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FAST ALGORITHMS OF HEREDITARY DOUBLY CHORDAL GRAPHS

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Abstract. We give a characterization of hereditary doubly chordal graphs using weak decomposition. We also give a recognition algorithm for hereditary doubly chordal graphs and we determine the combinatorial optimization numbers in efficient time.

1. Introduction and preliminaries

The class of triangulated graphs (also called chordal graphs), that has many useful properties, has been intensively studied due to its applications and its connections with other classes of graphs.

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As specified in [20] it is known that the for doubly chordal graphs the clique problem can be solved in polynomial time, while the independent set problem and the recognition problem can be solved in linear time.

According to [1] , $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ (or $co - G$) is the complement of G. If $U \subseteq V$, we denote by $G(U)$ (or $[U]_G$, or $[U]$) the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use the property $x\tilde{\ }y$, and $x\aleph y$ whenever x, y are not adjacent in G. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A and B are *totally adjacent* and we denote by $A \sim B$, while by $A\aleph B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say A and B are *totally non-adjacent*.

The *neighborhood* of a vertex $v \in V$ is the set $NG(v) = \{u \in V : uv \in E\}$, while $NG[v] = NG(v) \bigcup \{v\}$; we denote these sets by $N(v)$ and $N[v]$, respectively, when G appears clearly from the context. The *degree* of a vertex v in G is $dG(v) = |NG(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighborhood of $S \subset V$ is the set $N(S) = \bigcup_{v \in S} N(v) - S$ and $N[S] = S \bigcup N(S)$. A graph is complete if every pair of distinct vertices is adjacent.

By Pn, Cn, Kn we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A *complete bipartite* graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted $K_{m,n}$.

Let F denote a family of graphs. A graph G is called *F-free* if none of its subgraphs is in F.

The *Zykov sum* of the graphs $G1, G2$ is the graph $G = G1 + G2$ having:

$$V(G) = V(G1) \bigcup V(G2) \text{ and}$$
$$E(G) = E(G1) \bigcup E(G2) \bigcup \{uv : u \in V(G1), v \in V(G2)\}.$$

Let G be a connected graph and u and v be two nonadjacent vertices of G. A *uv-separator* is a set $S \subseteq V(G)$ such that u and v are in distinct connected components of $G - S$. The separator is said to be *minimal* if no proper subset of S has the same property.

Let $G = (V, E)$ be a connected graph. A non-empty set of vertices T is called *star-cutset* ([2]) if $G - T$ is not connected and there exists a vertex v in T that is adjacent to any other vertex in T. The cutset is *minimal* if no proper subset of T has the same property.

We call a graph *Berge* if neither the graph nor its complement contains C_n, as an induced subgraph, for n odd and $n \geq 5$.

The *chromatic number* of a graph G, denoted by $\chi(G)$ is the least number of colors needed to color its vertices so that adjacent vertices have different colors. The *stability number* $\alpha(G)$ of a graph G is the cardinality of the largest stable set. Recall that a stable set of G is a subset of the vertices such that no two of them are connected by an edge. The *clique number* of a graph G is a number of the vertices in a maximum clique of G, denoted by $\omega(G)$.

A graph G is *perfect* if, for each induced subgraph S of G, the chromatic number of S is equal to the clique number of S. A graph is *minimal imperfect* if it is not perfect and yet every proper induced subgraph is perfect.

A class H of graphs is called *hereditary* if every induced subgraph of a graph in H is in H.

2. A new characterization of doubly chordal graphs

The notion of weak decomposition (a partition of the set of vertices in three classes A, B, C such that A induces a connected graph and C is totally adjacent to B and totally non-adjacent to A) and the study of its properties allow us to obtain several important results.

We recall a characterization of the weak decomposition of a graph.

Definition 1. ([15]) *A set $A \subset V(G)$ is called a weak set of the graph G if $NG(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted by A.*

Definition 2. ([15]) *Let $G = (V, E)$ be a connected and non-complete graph. If A is a weak set, then the partition $\{A, N(A), V - A \bigcup N(A)\}$ is called a weak decomposition of G with respect to A.*

The existence of a *weak component* of any connected and connected graph is guaranteed by the following result.

Theorem 1.([16]) *Every connected and non-complete graph $G = (V, E)$ admits a weak component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

We recall a characterization of a weak component of a connected and non-complete graph.

Theorem 2.([16]) *Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weak component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The following consequence of Theorem 2 ensures the existence of a weak decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weak decomposition (A, B, C), such that $G(A)$ is a weak component and $G(V - A) = G(B) + G(C)$.*

The following *O(n+m)* algorithm for building a weak decomposition for a non-complete and connected graph is provided by Theorem 2.

