_{cl}-space which is R_{cl}-locally connected is locally connected.*

Proof: In an R_{cl}-space every open set is r_{cl}-open.

4.2 Proposition. *Every R_{cl}-locally connected space is sum connected.*

Proof: This is immediate in view of [26, Proposition 3.5] and the fact that every clopen set is r_{cl}-open.

4.2 Proposition. *A T_0 $R_{cl}-$ space is totally disconnected.*

Proof: Let X be a T_0 R_{cl}-space. Let A be any subset of X having two or more points. We shall show that A is not connected. To this end, let $x, y \in A$, $x \neq y$.Since X is a T_0–space, there exists an open set U containing one of the points x and y but not both. To be precise, assume that $x \in U$.Since X is an R_{cl}-space, there exists a cl-closed set B such that $x \in B \subset U$. Let $B = \bigcap\{C_\alpha : \alpha \in \Lambda\}$,where each C_α is a clopen set. Then there exists an $\alpha_0 \in \Lambda$ such that $x \in C_{\alpha_0}$ and $y \notin C_{\alpha_0}$.Then $(A \bigcap C_{\alpha_0}) \bigcup (A \bigcap (X \backslash C_{\alpha_0}))$ is a partition of A and so A is disconnected. Thus the space X is totally disconnected.

4.4 Theorem: *A space X is R_{cl}-locally connected if and only if each component of every r_{cl}-open subset of X is open in X.*

Proof: Suppose that X is an R_{cl}-locally connected space and let U be an r_{cl}-open subset of X. Let C be a component of U and let $x \in C$. Since X is R_{cl}-locally connected, there exists an open connected set V such that $x \in V \subset U$. In view of maximality of C and connectedness of V, $V \subset C$. Thus C, being a neighbourhood of each of its points, is open in X.

Conversely, suppose that the components of r_{cl}-open sets are open in X. Let $x \in X$.Let W be an r_{cl}-open set in X containing x and let

C be the component of x in W. By hypothesis C is an open connected set and so X is R_{cl}-locally connected at x.

4.5 Proposition: *In an R_{cl}-locally connected space, components and quasicomponents coincide in every r_{cl}-open set.*

Proof: Let X be an R_{cl}-locally connected space and let U be an r_{cl}-open set in X. Then in view of Theorem 4.4 any component of U is open in U and hence in X. Thus every quasicomponent of U, being a union of components, is open. Since every open quasicomponent is a component, the result is immediate.

4.6 Proposition: *A T_0 sum connected, R_{cl}-space is discrete.*

Proof: This is immediate in view of Proposition 4.3 and fact that components are open in a sum connected space. (see also [26, proposition 3.5])

4.7 Corollary [26, Proposition 3.10]**:** *A T_0 zero dimensional sum connected space is discrete.*

Proof: Every zero dimensional space is an R_{cl}-space.

4.8 Definition ([38] [39])**:** *A space X is said to be mildly compact if every clopen cover of X has a finite subcover.*

Sostak refers mildly compact spaces as clustered spaces in [38].

4.9 Proposition: *Every mildly compact R_{cl}-locally connected space has at most finitely many components.*

Proof: By Proposition 4.2 every R_{cl}-locally connected space is sum connected, so the result follows in view of [26, Proposition3.6].

4.10 Corollary: *Every pseudocompact, R_{cl}-locally connected space has at most finitely many components.*

4.11 Corollary ([15, Proposition 2.16])**:** *Every pseudocompact sum connected space has at most finitely many components.*

4.12 Theorem: *Every r_{cl}-open subspace of an R_{cl}-locally connected space X is R_{cl}-locally connected space.*

Proof: Let Y be an r_{cl}-open subspace of an R_{cl}-locally connected X. Let U be an r_{cl}-open subset of Y. In view of r_{cl}-openness of Y in X, it is easily verified that U is r_{cl}-open subset of X. Let $x \in U$. Since X is R_{cl}-locally connected, there exists a connected open set C in X such that $x \in C \subset U$. Clearly, C is an open connected subset of Y and so Y is R_{cl}-locally connected.

