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GENERALIZATIONS OF TRIANGULATED GRAPHS

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Abstract. Using weak decomposition, we characterize in a unified manner the classes of doubly chordal, of hereditary dually chordal and of strongly chordal graphs. We also give a recognition algorithm that is applicable for doubly chordal graphs, for hereditary dually chordal and for strongly chordal graphs. In addition, we determine the combinatorial optimization numbers in efficient time for all the above classes of graphs.

1. Introduction

The class of triangulated graphs (also known as chordal graphs) has remarkable properties, among those we mention perfection, this class is algorithmically useful, possessing recognition algorithms and provide ways to solve some combinatorial optimization problems (such as determining the stability number and minimum number of covering cliques) with linear complexity algorithms.

Various generalizations of triangulated graphs were introduced. Among these, we mention the class of Weakly Triangulated Graphs, introduced by R. Hayward [10], see also [3], [2], which are the graphs containing no cycles of length less than or equal to five and the complements of these cycles (i. e. $\left\{C_k, \overline{C_k}\right\} \not\subset G$ for every $k \geq 5$).

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Another generalization is the class of Slightly Triangulated Graphs, introduced by F. Maire [10], which are known as the graphs containing no cycles of length less than or equal to five and having the propriety that for any induced subgraph there is a vertex whose neighborhood in the subgraph does not contain P4.

The strongly chordal graphs, another generalization for the triangulated graphs, are defined in terms of a stronger ordering condition (M. Farber [8], see [6], [12]). A graph G is strongly chordal if and only if every induced subgraph of G has a simple vertex. A vertex v of the graph G is simple in G if the set $\{N[u] : u \in N[v]\}$ is linearly ordered by inclusion. The problem of locating minimum weight dominating sets and minimum weight independent dominating sets in strongly chordal graphs with real vertex weights can be solved in polynomial time, whereas each of these problems is NP-hard for chordal graphs.

Graphs with maximum neighborhood orderings were characterized and turned out to be algorithmically useful. These graphs are dual to chordal graphs [4]. The graph G is *dually chordal* [4] if and only if G has a maximum neighborhood ordering.

Recall that a vertex $u \in N[v]$ is a *maximum neighbor* of v if and only if for all $w \in N[v]$ the inclusion $N[w] \subseteq N[v]$ holds (the case $u = v$ is not excluded). The ordering $(v_1, \cdots, v_n)$ is a *maximum neighborhood ordering* if for all $i \in \{1, \cdots, n\}$ there is a maximum neighbor $u_i \in N_i[v_i]$: for all $w \in N_i[v_i]$, $N_i[w] \subseteq N_i[u_i]$ holds.

It is specified in [19] that for doubly chordal graphs clique problem can be solved in polynomial time, while the independent set problem and the recognition problem can be solved in linear time.

Many problems that are efficiently solvable for strongly chordal graphs remain polynomial-time solvable for dually chordal graphs.

The *hereditary dually chordal* graphs [4] are the graphs for which each induced subgraph is a dually chordal graph.

The following complexity results are known. In [20] it is specified that the for dually chordal graphs the clique problem and the independent set problem are NP-complete, while the recognition problem can be solved in linear time. In [21] it is specified that for strongly chordal graphs the clique problem can be solved in polynomial time, the independent set problem in linear time, the recognition problem in polynomial time.

2. Preliminaries

Let $G = (V, E)$ be a connected, finite and undirected graph, without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ (or $co - G$) is the complement of G. If $U \subseteq V$, by $G(U)$ (or $[U]_G$, or $[U]$) we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use the notation $x \sim y$, while we denote $x \aleph y$ whenever x, y are not adjacent in G. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A and B are *totally adjacent* and we denote by $A \sim B$, while by $A \aleph B$ we mean that no edge of G joins some vertex of A to a vertex from B and in this case we say A and B are *totally non-adjacent*.

