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SOME GEOMETRICAL PROPERTIES OF PFAFF SYSTEMS ON MANIFOLDS

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Abstract. We continue our investigation [7] of Pfaff systems on manifolds by studying some geometrical properties and some consequences of these structures. We point out new geometrical invariants of a distributions on manifolds.

1. Introduction

As we previously shown in [7], if X is C^∞ manifold of dimension $n+m$ a distribution on X is a mapping $x \to H_x \subset T_xX$, $x \in X$ with the properties: H_x is a linear subspace of dimension n in T_xX and for every $x_0 \in X$, there exists an open neighborhood U and the vector fields $X_1, ..., X_n$ on U which are linearly independent on U and $H_x = \text{span}[X_1(x), ..., X_n(x)]$, $\forall x \in U$. Also, we shown in [7] that a Pfaff system is a mapping $\epsilon : x \to \epsilon_x \subset T^*_xX$, $x \in X$, with the properties ϵ_x is a linear subspace of dimension m and for any $x_0 \in X$, there exists an open neighborhood U and the 1-forms $\omega^1, ..., \omega^m$ on U which are linear independent and

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$\epsilon_{x_0} = span[\omega^1(x_0), ..., \omega^m(x_0)]$.

Let 1-forms (ω^a) with :

$$\omega^a(x) = \omega^a_\alpha(x)dx^\alpha, rank(\omega^a_\alpha) = m, \tag{1.1}$$

with a sum over $\alpha = 1, 2, ...d = dimX$ and $a = 1, 2, ...m$ according to the Einstein convention on summation.

Assume that the first m lines in the matrix $(\omega^a\alpha)$ are linear independent and denote by (ϕ^b_c) the inverse of the matrix (ω^a_b) from the expression $(\omega^a\alpha) = (\omega^a_b, \pi\alpha_i)$.

Then, $(\phi^a_c)(\omega^c_b, \pi^c_i) = (\delta^a_b, N^a_i)$.

The lines of this last matrix define the 1-forms

$$\tilde{\omega}^a = dx^a + N^a_i(x)dx^i, \tag{1.2}$$

that satisfy $\tilde{\omega}^a = \phi^a_c\omega^c$.

Thus $(\tilde{\omega}^a)$ span the same Pfaff system as (ω^a).

In [7] we shown that the 1- forms (ω^a) may be replaced with the 1-forms $\delta y^a = dy^a + N^a_i(x, y)dx^i$ when the coordinates (x^α) on X are separated in (x^i, x^a) and (x^a) are denoted as (y^a) and we compute $d(\delta y^a)$ and put the result into the form

$$d(\delta y^a) + \omega^a_b \wedge \delta y^b = \Omega^a \tag{1.3}$$

that provides a first structure equation and introduces the 2- forms of torsion Ω^a.

Exterior differentiating again we get the first Bianchi identity

$$d\Omega^a + \omega^a_b \wedge \Omega^b = \theta^a_c \wedge \delta y^c, \tag{1.4}$$

where the equality

$$\theta^a_c = d\omega^a_c + \omega^a_b \wedge \omega^b_c \tag{1.5}$$

represents a second structure equation.

A new exterior differentiation leads to the second Bianchi identity:

$$d\theta^a_b + \omega^a_c \wedge \theta^c_b = \theta^a_c \wedge \omega^c_b. \tag{1.6}$$

2. Main result

We know that the structure equations (1.3) and (1.5) introduce the 2-forms of torsion Ω^a and the 2-forms of curvature θ^a_b. We notice that for certain particular distributions that are used in Lagrange geometry [4] as well as in the general setting from [6] the 2-forms Ω^a

$$\Omega^a_{ij} = \delta_i N^a_j - \delta_j N^a_i, \tag{2.1}$$

are called the curvature of the distribution H.