4.13 Theorem: *Every r_{cl}-open cover of an R_{cl}-locally connected space has a refinement consisting of open connected sets.*

Proof: Let X be an R_{cl}-locally connected space and let $\wp = \{V_\alpha : \alpha \in \Lambda\}$be an r_{cl}-open cover of X. Since by Theorem 4.4 components of r_{cl}-open sets in an R_{cl}-locally connected space are open, components of members of $\wp$ constitute a refinement of $\wp$ consisting of open connected sets.

4.14 Theorem: *Every clopen cover of a sum connected space has a refinement consisting of clopen connected sets.*

Proof: This is immediate in view of the fact that in a sum connected space components are open.

4.15 Theorem: *The disjoint topological sum of any family of R_{cl}-locally connected spaces is R_{cl}-locally connected.*

Proof: Let $\{X_\alpha : \alpha \in \Lambda\}$ be any family of R_{cl}-locally connected spaces and let $X = \underset{\alpha\in\Lambda}{\oplus} X_\alpha$ be their disjoint topological sum. Let U be any r_{cl}-open set in X and let $x \in U$. Then there exists $\beta \in \Lambda$ such that $x \in U \bigcap X_\beta$. It is easily verified that $U \bigcap X_\beta$ is an r_{cl}-open set in X_β. Since X_β is R_{cl}-locally connected, there exists a connected open set C in $X_\beta$containing x such that $C \subset U \bigcap X_\beta$.Clearly C is a connected open set in X and so X is R_{cl}-locally connected.

4.16 Lemma: *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces and let $X = \prod X_\alpha$ be the product space. If $x = (x_\alpha) \in X$ and V is an r_{cl}-open set in the product space containing x, then there exists a basic r_{cl}-open set $\prod V_\alpha$ such that $x \in \prod V_\alpha \subset V$, where V_α is an r_{cl}-open set in X_α for each $\alpha \in \Lambda$ and $V_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$.*

Proof: Since V is an open set in the product space X containing $x = (x_\alpha)$, there exists a basic open set $B = \prod B_\alpha$ in X containing x, where $B_\alpha = X_\alpha$for $\alpha \neq \alpha_1, ..., \alpha_n$ and each B_α is open in $X_\alpha$and $B \subset V$.

Since V is an r_{cl}-open set in the product space containing x, for each $\alpha = \alpha_i$ $(i = 1, ..., n)$, there exists an r_{cl}-open set V_{α_i} in X_{α_i} such that $x_{\alpha_i} \in V_{\alpha_i} \subset B_{\alpha_i} \subset p_{\alpha_i}(V)$. Then $\prod V_\alpha = \prod_{\alpha\neq\alpha_1,...,\alpha_n} X_\alpha \times V_{\alpha_1} \times .... \times V_{\alpha_n}$ is the required basic r_{cl}-open set in the product space containing x and contained in V.

4.17 Theorem: *Let$\{X_\alpha : \alpha \in \Lambda\}$ be any family of R_{cl}-locally connected spaces. Then their product $X = \prod X_\alpha$ is R_{cl}-locally connected if and only if all except finitely many coordinate spaces are connected,*

Proof: Suppose that the product space $X = \prod X_\alpha$ is R_{cl}-locally connected. Since projection maps are continuous open maps and in view of the fact that continuous open images of R_{cl}-locally connected spaces are R_{cl}-locally connected (see Corollary 5.2), each X_α is an R_{cl}-locally connected space. Again, since if W is an open connected set in X, then $p_\alpha(W) = X_\alpha$ for all but finitely many α, where $p_\alpha : \prod X_\alpha \to X_\alpha$ denotes the projection map, and so all except finitely many X_α are connected.

Conversely, let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of R_{cl}-locally connected spaces such that each X_α is connected except for $\alpha \neq \alpha_1, ..., \alpha_n$ and let $X = \prod X_\alpha$ be the product space. Let $S = \{\alpha_1, ..., \alpha_n\}$. Let $x = (x_\alpha) \in X$ and let G be a r_{cl}-open set in the product space X containing x. By Lemma 4.16, there is a basic r_{cl}-open set $\prod G_\alpha$ such that $x \in \prod G_\alpha \subset G$, where G_α is an r_{cl}-open set and $G_\alpha = X_\alpha$ except for finitely many $\alpha \neq \beta_1, ..., \beta_m$.Let $T = \{\beta_1, ..., \beta_m\}$. Since each X_α is R_{cl}-locally connected, for each $\alpha \in T$ there exists an open connected set V_α such that $x_\alpha \in V_\alpha \subset G_\alpha$. Similarly, for each $\alpha \in S$,let U_α be an open connected set such that $x_\alpha \in U_\alpha \subset G_\alpha$. Let

$$W_\alpha = \begin{cases} X_\alpha \text{ if } \alpha \notin S \bigcup T, \\ U_\alpha \text{ if } \alpha \in S, \\ V_\alpha \text{ if } \alpha \in T \backslash S \end{cases}$$

