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PROPERTIES OF α-GENERALIZED REGULAR WEAKLY CONTINUOUS FUNCTIONS AND PASTING LEMMA

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Abstract: In this paper, some properties of αgrw-continuous functions are discussed and the notion of αgrw-closed graph is introduced. Also, the pasting lemma for αgrw-continuous functions is proved.

1. Introduction

The classical pasting lemma shows that the notion of a function restricted to a set being continuous without specifying the topologies with respect to which this continuity holds. This result is important by its applications in algebraic topology and in other applications involving topological concepts and methods. The pasting lemma for α-continuous maps have been introduced and investigated by Maki and Noiri[9]. Anitha et al. [2] establised pasting lemmas for g-continuous functions. Vidhya and Parimelazhagan[17] introduced the concepts of g*b-continuous maps and pasting lemma in topological spaces. Caldas et al. [4] studied the characterizations of functions with strongly alpha-closed graphs. In this paper pasting lemma for αgrw-continuous functions is proved and also αgrw-closed graph functions are introduced.

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Throughout this paper, the space (X,τ) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $cl(A)$, $int(A)$ and $X-A$ (or A^c)denote the closure of A, the interior of A and the complement of A in X respectively.

2. Preliminaries

Definition 2.1. *A subset A of a topological space (X,τ) is called*

(1) *regular open* [16] *if $A = int(cl(A))$ and regular closed if $A = cl(int(A))$.*
(2) *pre-open* [11] *if $A \subseteq int(cl(A))$ and pre-closed if $cl(int(A)) \subseteq A$*
(3) *β-open* [1] *if $A \subseteq cl(int(cl(A)))$ and β-closed if $int(cl(int(A))) \subseteq A$.*
(4) *α-open* [12] *if $A \subseteq int(cl(int(A)))$ and α-closed* [10] *if $cl(int(cl(A))) \subseteq A$*
The α-closure of a subset A of X denoted by $\alpha cl(A)$ is defined to be the intersection of all α-closed sets containing A. $\alpha cl(A)$ is α-closed. The α-interior of a subset A of X denoted by $\alpha int(A)$ is defined to be the union of all α-open sets containing A. $\alpha int(A)$ is α-open.

Definition 2.2. [5] *A subset A of a space (X,τ) is called regular semi-open if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$. The family of all regular semi-open set of X is denoted by $RSO(X)$.*
The complement of regular semi-open set is regular semi-closed set.

Definition 2.3. [13] *A subset A of a topological space (X,τ) is said to be α-generalized regular weakly closed (briefly αgrw-closed) set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open.*

A subset A of a topological space (X,τ) is said to be α-generalized regular weakly open *(briefly αgrw-open)* [15] if A^c is αgrw-closed.
The set of all αgrw-closed sets and αgrw-open sets are denoted by $\alpha grwC(X)$ and $\alpha grwO(X)$ respectively.

Definition 2.4. [14] *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be α-generalized regular weakly continuous (briefly αgrw-continuous) if $f^{-1}(V)$ is an αgrw-closed set of (X,τ) for every closed set V of (Y,σ).*

Definition 2.5. [14] *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be αgrw-irresolute if $f^{-1}(V)$ is an αgrw-closed set of (X,τ) for every αgrw-closed set V of (Y,σ).*

Definition 2.6. [12] *A topological space (X,τ) is an α-space if every α-closed subset of (X,τ) is closed in (X,τ).*

If $f:(X,\tau)\to(Y,\sigma)$ is any function, then the subset $G(f)=\{(x,f(x)):x\in X\}$ of the product space $(X\times Y,\tau\times\sigma)$ is called the graph of f [8].

Definition 2.7. [7] *A function $f:(X,\tau)\to(Y,\sigma)$ has an α-closed graph if for each $(x,y)\in(X\times Y)-G(f)$, there exists an α-open set U and an open set V containing x and y respectively such that $(U\times cl(V))\cap G(f)=\emptyset$.*

Lemma 2.8. [3] *Let $A\subseteq Y\subseteq X$, where X is a topological space and Y is open subspace of X. If A is regular semi-open in X, then A is regular semi-open in Y.*

Lemma 2.9. [13] *Every α-closed (α-open) set is αgrw-closed (αgrw-open).*

The above lemma implies that $\alpha O(X)\subseteq\alpha grwO(X)$ and $\alpha C(X)\subseteq\alpha grwC(X)$.

