"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 26(2016), No. 1, 39-46

FINSLERIAN MECHANICAL SYSTEMS IN CONFORMAL FINSLER SPACES

OTILIA LUNGU

Abstract. In [2] Hashiguchi studied the conformal change of a Finsler metric, namely $\overline{F}(x,y) = e^{c(x)}F(x,y)$. Since that moment other authors studied different kind of changes. In this paper we investigate the effect of the conformal change on Finslerian mechanical systems. We established the difference between the coefficients of the cannonical d-connection .

1. Introduction

Let M^n be an n-dimensional differentiable manifold and $F^n = (M^n, F(x,y))$, $\overline{F}^n = \left(M^n, \overline{F}(x,y)\right)$ be two Finsler Spaces with F and $\overline{F}$ fundamental Finsler functions. If the agle in F^n is equal to that in $\overline{F}^n$ for ay tangent vectors, than F^n is called conformal to $\overline{F}^n$ and the change $F \to \overline{F}$ is called a conformal change.

In [1] is shown

Proposition 1.1. A Finsler space F^n is conformal to a Finsler space $\overline{F}^n$ if and only if there exists a scalar field $c(x)$ satisfying

$$\overline{F}(x,y) = e^{c(x)}F(x,y)\,. \tag{1.1}$$

Keywords and phrases: Finsler Mechanical system, Finsler space, conformal change
(2010) Mathematics Subject Classification: 53C60, 53B40.

Let (x^i)be the coordinates of any point of the base manifold M^n and (y^i) a supporting element at the same point. We use the following notations:

∂_i: partial differentiation with respect to x^i;

$\underset{i}{\dot{\partial}}$: partial differentiation with respect to y^i;

$g_{ij} = \frac{1}{2}\underset{i}{\dot{\partial}}\,\underset{j}{\dot{\partial}}F^2$: the Finsler metric tensor ;

$l_i = \underset{i}{\dot{\partial}}F$: the normalized supporting element;

$h_{ij} = g_{ij} - l_i l_j$: the angular metric tensor;

$C_{ijk} = \frac{1}{2}\underset{k}{\dot{\partial}}g_{ij}$: the Cartan tensor;

G^i: the components of the canonical spray associated with $F^n = (M^n, F(x,y))$;

$\overset{\circ}{N}{}^i_j$: the components of the Cartan nonlinear connection associated with $F^n = (M^n, F(x,y))$;

$\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{\circ}{N}{}^j_i \frac{\partial}{\partial y^j}$: the basis vector fields of the horizontal bundle.

Let consider F^n conformal to $\overline{F}^n$ .The geometric objects associated with $\overline{F}^n$ will be denoted by barred symbols. Consequently we have

$$\overline{g}_{ij} = e^{2c(x)}g_{ij} \tag{1.2}$$

and

$$\overline{g}^{ij} = e^{-2c(x)}g^{ij}. \tag{1.3}$$

Between the Cartan tensors C_{ijk} and $\overline{C}_{ijk}$ we have

$$\overline{C}_{ijk} = e^{2c(x)}\left(C_{ijk} + g_{ij}\dot{\partial}_k c(x)\right). \tag{1.4}$$

Since the Cartan tensors are completely symmetric we have

$$g_{ij}\dot{\partial}_k c(x) = g_{ik}\dot{\partial}_j c(x), \tag{1.5}$$

which implies

$$\dot{\partial}_j c = 0. \tag{1.6}$$

Now, from 1.4 we get

$$\overline{C}_{ijk} = e^{2c(x)}C_{ijk} \tag{1.7}$$

and

$$\overline{C}^j_{ik} = C^j_{ik}, \tag{1.8}$$

that is the (h)v-torsion tensor C^j_{ik} of the Cartan connection is invariant under the conformal change.

From Izumi [3] we know that

$$\begin{cases} \overline{G}^i = G^i + B^{ih} c_h \\ \overline{G}^i_j = G^i_j + b^i_j \\ \overline{G}^i_{jk} = G^i_{jk} + b^i_{jk} \end{cases}, \tag{1.9}$$

Where

$$\begin{cases} B^{ih} = y^i y^h - \frac{1}{2} F^2 g^{ih} \\ b^i_j = \left(\dot{\partial}_j B^{ih} \right) c_h \\ b^i_{jk} = \dot{\partial}_k \left(\dot{\partial}_j B^{ih} \right) c_h. \end{cases} \tag{1.10}$$

2. Preliminaries

Radu Miron introduced the notion of Finslerian mechanical system. The author have studied different kind of such mechanical systems.[7]

A Finslerian mechanical system Σ_F is a triple $\Sigma_F = (M, F^2, F_e)$ where $F_e = F^i(x,y)\,\dot{\partial}\,y^i$ are the external forces given as a vertical vector field on the tangent manifold TM. One considers F^n endowed with Cartan nonlinear connection $N$and $F_i(x,y) = g_{ij}F^j(x,y)$ the covariant components of the external forces F_e. The canonical nonlinear connection of Σ_F has the coefficients

$$N^i_j = \overset{\circ}{N}{}^i_j - \frac{1}{4}\frac{\partial F^i}{\partial y^i}. \tag{2.1}$$

This nonlinear conection determines the horizontal distribution with the property $T_uTM = N_u \oplus V_u$, $\forall u \in TM$.