Then $W = \prod W_\alpha$is a connected open set in X containing x such that $W \subset G$. Thus the product space $X = \prod X_\alpha$is R_{cl}-locally connected.

5. Preservation of R_{cl}-local connectedness under mappings

5.1 Theorem: *Every quotient of an R_{cl}-locally connected space is R_{cl}-locally connected.*

Proof: Let $f : X \to Y$ be a quotient map of an R_{cl}-locally connected space X onto Y. To prove that Y is an R_{cl}-locally connected space, let V be an r_{cl}-open set in Y. Then $f^{-1}(V)$ is a r_{cl}-open set in X. Let C be a component of V. In view of Theorem 4.4, it suffices to show that C is open in Y or equivalently $f^{-1}(C)$ is open in X. To this end, suppose that $x \in f^{-1}(C)$.Let C_x be the component of $f^{-1}(V)$ containing x. Then $f(C_x)$ is a connected set containing $f(x)$, and so $f(C_x) \subset C$. Thus $x \in C_x \subset f^{-1}(C)$.Since X is R_{cl}-locally connected, and since $f^{-1}(V)$ is r_{cl}-open, by Theorem 4.4 C_x is open in X and so $f^{-1}(C)$ is open being a neighbourhood of each of its points.

5.2 Corollary: *Continuous open (closed) images, adjunctions and inductive limits of R_{cl}-locally connected spaces are R_{cl}-locally connected.*

Proof: Since continuous open (closed) maps are quotient maps, the corollary is immediate in view of Theorems 4.15 and 5.1.

5.3 Corollary: *If a product space is R_{cl}-locally connected, then so is each of its factors.*

We recall that a function $f : X \to Y$ is said to be R_{cl}-supercontinuous [41] if for $x \in X$ and each open set V in Y containing $f(x)$ there exists an r_{cl}-open set U containing x such that $f(U) \subset V$.

5.4 Theorem: *Let $f : X \to Y$ be a quotient map which is an R_{cl}-supercontinuous surjection. If X is an R_{cl}-locally connected space, then Y is locally connected.*

Proof: Let $y \in Y$ and let V be an open set containing y. Since f is R_{cl}-supercontinuous, $f^{-1}(V)$ is an r_{cl}-open set in X. Again, since X is R_{cl}-locally connected, for each $x \in f^{-1}(y)$ there exists an open connected set N_x containing x such that $N_x \subset f^{-1}(V)$. Then each $f(N_x)$ is a connected set containing y. Let $N = \bigcup\{f(N_x) : x \in f^{-1}(y)\}$.Then $N \subset V$ and N is a connected set, being union of connected sets having the common point y. Since f is a quotient map, N is a connected open set containing y and so Y is locally connected.

5.5 Definition [41]: Let $p : X \to Y$ be a function from a topological space X onto a set Y. The collection of all subsets $A \subset Y$ such that $p^{-1}(A)$ is an $r_{cl}-$open set in X is a topology on Y, called the $r_{cl}-$***-quotient topology***. *The map p is called the r_{cl}**-quotient map** and the space Y with r_{cl}-quotient topology is called an r_{cl}**-quotient space.***

5.6 Remark: *The function f in the above theorem is precisely the r_{cl}-quotient map and Y is equipped with r_{cl}-quotient topology (see* [41]*). Thus paraphrasing Theorem 5.4 we can say that an r_{cl}-quotient of an R_{cl}-locally connected space is locally connected.*

Since co-product and extremal quotient objects in TOP are disjoint topological sums and quotients maps, respectively, a characterization of mono-coreflective subcategories of TOP [10, Theorem 6] together with Theorems 4.15 and 5.1 yield the following.