The *neighborhood* of the vertex $v \in V$ is the set $NG(v) = \{u \in V : uv \in E\}$, while $NG[v] = NG(v) \bigcup \{v\}$; we denote these sets by $N(v)$ and $N[v]$, respectively, when G appears clearly from the context. The *degree* of v in G is $dG(v) = |NG(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighborhood of $S \subset V$ is the set $N(S) = \bigcup_{v \in S} N(v) - S$ and $N[S] = S \bigcup N(S)$. A graph is complete if every pair of distinct vertices is adjacent.

By Pn, Cn, Kn we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

Let F denote a family of graphs. A graph G is called *F-free* if none of its subgraphs is in F.

The *Zykov sum* of the graphs $G1, G2$ is the graph $G = G1 + G2$ having $V(G) = V(G1) \bigcup V(G2)$ and

$$E(G) = E(G1) \bigcup E(G2) \bigcup \{uv : u \in V(G1), v \in V(G2)\}.$$

Let G be connected and u and v be two nonadjacent vertices of G. A *uv-separator* is a set $S \subseteq V(G)$ such that u and v are in different connected components of $G - S$. The separator is *minimal* if no proper subset of S has the same property.

Let $G = (V, E)$ be a connected graph. A non-empty set of vertices T is called *star-cutset* ([2]) if $G - T$ is not connected and there exists a vertex v in T that is adjacent to any other vertex in T. The cutset is *minimal* if no proper subset of T has the same property.

We call a graph *Berge* if neither the graph nor its complement contains C_n, as an induced subgraph, for n odd and $n \geq 5$.

The *chromatic number* of a graph G, denoted by $\chi(G)$) is the least number of colors needed to color its vertices so that adjacent vertices have different colors. The *stability number* $\alpha(G)$ of a graph G is the cardinality of the largest stable set. Recall that a stable set of G is a subset of the vertices such that no two of them are connected by an edge. The *clique number* of a graph G is a number of the vertices in a maximum clique of G, denoted by $\omega(G)$.

A graph G is *perfect* if, for each induced subgraph S of G, the chromatic number of S is equal to the clique number of S. A graph is *minimal imperfect* if it is not perfect and yet every proper induced subgraph is perfect.

A class H of graphs is called *hereditary* if every induced subgraph of a graph in H is in H.

3. A new characterization of {doubly, hereditary dually and strongly} chordal graphs

The notion of weak decomposition (a partition of the set of vertices in three classes A, B, C such that A induces a connected graph and C is totally adjacent to B and totally non-adjacent to A) and the study of its properties allow us to obtain several important results.

Many problems efficiently solvable for strongly chordal and doubly chordal graphs remain efficiently solvable for dually chordal graphs too [5].

A. Brandstädt, V. Chepoi, F. Dragan [5] give an algorithm for solving the connected r-domination problem and Steiner tree problem in linear time on doubly chordal graphs and in quadratic time on dually chordal graphs.

We say that a vertex v is *simplicial* in G if $NG[v]$ is complete.

The graph G is *dually chordal* [4] if and only if G has a maximum neighborhood ordering.

A vertex v of a graph G is *doubly simplicial* [13] if v is simplicial and has a maximum neighbor. A linear ordering $(v_1, \cdots, v_n)$ of the vertices of G is *doubly perfect* if for all $i \in \{1, \cdots, n\}$ v_i is a doubly simplicial vertex of G_i.

A graph G is *doubly chordal* [13] if it admits a doubly perfect ordering. A graph G is *hereditary doubly chordal* graph if any induced subgraph of G is doubly chordal.

Theorem 1. [7, 9] *Let G be a graph. Then G is chordal if and only if every minimal vertex separator of G is complete.*

Theorem 2. [14] *A graph G is triangulated if and only if for any weak decomposition $(A; N; R)$ with $G(A)$ weak component the following hold:*

1) N is a clique

2) $[R]$ and $G - R$ are triangulated.

Corollary 1. [7] *For a graph G the following conditions are equivalent:*

(i) G is a strongly chordal graph;

(ii) G is a hereditary dually chordal graph.