A local adapted basis to the horizontal and vertical vector spaces N_u and V_u is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right)$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \tag{2.2}$$

or, from (2.1)

$$\frac{\delta}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4}\frac{\partial F^i}{\partial y^j}\frac{\partial}{\partial y^j}. \tag{2.3}$$

The adapted cobasis is $(dx^i, \delta y^i)$ with

$$\delta y^i = dy^i + N^i_j dx^j, \tag{2.4}$$

or, equivalently,

$$\delta y^i = \overset{\circ}{\delta}\, y^i - \frac{1}{4}\frac{\partial F^i}{\partial y^j} dx^j. \tag{2.5}$$

Theorem 2.1.[6] . The canonical metrical d-connection has the coefficients expressed by the generalized Christoffel symbols:

$$\begin{cases} F^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right) \\ C^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right) \end{cases} \tag{2.6}$$

Theorem 2.2. The coefficients F^i_{jk} and C^i_{jk} of the canonical metrical d-connection are

$$\begin{cases} F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \frac{1}{4} g^{is}\left(\overset{\circ}{C}_{skh}\frac{\partial F^h}{\partial y^j} + \overset{\circ}{C}_{jsh}\frac{\partial F^h}{\partial y^k} + \overset{\circ}{C}_{jkh}\frac{\partial F^h}{\partial y^s}\right) \\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} \end{cases} \tag{2.7}$$

3. Main results

In this section we investigate the effect of the conformal change 1.1 on the Finslerian mechanical systems.

Let consider $\Sigma_F = (M, F^2, F_e)$ and $\Sigma_{\overline{F}} = \left(M, \overline{F}^2, F_e\right)$ two Finslerian mechanical systems in conformal Finsler spaces F^n and $\overline{F}^n$. In $\Sigma_{\overline{F}}$the covariant components of the external forces are given by

$$\overline{F}_i(x,y) = \overline{g}_{ij} F^j(x,y). \tag{3.1}$$

From (1.2) we get

$$\overline{F}_i\,(x,y) = e^{2c(x)} F_i\,(x,y)\,. \tag{3.2}$$

The canonical nonlinear connection of $\Sigma_{\overline{F}}$ has the coefficients

$$\overline{N}^i_j = \frac{\partial \overline{G}^i}{\partial y^j}. \tag{3.3}$$

Using 1.9 we obtain

$$\overline{N}^i_j = \frac{\partial G^i}{\partial y^j} + \frac{\partial}{\partial y^j}\left(B^i_h c_h\right) = N^i_j + \frac{\partial}{\partial y^j}\left(B^i_h c_h\right) \tag{3.4}$$

and from 2.1,

$$\overline{N}^i_j = \overset{\circ}{N}{}^i_j - \frac{1}{4}\frac{\partial F^i}{\partial y^i} + \frac{\partial}{\partial y^j}\left(B^i_h c_h\right). \tag{3.5}$$

This nonlinear connection determines the horizontal distribution which is supplementary to the natural vertical distribution V on the tangent manifold.

A local adapted basis to these distributions is $\left(\frac{\overline{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ where

$$\frac{\overline{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overline{N}^j_i \frac{\partial}{\partial y^j}. \tag{3.6}$$

Using (3.5) we get

(3.7)

$$\frac{\overline{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{\circ}{N}{}^i_j + \frac{1}{4}\frac{\partial F^j}{\partial y^j}\frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^i}\left(B^j_h c_h\right)\frac{\partial}{\partial y^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4}\frac{\partial F^j}{\partial y^j}\frac{\partial}{\partial y^i} - c_h \frac{\partial B^j_h}{\partial y^i}\frac{\partial}{\partial y^i},$$

or

$$\frac{\overline{\delta}}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + D^j_{ih}\frac{\partial}{\partial y^i}, \tag{3.8}$$

with

$$D^j_{ih} = \frac{1}{4}\frac{\partial F^j}{\partial y^j} - c_h \frac{\partial B^j_h}{\partial y^i}. \tag{3.10}$$