5.7 Theorem: *The full subcategory of R_{cl}-locally connected spaces is a mono-coreflective subcategory of TOP containing the full subcategory of almost locally connected spaces.*

6. Function spaces and R_{cl}-locally connected spaces.

It is a well known fact in the theory of function spaces that the function space $C(X, Y)$ of all continuous functions from a topological space X into a uniform space Y is not closed in the topology of pointwise convergence; however, it is closed in the topology of uniform convergence. Moreover, it is of fundamental importance in analysis, topology and other branches of mathematics to know that whether a given function space is closed / compact / complete in the topology of uniform convergence / pointwise convergence. So it is of considerable significance for intrinsic interest as well as from applications view point to formulate conditions on the spaces X, Y and subsets of $C(X, Y)$ to be closed / compact in the topology of pointwise convergence / uniform convergence. Results of this nature and Ascoli type theorems abound in the literature (see [2] [13]). In this direction, Naimpally [31] showed that if X is locally connected and Y is Hausdorff, then the set $S(X, Y)$ of all strongly continuous functions from X into Y is closed in Y^X in the topology of pointwise convergence. Kohli and Singh [19] extended Naimpally's result to a larger framework, wherein it is shown that if X is sum connected (in particular X is connected or locally connected) and Y is Hausdorff, then the function space $P(X, Y)$ of all perfectly continuous functions as well as the function space $L(X, Y)$ of all cl-supercontinuous functions is closed in Y^Xin the topology of pointwise convergence.

In view of Proposition 4.2 every R_{cl}-locally connected space is sum connected. So by [19, Theorem 3.7 and Proposition 3.8] we conclude with the following.

6.1 Corollary: *If X is an R_{cl}-locally connected space and Y is Hausdorff, then $S(X, Y) = P(X, Y) = L(X, Y)$ is closed in Y^X in the topology of pointwise convergence. Furthemorer, if Y is compact, then $S = P = L$ is a compact Hausdorff subspace of Y^X in the topology of pointwise convergence.*

6.2 Remark: *More recently results of Kohli and Singh* [19] *have been considerably strengthened to a larger framework of spaces and functions. For examples see* ([1] [17] [22] [25]). *In the light of these developments Corollary 6.1 can be extended / strengthened accordingly to a larger framework of spaces and functions.*

7. Change of Topology and R_{cl}-locally connected spaces.

The technique of change of topology of a space is frequently employed in mathematics and is of considerable significance. It is widely used in functional analysis, topology and several other branches of mathematics. For example, weak and weak* topologies of a Banach space and its dual, weak and strong operator topologies on $B(H)$ the space of operators on a Hilbert space H, the hull kernel topology and the multitude of other topologies on $Id(A)$ the space of closed two sided ideals of a Banach algebra A (see [3] [4] [5] [37]). Moreover, to taste the flavour of applications of the technique in the topology see ([8] [9] [14] [16] [21] [43]).

In this section we retopologize the topology of an R_{cl}-locally connected space (X, τ) such that it transforms into a locally connected space.

7.1 Let (X, τ) be a topological space. Let B_{cl} denote the collection of all clopen subsets of the space (X, τ). Since the intersection of two clopen sets is again a clopen set, the collection B_{cl} is a base for a topology τ_{cl} on X. The member of τ_{cl} are called cl-open sets. The topology τ_{cl} has been adequately used in literature (see [36] [39]). Staum calls 'cl-open' sets as 'quasi open' in [39].

7.2 Let (X, τ) be a topological space. Let $\tau_{r_{cl}}$ denote the set of all r_{cl}-open subsets of the space (X, τ). Since arbitrary unions and finite intersections of r_{cl}-open sets is r_{cl}-open, the collection $\tau_{r_{cl}}$ is indeed a topology for X. Clearly $\tau_{r_{cl}} \subset \tau$. The topology $\tau_{r_{cl}}$ has been defined and used in ([20] [41]).

7.3 Lemma: Since every cl-open set is r_{cl}-open which in turn is open, we have $\tau_{cl} \subset \tau_{r_{cl}} \subset \tau$.

7.4 Lemma: The spaces $(X, \tau), (X, \tau_{r_{cl}})$ and (X, τ_{cl}) have the same class of clopen sets and so have the same class of cl-open sets as well as the same class of connected sets.