Theorem 3. ([17]) *A graph G is hereditary dually chordal if and only if for any weak decomposition (A, N, R) with $G(A)$ a weak component:*

(1) *N is a clique*

(2) *$G(R)$ and $G(V - R)$ are hereditary dually chordal.*

Theorem 4. ([18]) *Let $G = (V, E)$ be a connected and non-complete graph with $G(A)$ a weak component. G is hereditary doubly chordal if and only if the following hold:*

(1) *N is a clique;*

(2) *$G(R)$ and $G(V - R)$ are hereditary doubly chordal graphs.*

Corollary 2. [16]. *If G is a connected graph and (A, N, R) is a weak decomposition with A a weak component then the following hold:*

$$\alpha(G) = \max\left\{\alpha[A] + \alpha[R], \alpha(A \bigcup N)\right\};$$

$$\omega(G) = max\{\omega([N]) + \omega([R]), \omega([A \bigcup N])\}.$$

Definition 1. ([8]) *A strong elimination ordering of a graph $G = (V, E)$ is an ordering $v_1, v_2, \cdots, v_n$ of V with the property that for each i, j, k and l, if $i < j$,$k < l$, $v_k, v_l \in N[v_i]$ and $v_k \in N[v_j]$, then $v_l \in N[v_j]$.*

Definition 2. ([8]) A graph is strongly chordal if it admits a strong elimination ordering.

Remark 1. ([8]) *Every induced subgraph of a strongly chordal graph is strongly chordal.*

If C is a cycle of even length in the graph G, then a *strong chord* of C is an edge of G joining two vertices, u and v of C such that $d_C(u, v)$ is odd and greater than 1.

Theorem 5. ([8]) *A graph G is strongly chordal if and only if it is chordal and every even cycle of length at least 6 in G has a strong chord.*

In the following we will unify several known results.

Let M be one of the following classes of graphs: hereditary dually chordal graphs, hereditary doubly chordal graphs, strongly chordal graphs.

Theorem 6. *$G \in M$ if and only if*

(1) *N is a clique*
(2) *$G(V-R)$ and $G(R)$ are of the same type as G, i. e. belong to M.*

Proof.

(1) Theorem 3. ([17]) refers to the case of hereditary dually chordal graphs.
(2) Theorem 4. ([18]) refers to the case of hereditary doubly chordal graphs.
(3) We consider the case of strongly chordal graphs.

Theorem 7. *A graph G is strongly chordal if and only if for any weak decomposition (A, N, R) with $G(A)$ a weak component:*

(1) *N is a clique*
(2) *$G(R)$ and $G(V-R)$ are strongly chordal.*

Proof. I. We suppose that G is strongly chordal. If G is strongly chordal then, according to Remark 1, it follows that $G(R)$ and $G(V-R)$ are strongly chordal. So, (2) holds. Since G is strongly chordal, according to Theorem 5, it follows that G is chordal. From Theorem 2 it follows that N is a clique. So, the necessity part is proved.

II. Suppose N is a clique and $G(R)$ and $G(V-R)$ are strongly chordal.. We show that G is strongly chordal. Since $G(R)$ and $G(V-R)$ are strongly chordal, it follows from Theorem 5 that $G(R)$ and $G(V-R)$ are chordal graphs. Then, applying Theorem 2, it follows that G is chordal. If G is not strongly chordal, then, according to Theorem 5, there is an even cycle C, of length at least 6, which has no strong chord.

Because N is a clique and $R \sim N$ it follows that either $C \subseteq G(R)$ or $C \subseteq G(N \bigcup A)$. So, either $G(R)$ or $G(V-R)$ is not strongly chordal, a contradiction.

The above results lead to the following recognition algorithm.

Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.