The 2-forms of curvature θ^a_b do not belong to the ideal δy^a if and only if $\frac{\partial^2 N^a_i}{\partial y^b \partial y^c} = 0$, that is if and only if

$$N^a_i(x,y) = \Gamma^a_{ci}(x)y^c + K^a_i(x). \tag{2.2}$$

It follows
(2.3)

$$\omega^a_b = \Gamma^a_{bi}(x)dx^i,$$
$$\theta^a_b = \frac{1}{2}\left(\frac{\partial\Gamma^a_{bj}}{\partial x^i} - \frac{\partial\Gamma^a_{bi}}{\partial x^j} + \Gamma^a_{ci}\Gamma^c_{bj} - \Gamma^a_{cj}\Gamma^c_{bi}\right)dx^i \wedge dx^j := \frac{1}{2}R^a_{b\ ij}dx^i \wedge dx^j$$
$$\Omega^a = \frac{1}{2}\left(R^a_{b\ ij}y^b + \frac{\partial K^a_i}{\partial x^i} + K^c_j\Gamma^a_{ci} - K^c_i\Gamma^a_{cj}\right)dx^i \wedge dx^j.$$

where (θ^a_b) deserve the name of curvature 2-forms.

For $K^a_i = 0$, the coefficients of ω^a reduce to $R^a_{bij}y^b$ one of the torsions of the Cartan connection in a Finsler space.

If we keep the splitting of coordinates on X in the form (x^i, y^a), a change $(x^i, y^a) \to (\widetilde{x}^i, \widetilde{y}^a)$ of these coordinates has the form

$$\begin{aligned} \widetilde{x}^i &= \widetilde{x}^i(x^i, y^a) \\ \widetilde{y}^a &= \widetilde{y}^a(x^i, y^b), \end{aligned} \tag{2.4}$$

with

$$\frac{\partial(\widetilde{x}^i, \widetilde{y}^a)}{\partial(x^i, y^a)} \neq 0. \tag{2.5}$$

We obtain:

$$\begin{aligned} d\widetilde{x}^i &= \frac{\delta\widetilde{x}^i}{\delta x^j}dx^j + \frac{\partial\widetilde{x}^i}{\partial y^a}\delta y^a = A^i_j dx^j + A^i_a \delta y^a, \\ d\widetilde{y}^a &= \frac{\delta\widetilde{y}^a}{\delta x^j}dx^j + \frac{\partial\widetilde{y}^a}{\partial y^b}\delta y^b = A^a_j dx^j + A^a_b \delta y^b, \end{aligned} \tag{2.6}$$

where $A^i_j = \frac{\delta\widetilde{x}^i}{\partial x^j}$, $A^i_a = \frac{\partial\widetilde{x}^i}{\partial y^a}$

The mapping given by (2.4) may be viewed as a local diffeomorphism $\phi : X \to X$. This induces an isomorphism $\phi_* : TX \to TX$ which has to satisfy

$$\phi_*(H) = H, \tag{2.7}$$

for any distribution H on X.

Let H be spanned by $\delta_i = \frac{\partial}{\partial x^i} - N^a_i(x,y)\frac{\partial}{\partial y^a}$

and let $\widetilde{\delta}_i = \frac{\partial}{\partial \widetilde{x}^i} - \widetilde{N}^a_i(\widetilde{x}, \widetilde{y})\frac{\partial}{\partial \widetilde{y}^a}$ be the local vector fields which span H when it is regarded with respect to the coordinates $(\widetilde{x}, \widetilde{y})$.

We know that the condition (2.7) is equivalent to the equalities $\phi_*(\delta_i) = \lambda^j_i\widetilde{\delta}_j$, $i = 1, 2, ...n$ for a matrix (λ^j_i) with $\tilde{\delta}(\lambda^j_i) \neq 0$.

We compute $\phi_*(\delta_i) = A_i^j \frac{\partial}{\partial \widetilde{x}^j} + A_i^a \frac{\partial}{\partial \widetilde{y}^a} = A_i^j \widetilde{\delta}_j + (A_i^a + A_i^j \widetilde{N}_j^a) \frac{\partial}{\partial \widetilde{y}^a}$.