7.5 Lemma: Every $\tau_{r_{cl}}$-connected set is τ-connected.

Proof: Let C be a connected subspace of $(X, \tau_{r_{cl}})$. To show that C is a connected subspace of (X, τ), assume the contrary. Let $C = A \bigcup B$ be a partition of C in (X, τ), i.e., A and B are nonempty relatively clopen subsets of (X, τ). In view of Lemma 7.4 it follows that A and B are nonempty relatively clopen subsets of $(X, \tau_{r_{cl}})$, which contradicts the fact that C is a connected subspace of (X, τ_{rcl}).

7.6 Theorem: If the space $(X, \tau_{r_{cl}})$ is locally connected, then the space (X, τ) is R_{cl}-locally connected.

Proof: Suppose that the space $(X, \tau_{r_{cl}})$ is locally connected. Let $x \in X$ and let U be an open set in $(X, \tau_{r_{cl}})$ containing x. Since the space $(X, \tau_{r_{cl}})$ is locally connected, there exists an open connected set V in $(X, \tau_{r_{cl}})$ such that $x \in V \subset U$. By Lemma 7.3, V is open in (X, τ). Again in view of Lemma 7.5 it follows that V is connected in the space (X, τ). Thus the space (X, τ) is R_{cl}-locally connected.

We conclude the section by raising the following question for the interested reader.

7.7 Question: Is the converse of Theorem 7.6 true?

References

[1] J. Aggarwal, D. Singh and J.K. Kohli, **Pseudo cl-supercontinuous functions and closedness / compactness of their function spaces** (preprint).

[2] A.V. Arhangel'skii, **General Topology** III, Springer Verlag, Berlin Heidelberg, 1995.

[3] F. Beckhoff, **Topologies on the space of ideals of a Banach algebra**, Stud. Math. 115(1995), 189-205.

[4] F. Beckhoff, **Topologies of compact families on the ideal space of a Banach alegebra**, Stud. Math. 118(1996), 63-75.

[5] F. Beckhoff, **Topologies on the ideal space of a Banach algebra and spectral synthesis**, Proc. Amer. Math. Soc. 125(1997), 2859-2866.

[6] N. Bourbaki, **Elements of Mathematics, General Topology,** Part I, Hermann / Addison Wesely, Paris / Reading, Massachusets, 1966.

[7] Eduard Čech, **Topological spaces**, Czechoslovak Akademy of Sciences, Prague, 1966.

[8] D. Gauld, M. Mrsevic, I. L. Reilly and M. K. Vamanamurthy, **Continuity properties of functions**, Coll. Math. Soc. Janos Bolyai 41(1983), 311-322.

[9] A. M. Gleason, **Universal locally connected refinements**, Illinois J. Math. 7(1963), 521-531.

[10] H. Herrlich and G. E. Strecker, **Coreflective subcategories**, Trans. Amer. Math. Soc. 157(1971), 205-226.

[11] H. Herrlich and G. E. Strecker, **Category Theory**, Allyn and Bacon Inc., Boston, 1973.

[12] J. G. Hocking and G. S. Young, **Topology**, Addison-Wesley Publishing Company Inc. Reading, Massachusetts, 1961.

[13] J. L. Kelly, **General Topology**, Van Nostrand, New York, 1955.

[14] J. K. Kohli, **A class of mappings containing all continuous and all semiconnected mappings**, Proc. Amer. Math. Soc. 72 (1) (1978), 175-181.

[15] J. K. Kohli, **A class of spaces containing all connected and all locally connected spaces**, Math.Nachrichten 82(1978), 121-129.

[16] J. K. Kohli, **Change of topology, characterizations and product theorems for semilocally** P**-spaces**, Houston J. Math. 17(3) (1991), 335-350.

[17] J. K. Kohli and J. Aggarwal, **Quasi** cl**-supercontinuous functions and their function spaces**, Demonstratio Math. 45(3) (2012), 677-697.

[18] J. K. Kohli and D. Singh, **Between weak continuity and set connectedness**, Studii Si Cercetari Stintifice Seria Mathematica, Nr. 15(2005), 55-65.

[19] J. K. Kohli and D. Singh, **Function spaces and strong variants of continuity**, Applied Gen. Top. 9(1) (2008), 33-38.