Algorithm for the weak decomposition of a graph ([13], see [18])

Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.

Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$ and $A \aleph R = \overline{N}(A)$.

Begin
A:= any set of vertices such that $A \bigcup N(A) \neq V$
$N := N(A)$
$R := V - A \bigcup N(A)$
While ($\exists\ n \in N$, $\exists\ r \in R$ such that $nr \notin E$) *do*
Begin
$A := A \bigcup \{n\}$
$N := (N - \{n\}) \bigcup (N(n) \bigcap R)$
$R := R - (N(n) \bigcap R)$
end
end

Many problems efficiently solvable for doubly chordal graphs remain efficiently solvable for dually chordal graphs too [5].

A. Brandstädt, V. Chepoi, F. Dragan [5] give an algorithm for solving the connected r-domination and Steiner tree problem in linear time on doubly chordal graphs and in quadratic time on dually chordal graphs.

The *k-th power* of a graph G is a graph on the same vertex set as G, where a pair of vertices is connected by an edge if they have distance at most k in G.

As was shown in [4] all powers of doubly chordal graphs are doubly chordal.

We say that a vertex v is *simplicial* in G if $N_G[v]$ is complete.

A vertex $u \in N[v]$ is a *maximum neighbor* of v if and only if for all $w \in N[v]$ the inclusion $N[w] \subseteq N[v]$ holds (note that the case $u = v$ is not excluded). The ordering $(v_1, \cdots, v_n)$ is a *maximum neighborhood ordering* if for all $i \in \{1, \cdots, n\}$ there is a maximum neighbor $u_i \in N_i[v_i]$:

$$\text{for all } w \in N_i[v_i],\ N_i[w] \subseteq N_i[u_i] \text{ holds.}$$

The graph G is *dually chordal* [4] if and only if G has a maximum neighborhood ordering. A vertex v of a graph G is *doubly simplicial* [12] if v is simplicial and has a maximum neighbor. A linear ordering $(v_1, \cdots, v_n)$ of the vertices of G is *doubly perfect* if for all $i \in \{1, \cdots, n\}$ v_i is a doubly simplicial vertex of G_i. A graph G is *doubly chordal* [12] if it admits a doubly perfect ordering.

Theorem 3. *(Dragan [8], Moscarini [12] and Brandstädt [4]). For a graph G the following condition are equivalent:*

(1) *G is double chordal;*
(2) *G is chordal and dually chordal.*

Moscarini [12] presented a polynomial algorithm for the connected domination problem on doubly chordal graphs.

A graph G is called *hereditary dually chordal* graph if any induced subgraph of G is dually chordal. Similarly, a graph G is called *hereditary doubly chordal* if any induced subgraph of G is doubly chordal.

The following characterization of chordal graphs using minimal separators is well-known.

Theorem 4. [7, 9] *Let G be a graph. Then G is chordal if and only if every minimal vertex separator of G is complete.*

We recall the following characterizations in terms of weak decompositions.

Theorem 5. [13] A graph *G is triangulated if and only if for any weak decomposition $(A; N; R)$ with $G(A)$ weak component the following hold:*

1) N is a clique
2) $[R]$ and $G - R$ are triangulated.

Theorem 6. ([19]) *A graph G is hereditary dually chordal if and only if for any weak decomposition (A, N, R) with $G(A)$ weak component the following hold:*

(1) *N is clique*
(2) *$G(R)$ and $G(V - R)$ are hereditary dually chordal.*

We prove a new characterization of *hereditary doubly chordal* graphs.

Theorem 7. *Let $G = (V, E)$ be a connected and non-complete graph. G is hereditary doubly chordal if and only if for any weak decomposition $(A; N; R)$ with $G(A)$ weak component the following hold:*

(1) *N is clique;*
(2) *$G(R), G(V - R)$ are hereditary doubly chordal graphs.*

Proof. I) Suppose that G is a hereditary doubly chordal graph. From Theorem 3 it follows that G is chordal and hereditary dually chordal. Because G is chordal, from Theorem 5, it follows that N is a clique in G and $G(R)$ and $G(V - R)$ are chordal graphs. Because G is hereditary dually chordal, it follows that $G(R)$ and $G(V - R)$ are hereditary dually chordal graphs. Then $G(R)$ and $G(V - R)$ are hereditary doubly chordal graphs.

II) Suppose that (1) and (2) hold. We show that G is hereditary doubly chordal. Since $G(R)$ and $G(V - R)$ are hereditary doubly chordals graphs, it follows from Theorem 3 that $G(R), G(V - R)$ are chordal graphs and hereditary dually chordal graphs. Since $G(R), G(V - R)$ are chordal graphs and N is a clique, it follows from Theorem 5 that G is chordal. Since $G(R), G(V - R)$ are hereditary dually chordal graphs and N is a clique, Theorem 6 shows that G is hereditary dually chordal. Because G is chordal and hereditary dually chordal, according to Theorem 3, it follows that G is a hereditary doubly chordal graph.