3. αgrw-CONTINUOUS FUNCTIONS

Definition 3.1. *A function $f:(X,\tau)\to(Y,\sigma)$ is called regular semi-open* (resp. regular semi-closed*) if $f(V)$ is regular semi-open(resp. regular semi-closed) in (Y,σ) for every regular semi-open(resp. regular semi-closed) set V in (X,τ).*

Definition 3.2. *A function $f:(X,\tau)\to(Y,\sigma)$ is called regular semi-irresolute if $f^{-1}(V)$ is regular semi-open in (X,τ) for every regular semi-open V in (Y,σ).*

Proposition 3.3. *If A is αgrw-closed in a α-space (X,τ) and if $f:(X,\tau)\to(Y,\sigma)$ is regular semi-irresolute and α-closed, then $f(A)$ is αgrw-closed in (Y,σ).*

Proof. Let U be any regular semi-open in (Y,σ) such that $f(A)\subseteq U$. Then $A\subseteq f^{-1}(U)$ and by assumption, $\alpha cl(A)\subseteq f^{-1}(U)$. This implies $f(\alpha cl(A))\subseteq U$ and $f(\alpha cl(A))$ is α-closed.

Now, $\alpha cl(f(A)) \subseteq \alpha cl(f(\alpha cl(A))) = f(\alpha cl(A)) \subseteq U$. Therefore $\alpha cl(f(A)) \subseteq U$ and hence $f(A)$ is αgrw-closed in (Y,σ).

The following lemma is a characterization of αgrw-open sets.

Lemma 3.4. [15] *A subset A of (X,τ) is αgrw-open if and only if $F \subseteq \alpha int(A)$ whenever F is regular semi-closed and $F \subseteq A$.*

Lemma 3.5. [15] *If $A \subseteq X$ is αgrw-closed, then $\alpha cl(A) - A$ is αgrw-open.*

Theorem 3.6. *Let f be an αgrw-continuous and regular semi-closed* function from a space (X,τ) to an α-space (Y,σ). Then f is an αgrw-irresolute function.*

Proof. Let A be an αgrw-open subset in (Y,σ) and let F be any regular semi-closed set in (X,τ) such that $F \subseteq f^{-1}(A)$. Then $f(F) \subseteq A$. Since f is regular semi-closed*, $f(F)$ is regular semi-closed. Therefore $f(F) \subseteq \alpha int(A)$ by Lemma 3.4 and so $F \subseteq f^{-1}(\alpha int(A))$. Since f is αgrw-continuous and Y is an α-space, $f^{-1}(\alpha int(A))$ is αgrw-open in (X,τ). Thus $F \subseteq \alpha int(f^{-1}(\alpha int(A)) \subseteq \alpha int(f^{-1}(A))$ and so $f^{-1}(A)$ is αgrw-open in (X,τ) by Lemma 3.4. The proof is similar for αgrw-closed set.

Corollary 3.7. *If $f : (X,\tau) \to (Y,\sigma)$ is αgrw-continuous and regular semi-closed* and if A is αgrw-closed (or αgrw-open) subset of an α-space (Y,σ), then $f^{-1}(A)$ is αgrw-closed (or αgrw-open) in (X,τ).*

Proof. Follows from Lemma 3.5 and Theorem 3.6.

Corollary 3.8. *Let $(X,\tau),(Z,\eta)$ be a topological spaces and (Y,σ) be an α-space. If $f : (X,\tau) \to (Y,\sigma)$ is αgrw-continuous and regular semi-closed* and $g : (Y,\sigma) \to (Z,\eta)$ is αgrw-continuous, then $g \circ f : (X,\tau) \to (Z,\eta)$ is αgrw-continuous.*

Proof. Let F be any closed set in (Z,η). Since g is αgrw-continuous, $g^{-1}(F)$ is αgrw-closed. By assumption and by Theorem 3.6, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is αgrw-closed in (X,τ) and so $g \circ f$ is αgrw-continuous.

Proposition 3.9. *If $f : (X,\tau) \to (Y,\sigma)$ is αgrw-continuous then for each point x in X and each open set V in Y with $f(x) \in V$, there is an αgrw-open set U in X such that $x \in U$ and $f(U) \subseteq V$.*

Proof. Let V be an open set in (Y,σ) and let $f(x) \in V$. Then $x \in f^{-1}(V) \in \alpha grwO(X)$, since f is αgrw-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$.

Lemma 3.10. [13] *If A and B are αgrw-closed then $A \cup B$ is αgrw-closed.*

Lemma 3.11. [6] *Suppose $B \subseteq A \subseteq X$, B is αgrw-closed relative to A and A is both regular open and αgrw-closed subset of X. Then B is αgrw-closed in X.*

The following theorem is the pasting lemma for αgrw-continuous functions.