Output: An answer to the question: is G a *doubly, hereditary dually, strongly chordal* graph, respectively

begin

$L := \{G\}$;*//L a list of graphs*

While ($L \neq \emptyset$)

Extract an element F from L;

Find a weak decomposition (A, N, R) *for F;*

If (N not clique in F) then

Return: G is not (doubly, hereditary dually, strongly) chordal graph, respectively

else introduce in L the connected components of $G(R), G(V - R)$ *incomplete*

Return: G is (doubly, hereditary dually, strongly) chordal graph, respectively

end

The fact that N is a clique is proved in the following: If there is a vertex v in N with the degree in $G(N)$ less than $|N| - 1$ (thus we determine the grades of the vertices in the subgraph $G(N)$), then N is not a clique. So the algorithm is $O(n(n + m))$ time (just because the weak decomposition is $O(n + m)$).

The following holds.

Corollary 3. *Let $G = (V, E)$ be a connected and non-complete graph with $G(A)$ a weak component in G. If G is (doubly, hereditary dually, strongly) chordal graph, respectively then the following hold:*

$$\alpha(G) = \alpha(G(A)) + \alpha(G(R)); \tag{1}$$

$$\omega(G) = \max\left\{|N| + \omega(G(R)), \omega(G(A \bigcup N))\right\}. \tag{2}$$

Proof. From Corollary 2, because N is a clique in G we obtain (2). Let $T \subset A \bigcup N$ such that T is a stable set and $|T| = \alpha([A \bigcup N]_G)$. Because N is a clique in G, it follows that $|T \bigcap N| \leq 1$. If $T \bigcap N = \emptyset$, then $T \bigcup \{r\}$ is a stable set in $[A \bigcup R]_G$; else, if $T \bigcap N = \{v\}$, then $(T - \{v\}) \bigcup \{r\}$ is a stable set in $[A \bigcup R]_G$, for every $r \in R$. So, (1) holds.

Corollary 3 implies an algorithm for the construction of a stable set of maximum cardinal and a clique of maximum cardinal in a *(doubly, hereditary dually, strongly) chordal* graph, respectively.

Input: $G = (V, E)$ a connected and noncomplete graph satisfying conditions in Corollary 3

Output: A stable set S with $|S| = \alpha(G)$

begin

$S = \emptyset$

$L = \{G\}//L$ is a list of graphs

while $(L \neq \emptyset)$

begin

extract an element F from L

if (F is complete) *then*

$$S = S \bigcup \{v\}\,, \forall v \in V(F)$$

else

Determine a weak decomposition (A, N, R) for F

Put $[A]F$ and the connected components of $[R]F$ in L

end

end.

Input: $G = (V, E)$ a connected and noncomplete graph satisfying conditions in Corollary 3

Output: A clique Q with$|Q| = \omega(G)$

Begin

$$Q = \emptyset$$

$k = 0$

$L = \{G\}//L$ is a list of graphs

while $(L \neq \emptyset)$

begin

extract an element F from L

if (F is complete) *then*

$$Q = Q \bigcup V(F)$$

$$k = k + 1$$

$$Q_k = Q$$

$$Q = \emptyset$$

else
Determine a weak decomposition (A,N, R) for L
Put $[A \bigcup N]_F$ and $[N \bigcup R]_F$ in L
end
Let m be so that $|Q_m| = \max\{|Q_1|, \cdots, |Q_k|\}$and $Q = Q_m$
Then $\omega(G) = |Q|$
end

The complexity of α and ω is *O(n(n+m))*. The fact that $V(F)$ is a clique reads as follows: If there is a vertex v in $V(F)$ with the degree in F less than $|V(F)|-1$ (thus we determine the grades of the vertices in F), then $V(F)$ is not a clique. This way, the algorithm is *O(n(n+m))* (as long as the weak decomposition has the complexity *O(n+m)*).

4. Conclusions

Using weak decomposition, we give a characterization for each of the following classes: hereditary dually chordal graphs, hereditary doubly chordal graphs and strogly chordal graphs, in a unified manner. We also give recognition algorithms for doubly chordal, of hereditary dually chordal and strongly chordal graphs in *O(n(n+m))* time. Finally, we determine the combinatorial optimization numbers in *O(n(n+m))* time.

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