[20] J. K. Kohli and D. Singh, R_{cl}**-spaces and closedness / completeness of certain function spaces in the topology of uniform convergence**, Applied Gen. Top. 15(2) (2014), 155-166.

[21] J. K. Kohli, D. Singh and J. Aggarwal, R-**supercontinuous functions**, Demonstratio Math. 33(3) (2010), 703-723.

[22] J. K. Kohli, D. Singh, J. Aggarwal and M. Rana, **Pseudo perfectly continuous functions and closedness / compactness of their function spaces**, Applied Gen. Top. 14(1) (2013), 115-134.

[23] J. K. Kohli, D. Singh and C. P. Arya, **Perfectly continuous functions**, Stud. Cerc. St. Ser. Mat. Nr. 18 (2008), 99-110.

[24] J. K. Kohli, D. Singh and B. K. Tyagi, **Quasi locally connected spaces and pseudo locally connected spaces**, Commentation Math. 50(2) (2010), 183-199.

[25] J. K. Kohli, D. Singh and B. K. Tyagi, **Quasi perfectly continuous functions and their function spaces**, Sci. Stud. Res., Ser. Math. Inform., 21(2) (2011), 23-40.

[26] J. K. Kohli, D. Singh and B. K. Tyagi, **Between local connected spaces and sum connectedness**, Commentat. Math. Univ. Carol. 53(1) (2013), 3-16.

[27] N. Levine, **Strong continuity in topological spaces**, Amer. Math. Monthly, 67 (1960), 269.

[28] J. Mack, **Countable paracompactness and weak normality properties**, Trans. Amer. Math. Soc.148 (1970), 265-272.

[29] V. J. Mancuso, **Almost locally connected spaces**, J. Austral. Math. Soc. 31(1981), 421-428.

[30] J. R. Munkres, **Topology**, Second Edition, Pearson Prentice Hall, 2008.

[31] S. A. Naimpally, **On strongly continuous functions**, Amer. Math. Monthly, 74(1967), 166-168.

[32] T. Nieminen, **On ultra pseudo compact and related spaces**, Ann. Acad. Sci. Fenn. A I Math. 3(1977), 185-205.

[33] T. Noiri, **On almost locally connected spaces**, J. Austral. Math. Soc. 34(1984), 258-264.

[34] T. Noiri, **Supercontinuity and some strong forms of continuity**, Indian J. Pure. Appl.Math.15(3) (1984), 241-250.

[35] I. L. Reilly and M. K. Vamanamurthy, **On super-continuous mappings**, Indian J.Pure. Appl.Math.14(6) (1983), 767-772.

[36] D. Singh, cl-**supercontinuous functions**, Applied General Topology, 8(2) (2007), 293-300.

[37] D. W. B. Somerset, **Ideal spaces of Banach algebras**, Proc. London Math. Soc. 78(3) (1999), 369-400.

[38] A. Sostak, **On a class of topological spaces containing all bicompact and connected spaces**, General Topology and its Relations to Modern Analysis and Algebra IV: Proceedings of the 4th Prague Topological Symposium 1976, Part B, 445-451.

[39] R. Staum, **The Algebra of bounded continuous functions into a nonarchimedean field**, Pacific J. Math. 50(1) (1974), 169-185.

[40] L. A. Steen and J. A. Seebach, Jr., **Counterexamples in Topology**, Springer Verlag, New York, 1978.

[41] B. K. Tyagi, J. K. Kohli and D. Singh, R_{cl}-**supercontinuous functions**, Demonstratio Math. 46(1) (2013), 229-244.

[42] N. K. Velićko, H-**closed topological spaces**, Amer. Math. Soc. Transl.78(2) (1968), 103-118.

[43] G. S. Young, **Introduction of local connectivity by change of topology**, Amer. J. Math. 68 (1946), , 479-494.

J. K. Kohli

Department of Mathematics, Hindu College, University of Delhi, Delhi-110007, INDIA, e-mail: jk_kohli@yahoo.co.in

D. Singh

Department of Mathematics, Sri Aurobindo College, University of Delhi, New Delhi-110017, INDIA,

e-mail: dstopology@gmail.com

B. K. Tyagi

Department of Mathematics, A.R.S.D. College, University of Delhi, New Delhi-1100021, INDIA,

e-mail: brijkishore.tygi@gmail.com