Theorem 3.12. *Let $X = A \cup B$, where A and B are αgrw-closed and regular open in X. Let $f : (A, \tau_A) \to (Y, \sigma)$ and $g : (B, \tau_B) \to (Y, \sigma)$ be αgrw-continuous such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the combination $f \nabla g : (X, \tau) \to (Y, \sigma)$ defined by $(f\nabla g)(x) = f(x)$ if $x \in A$ and $(f\nabla g)(x) = g(x)$ if $x \in B$ is αgrw-continuous.*

Proof. Let U be any closed set in Y. Then $(f\nabla g)^{-1}(U) = [(f\nabla g)^{-1}(U) \cap A] \cup [(f\nabla g)^{-1}(U) \cap B] = f^{-1}(U) \cup g^{-1}(U) = C \cup D$, where $C = f^{-1}(U)$ and $D = g^{-1}(U)$. Since f is αgrw-continuous, we have C is αgrw-closed in (A, τ_A) and also since A is αgrw-closed and regular open in X, C is αgrw-closed in X by Lemma 3.11. Similarly, D is αgrw-closed in X and by Lemma 3.10, $(f\nabla g)^{-1}(U) = C \cup D$ is αgrw-closed in X. Hence $f\nabla g$ is αgrw-continuous.

4. αgrw-CLOSED GRAPH FUNCTIONS

Definition 4.1. *A function $f : (X, \tau) \to (Y, \sigma)$ has an αgrw-closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exists an αgrw-open set U and an open set V containing x and y respectively such that $(U \times cl(V)) \cap G(f) = \emptyset$.*

Example 4.2. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$ and $Y = \{p, q, r\}$ with topology $\sigma = P(Y)$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p, f(b) = q$ and $f(c) = r$. Then f has an αgrw-closed graph.*

Proposition 4.3. *A function with α-closed graph has an αgrw-closed graph.*

Proof. The proof follows from the fact that every α-open set is an αgrw-open set[13]. Hence f has an αgrw-closed graph.

Remark 4.4. *The converses of the above proposition need not be true in general. In Example 4.2, the function f has an αgrw-closed graph but not has an α-closed graph.*

Lemma 4.5. *The function $f:(X,\tau)\to(Y,\sigma)$ has an αgrw-closed graph if and only if for each $(x,y)\in X\times Y$ such that $f(x)\neq y$, there exist an αgrw-open set U and an open set V containing x and y respectively, such that $f(U)\cap cl(V)=\emptyset$.*

Proof. Necessity. Let for each $(x,y)\in X\times Y$ such that $f(x)\neq y$. Then there exist an αgrw-open set U and an open set V containing x and y, respectively, such that $(U\times cl(V))\cap G(f)=\emptyset$, since f has an αgrw-closed graph. Hence for each $x\in U$ and $y\in cl(V)$ with $y\neq f(x)$, we have $f(U)\cap cl(V)=\emptyset$.

Sufficiency. Let $(x,y)\notin G(f)$. Then $y\neq f(x)$ and so there exist an αgrw-open set U and an open set V containing x and y, respectively, such that $f(U)\cap cl(V)=\emptyset$. This implies, for each $x\in U$ and $y\in cl(V), f(x)\neq y$. Therefore $(U\times cl(V))\cap G(f)=\emptyset$. Hence f has an αgrw-closed graph.

Theorem 4.6. *If f is an αgrw-continuous function from a space X into a Hausdorff space Y, then f has an αgrw-closed graph.*

Proof. Let $(x,y)\notin G(f)$. Then $y\neq f(x)$. Since Y is Hausdorff space, there exist two disjoint open sets V and W such that $f(x)\in W$ and $y\in V$. Since f is αgrw-continuous, there exists an αgrw-open set U such that $x\in U$ and $f(U)\subseteq W$ by Proposition 3.9. Thus $f(U)\subseteq Y-cl(V)$. Therefore $f(U)\cap cl(V)=\emptyset$ and so f has an αgrw-closed graph.

Theorem 4.7. *If f is a surjective function with an αgrw-closed graph from a space X onto a space Y, then Y is Hausdorff.*

Proof. Let y_1 and y_2 be two distinct points in Y. Then there exists a point $x_1\in X$ such that $f(x_1)=y_1\neq y_2$. Thus $(x_1,y_2)\notin G(f)$. Since f has an αgrw-closed graph, there exist an αgrw-open set U and an open set V containing x_1 and y_2, respectively, such that $f(U)\cap cl(V)=\emptyset$ and so $f(x_1)\notin cl(V)$. But $Y-cl(V)$ is an open set containing $f(x_1)=y_1$. Thus $V\cap(Y-cl(V))=\emptyset$. Hence Y is Hausdorff.

Proposition 4.8. *The space X is Hausdorff if and only if the identity mapping $f:X\to X$ has an αgrw-closed graph.*

Proof. Obvious from Theorems 4.6 and 4.7.

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