It comes out that (2.7) holds if and only if the following two conditions are satisfied:

$$\tilde{\delta}(A_i^j) \neq 0, \tag{2.8}$$

$$A_i^a + A_i^j \widetilde{N}_j^a = 0. \tag{2.9}$$

Assume for a moment that (2.8) holds. Then the functions $\widetilde{N}_j^a$ are completely determined from (2.9) in the form $\widetilde{N}_j^a = (A^{-1})_j^i A_i^a$.

Using again (2.9) we get

$$\delta \widetilde{y}^a = (A_b^a + \widetilde{N}_i^a A_b^i) \delta y^b := B_b^a \delta y^b$$

and we must have

$$\tilde{\delta}(B_b^a) \neq 0.$$

Denoting the inverse of the matrix $A_j^i = (\frac{\tilde{\delta} x^i}{\delta x^j})$ by $(\frac{\delta x^i}{\tilde{\delta} x^j})$, the condition (2.9) takes the form

$$\widetilde{N}_j^a = \frac{\delta x^i}{\delta \widetilde{x}^j} N_i^b \frac{\partial \widetilde{y}^a}{\partial y^b} - \frac{\delta x^i}{\delta \widetilde{x}^j} \frac{\partial \widetilde{y}^a}{\partial x^i}. \tag{2.10}$$

The equation (2.10) simplifies if $(\frac{\tilde{\partial} x^j}{\partial y^a}) = (0)$, that is in (2.4) we have $\tilde{x}^i = \tilde{x}^i(x^j)$. In this case $A_j^i = \frac{\partial \tilde{x}^i}{\partial x^j}$ and (2.10) reduces to

$$\tilde{N}_j^a = \frac{\partial x^i}{\partial \tilde{x}^j} N_i^b \frac{\partial \tilde{y}^a}{\partial y^b} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{y}^a}{\partial x^i}. \tag{2.11}$$

which is the law of transformation of the coefficients of a nonlinear connection in a vector bundle,[4].

Now we search for the behavior of ω_{ij}^a under the local diffeomorphism ϕ. We have seen that $\delta \tilde{y}^a - B_b^a \delta y^b = 0$.

We exterior differentiate this equality and use in the result the followings:

$$\begin{aligned} d(\delta y^a) &= \tfrac{1}{2} \Omega_{ij}^a dx^i \wedge dx^j - \omega_{bi}^a dx^i \wedge \delta y^b, \omega_{bi}^a = \tfrac{\partial N_i^a}{\partial y^b} \\ d(\delta \tilde{y}^a) &= \tfrac{1}{2} \tilde{\Omega}_{ij} d\tilde{x}^i \wedge d\tilde{x}^j - \tilde{\omega}_{bi}^a d\tilde{x}^i \wedge \delta \tilde{y}^b, \end{aligned} \tag{2.12}$$

as well as (2.6).

The coefficients of the exterior products $dx^i \wedge dx^j$, $\delta y^b \wedge dx^k$, $\delta y^c \wedge \delta y^b$ have to be zero.

Therefore, we have

(2.12')
$$\tilde{\omega}^a_{kh} A^k_i A^h_j = B^a_b \omega^b_{ij},$$

$$\tilde{\omega}^a_{ci} B^c_b A^i_k + \frac{1}{2}\tilde{\omega}^a_{ij}(A^i_b A^j_k - A^i_k A^j_b) = B^a_c \omega^c_{bk} - \frac{\delta B^a_b}{\delta x^k}$$

$$\frac{1}{2}\tilde{\omega}^a_{ij}(A^i_c A^j_d - A^j_d A^i_c) - \tilde{\omega}^a_{bi}(A^i_c B^b_d - A^i_d B^b_c) = \tilde{\omega}^a_{ci} A^i_d - \tilde{\omega}^a_{di} A^i_c.$$

For the horizontal vector field $U = U^i\delta_i$ and $V = V^j\delta_j$, we have $[U,V] = U^i\delta_i(V^j)\delta_j - V^j\delta_j(U^i)\delta_i - U^iV^j\Omega^a_{ij}\frac{\partial}{\partial y^a}$ and it follows that $[U,V]$ is horizontal if and only if $\Omega^a_{ij} = 0$.