The adapted cobasis is $\left(dx^i, \overline{\delta} y^i\right)$ with

$$\bar{\delta} y_i = dy^i + \overline{N}^i_j dx^j. \tag{3.11}$$

From (3.5) ,(3.11) and (3.10) we obtain

$$\bar{\delta} y_i = \overset{\circ}{\delta}\, y^i - D^i_{jh} dx^j. \tag{3.12}$$

Theorem 3.1. The local coefficients of the canonical metrical d-connection of $\Sigma_{\overline{F}}$ are

$$\begin{cases} \overline{F}^i_{jk} = \frac{1}{2}\overline{g}^{is}\left(\frac{\bar{\delta}\,\overline{g}_{sj}}{\delta x^k} + \frac{\bar{\delta}\,\overline{g}_{sk}}{\delta x^j} - \frac{\bar{\delta}\,\overline{g}_{jk}}{\delta x^s}\right) \\ \overline{C}^i_{jk} = \frac{1}{2}\overline{g}^{is}\left(\frac{\partial \overline{g}_{sj}}{\partial y^k} + \frac{\partial \overline{g}_{sk}}{\partial y^j} - \frac{\partial \overline{g}_{jk}}{\partial y^s}\right) \end{cases} \tag{3.13}$$

In order to calculate $\overline{F}^i_{jk}$ and $\overline{C}^i_{jk}$ we have

$$\begin{aligned} \frac{\bar{\delta}\,\overline{g}_{sj}}{\delta x^k} &= \frac{\bar{\delta}}{\delta x^k}\left(e^{2c(x)} g_{sj}\right) = \left(\frac{\overset{\circ}{\delta}}{\delta x^i} + D^j_{ih}\frac{\partial}{\partial y^i}\right)\left(e^{2c(x)} g_{sj}\right) = \\ &= e^{2c(x)} c_k g_{sj} + e^{2c(x)} \frac{\overset{\circ}{\delta}\, g_{sj}}{\delta x^i} + e^{2c(x)} D^j_{ih}\frac{\partial g_{sj}}{\partial y^i} \end{aligned} \tag{3.14}$$

and

$$\frac{\partial \overline{g}_{sj}}{\partial y^k} = \frac{\partial}{\partial y^k}\left(e^{2c(x)} g_{sj}\right) = e^{2c(x)}\frac{\partial g_{sj}}{\partial y^k}. \tag{3.15}$$

In conclusion the developped expression of the coefficients $\overline{F}^i_{jk}$ and $\overline{C}^i_{jk}$ is given in the next theorem:

Theorem 3.2. The canonical metrical d-connection of $\Sigma_{\overline{F}}$ has the coefficients

(3.16)

$$\begin{cases} \overline{F}^i_{jk} = F^i_{jk} + \frac{1}{2} g^{is}\left[\left(c_k g_{sj} + c_j g_{sk} - c_s g_{jk}\right) + \left(D^j_{kh}\frac{\partial g_{sj}}{\partial y^k} + D^k_{jh}\frac{\partial g_{sk}}{\partial y^j} - D^k_{sh}\frac{\partial g_{jk}}{\partial y^s}\right)\right] \\ \overline{C}^i_{jk} = C^i_{jk} \end{cases}$$

References

[1] P.L. Antonelli, R.S.Ingarden, M.Matsumoto, **The theory of sprays and Finsler spaces with applications in physics and biology,** Kluwer Academic Publisher, FTPH, no. 58 (1993).

[2] M. Hashiguchi, **On conformal transformations of Finsler metrics,** J.Math. CityplaceKyoto Univ , 16 (1976), 25-50.

[3] H. Izumi, **Conformal transformation of Finsler spaces**, Tensor (N.S.) 31(1977), 33-41.

[4] O. Lungu, V. Niminet , **General Randers Mechanical Systems,** Scientific Studies and Research. Series Mathematics and Informatics, Vol. 22, No.1, 25-30, (2012).

[5] O. Lungu , V. Niminet , **Ingarden spaces with a special nonlinear connection**, Scientific Studies and Research. Series Mathematics and Informatics.Vol.22, No.2, 2012, pp. 31-38.

[6] R. Miron, **Lagrangian and Hamiltonian Geometries. Applications to Analytical Mechanics,** Ed. Academiei Romne & Ed. Fair Partners, Buc., 2011.

[7] R. Miron, **Dynamical Systems of Lagrangian and Hamiltonian Mechanical Systems**, Advance Studies in Pure Mathematics, 48, 2007, pp. 309-340.

[8] R. Miron, V.Niminet, **The Lagrangian Geometrical model and The associated dynamical system of a nonholonomic mechanical system** , Facta universitatis-series Mechanics Automatic Control and Robotics, Vol 5, No.1, 1-24.

[9] S.C. Rastogi, A.K. Dwivedi, **On conformal transformation of certain Finsler spaces**, Tamkang Journal of mathematics, 38(2007),277-289.

[10] V. Nimineţ, **New geometrical properties generalized Lagrange spaces of relativistic optics,**, Tensor, New Series, 68(1), 66-70.

Department of Mathematics, Informatics and Education Sciences,
Faculty of Sciences,
"Vasile Alecsandri" University of Bacau ,
157 Calea Marasesti, 600115 Bacau , ROMANIA
E-mail address: otilas@yahoo.com ; otilia.lungu@ub.ro