The above results lead to the following recognition algorithm.

Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.

Output: An answer to the question: is G a *hereditary doubly chordal* graph

begin
$L := \{G\}$;//La list of graphs
While ($L \neq \emptyset$)
Extract an element F from L;
Find a weak decomposition (A, N, R) for F;

If ((N not clique in G) or (N not clique in G2)) then
Return: G is not hereditary doubly chordal
else introduce in L the connected components of $G(R)$, $G(V-R)$ *incomplete*
Return: G is hereditary doubly chordal
end

The fact that N is a clique is proved in the following. If there is a vertex v in N with the degree in $G(N)$ less than $|N|-1$ (thus we determine the grades of the vertices in the subgraph $G(N)$) then N is not a clique. So the algorithm is has the order of $O(n(n+m))$ time complexity (since the weak decomposition is $O(n+m)$).

We will give an analogue of the following result for the case of hereditary doubly chordal graphs.

Corollary 2. [14, 18]. *If G is a connected graph and (A, N, R) is a weak decomposition with A a weak component then the following holds:*

$$\alpha(G) = \max\left\{\alpha[A] + \alpha[R], \alpha(A \bigcup N)\right\};$$

$$\omega(G) = max\{\omega([N]) + \omega([R]), \omega([A \bigcup N])\}.$$

Corollary 3. *Let $G = (V, E)$ be a connected and non-complete graph with $G(A)$ a weak component in G. If G is hereditary doubly chordal then the following holds:*

$$\alpha(G) = \alpha(G(A)) + \alpha(G(R)); \tag{1}$$

$$\omega(G) = \max\left\{|N| + \omega(G(R)), \omega(G(A \bigcup N))\right\}. \tag{2}$$

Proof. From Corollary 2, because N is a clique in G we get (2). Let $T \subset A \bigcup N$ such that T is a stable set and $|T| = \alpha([A \bigcup N]_G)$. Because N is a clique in G it follows that $|T \bigcap N| \leq 1$. If $T \bigcap N = \emptyset$, then $T \bigcup \{r\}$ is a stable set in $[A \bigcup R]_G$, else if $T \bigcap N = \{v\}$ then $(T - \{v\}) \bigcup \{r\}$ is a stable set in $[A \bigcup R]_G$, $\forall r \in R$. So, (1) holds.

Corollary 3 implies an algorithm for the construction of a stable set of maximum cardinal and a clique of maximum cardinal in a *hereditary doubly chordal* graph.

Input: $G = (V, E)$ a connected and noncomplete graph satisfying conditions in Corollary 3
Output: A stable set S with $|S| = \alpha(G)$
begin
$S = \emptyset$
$L = \{G\}//L$ is a list of graphs
while $(L \neq \emptyset)$
begin
extract an element F from L
if (F is complete) *then*

$$S = S \bigcup \{v\}, \forall v \in V(F)$$

else
Determine a weak decomposition (A, N, R) for F
Put $[A]F$ and the connected components of $[R]F$ in L
end
end.
Input: $G = (V, E)$ a connected and noncomplete graph satisfying conditions in Corollary 3
Output: A clique Q with $|Q| = \omega(G)$
Begin

$$Q = \emptyset$$

$k = 0$
$L = \{G\}//L$ is a list of graphs
while $(L \neq \emptyset)$
begin
extract an element F from L
if (F is complete) *then*

$$Q = Q \bigcup V(F)$$

$$k = k + 1$$

$$Q_k = Q$$

$$Q = \emptyset$$

else
Determine a weak decomposition (A, N, R) for L

Put $[A\bigcup N]_F$ and $[N\bigcup R]_F$ in L
end
Let m be so that $|Q_m| = \max\{|Q_1|, \cdots, |Q_k|\}$and $Q = Q_m$
Then $\omega(G) = |Q|$
end

The complexity of α and $\omega$is *O(n(n+m))*. The fact that $V(F)$ is a clique reads as follows: if there is a vertex v in $V(F)$ with the degree in F less than $|V(F)|-1$ (thus we determine the grades of the vertices in F), then $V(F)$ is not a clique. This way, the algorithm is *O(n(n+m))* (as long as the weak decomposition has the complexity $O(n+m)$).

3. Conclusions

We give a characterization of *hereditary doubly* chordal graphs using weak decomposition. We also give recognition algorithms for *hereditary doubly* chordal graphs in *O(n(n+m))* time. Finally, we determine the combinatorial optimization numbers in *O(n(n+m))* time.

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