If the distribution H is a subbundle of TX we may consider the quotient vector bundle TX/H as well as the vector bundle $\Lambda^2 H$.

Then $\delta_i \wedge \delta_j$ is a basis of local sections in $\Lambda^2 H$.

In [5] one says that the curvature of the distribution H is the vector bundle morphism $F : \Lambda^2 H \to TX/H$ given by $F(U \wedge V) = -[U,V]$ mod H for every sections U, V in H (horizontal vector fields).

The mapping F is indeed a tensorial one since $[fU, gV]$ mod $H = fg[U,V]$ mod H. We have:

$$F(U,V) = -[U,V]\mathrm{mod}H = U^iV^j[\delta_i, \delta_j].$$

Hence F is completely determined by Ω^a_{ij} and it vanishes if and only if $\Omega^a_{ij} = 0$.

In order to study the behavior of θ by the local diffeomorphism ϕ we shall use a matrix notation. We write $\delta y = (\delta y^a)$ as a column matrix and rewrite (1.1) - (1.3) as follows:

$$d(\delta y) + \omega \wedge \delta y = \omega,$$

$$d\Omega + \omega \wedge \omega = \theta \wedge \delta y,$$

$$\theta = d\omega + \omega \wedge \omega,$$

(2.13)
$$d\theta + \omega \wedge \theta = \theta \wedge \omega.$$

By ϕ we get $\delta\tilde{y} = B\delta y$.

We exterior differentiate this equality as well as $d(\delta\tilde{y}) + \tilde{\omega} \wedge \delta\tilde{y} = \tilde{\Omega}$.

After some calculation we get

(2.14)
$$(B\Omega - dB - \tilde{\Omega}B) \wedge \delta y = B\Omega - \tilde{\Omega}.$$

Let J be a matrix of 1-forms such that $J \wedge \delta y$ to represent the component of $\tilde{\Omega}$ from the ideal $\{\delta y\}$.

We add $J \wedge \delta y$ to the both side of (2.14). It results

$$(B\Omega - dB - \tilde{\Omega} B + J) \wedge \delta y = B\Omega - (\tilde{\Omega} - J \wedge \delta y). \tag{2.15}$$

In (2.15), the left side is in ideal $\{\delta y\}$ and the right side is not in this ideal. The equality holds only if the both sides vanish. Thus we obtain

$$\tilde{\omega} B = B\omega - dB + J,\ \tilde{\Omega} = B\Omega + J \wedge \delta y. \tag{2.16}$$

We know:

$$\widetilde{\theta} = d\widetilde{\omega} + \widetilde{\omega} \wedge \widetilde{\omega}$$

Then

$$\widetilde{\theta} B = d\widetilde{\omega} B + \widetilde{\omega} \wedge \widetilde{\omega} B = d(\widetilde{\omega} B) + \widetilde{\omega} \wedge dB + \omega \wedge \widetilde{\omega} B =$$

$$= d(\widetilde{\omega} B) + \widetilde{\omega} \wedge (dB + \widetilde{\omega} B) = d(\widetilde{\omega} B) + \widetilde{\omega} \wedge (B\omega + J) =$$

$$= dB \wedge \omega + Bd\omega + dJ + \widetilde{\omega} \wedge J + \widetilde{\omega} BB^{-1} \wedge B\widetilde{\omega} =$$

$$= B(d\omega + \omega \wedge \omega) + dJ + J \wedge \omega + \widetilde{\omega} \wedge J = B\theta + dJ + J \wedge \omega + \widetilde{\omega} \wedge J,$$

where we have also used

$$d(\widetilde{\omega} B) = d\widetilde{\omega} B + \widetilde{\omega} \wedge dB$$

So, the connection between $\tilde{\theta}$ and θ is:

$$\widetilde{\theta} B = B\theta + dJ + J \wedge \omega + \widetilde{\omega} \wedge J.$$

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