"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 26(2016), No. 2, 5-30

SOME GENERALIZATIONS OF ULTRA CONTINUOUS MULTIFUNCTIONS

TAKASHI NOIRI AND VALERIU POPA

Abstract. We introduce the notion of upper/lower (τ, m)-continuous multifunctions and obtain many characterizations of such multifunctions. The notion of upper/lower (τ, m)-continuous multifunctions is a generalization of (τ, m)-continuous functions [34] and upper/lower ultra continuous multifunctions due to Navalagi et al. [26].

Work presented at "Gheorghe Vrănceanu" Conference on Mathematics and Informatics, December 6, 2016, "Vasile Alecsandri" University of Bacău

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity of functions and multifunctions in topological spaces and bitopological spaces. By using these sets, many authors introduced and studied various types of modifications of continuity in bitopological spaces.

The notions of (i, j)-semi-open sets [20], (i, j)-preopen sets [11], (i, j)-α-open sets [12], (i, j)-semi-preopen sets [14]; (i, j)-semi-continuity, (i, j)-precontinuity, (i, j)-α-continuity and (i, j)-semiprecontinuity are introduced and investigated in bitopological spaces.

Keywords and phrases: m-structure, m-space, bitopological space, multifunction, (τ, m)-continuous, ultra continuous.
(2010) Mathematics Subject Classification: 54C08, 54C60, 54E55.

On the other hand, the notions of quasi-open sets or $\tau_1\tau_2$-open sets [9], [21], [37], [38], quasi-semi-open sets [15], quasi preopen sets [30], quasi-α-open sets [39], quasi-semipreopen sets [39]; quasi-continuity, quasi-semi-continuity, quasi precontinuity, quasi α-continuity and quasi semiprecontinuity are introduced and studied in bitopological spaces.

As variations of quasi-open sets and quasi-continuity, the notions of (1,2)-semi-open sets, (1,2)-preopen sets, (1,2)-α-open sets; (1,2)-semi-continuity, (1,2)-precontinuity, and (1,2)-α-continuity are introduced in [16]. The notions of (1,2)-semi-preopen sets and (1,2)-semi-precontinuity are introduced and studied in [17].

Similarly, the notions of (1,2)*-semiopen sets, (1,2)*-preopen sets, (1,2)*-α-open sets, (1,2)*-semi-preopen sets; (1,2)*-semi-continuity, (1,2)*-precontinuity, (1,2)*-α-continuity, and (1,2)*-semi-precontinuity are introduced in [36].

The present authors introduced and investigated the notions of minimal structures, m-spaces [31] and [33], m-continuity [33], M-continuity [31] and (τ, m)-continuity [34] for functions. In this paper, we introduce the notion of upper/lower (τ, m)-continuous multifunctions as multifunctions from a topological space (X, τ) into an m-space (Y, m). The notion of (τ, m)-continuous multifunctions is a generalization of the notions of (τ, m)-continuous functions [34] and ultra continuous multifunctions due to Navalagi et al. [26]. In the last section, we transfer the study of a multifunction F from a topological space (X, τ) into a bitopological space $(Y, \sigma_1.\sigma_2)$ to the study of a (τ, m)-continuous multifunction $F : (X, \tau) \to (Y, m(\sigma_1.\sigma_2))$. Then such multifunctions enable us to obtain the unified theory of many generalizations of ultra-continuous multifunctions.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. The closure and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *semi-open* [18] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,

(2) *preopen* [23] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,

(3) *α-open* [27] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(4) *β-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all semi-open (resp. preopen, α-open, β-open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$).

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, β-open) set is said to be *semi-closed* [5] (resp. *preclosed* [23], *α-closed* [24], *β-closed* [1] or *semi-preclosed* [3]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed) sets of X containing A is called the *semi-closure* [5] (resp. *preclosure* [10], *α-closure* [24], *β-closure* [2] or *semi-preclosure* [3]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. singlevalued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

3. Minimal structures

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [31], [32] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty subset X with a minimal structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (briefly m-open) and the complement of an m_X-open set is said to be *m_X-closed* (briefly m-closed).

Remark 3.1. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$ are all m-structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X*-closure* of A and the m_X*-interior* of A are defined in [22] as follows:

(1) $\mathrm{mCl}(A) = \bigcap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \bigcup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{spInt}(A)$).

Lemma 3.1. (Maki et al. [22]) *Let X be a nonempty set and m_X an m-structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,

(2) If $(X \setminus A) \in m_X$, then $\mathrm{mCl}(A) = A$ *and if $A \in m_X$, then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2. (Popa and Noiri [32]) *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. An m-structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [22] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 3.3. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, and $\mathrm{SPO}(X)$ have property $\mathcal{B}$.

Lemma 3.3. (Popa and Noiri [35]) *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,

(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,

(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Definition 3.4. A function $f : (X, \tau) \to (Y, m_Y)$ is said to be (τ, m)-*continuous* [34] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in \tau$ containing x such that $f(U) \subset V$. The function $f : (X, \tau) \to (Y, m_Y)$ is said to be (τ, m)-*continuous* if it has the property at each point $x \in X$.

Theorem 3.1. (Popa and Noiri [34]) *For a function $f : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*
(1) f is (τ, m)-continuous;
(2) $f^{-1}(V)$ is open in X for every $V \in m_Y$;
(3) $f(\mathrm{Cl}(A)) \subset \mathrm{mCl}(f(A))$ for every subset A of X;
(4) $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{mCl}(B))$ for every subset B of Y;
(5) $f^{-1}(\mathrm{mInt}(B)) \subset \mathrm{Int}(f^{-1}(B))$ for every subset B of Y;
(6) $f^{-1}(K)$ is closed for every m-closed set K of Y.

4. (τ, m)-continuity for multifunctions

Definition 4.1. A multifunction $F : (X, \tau) \to (Y, m_Y)$ is said to be
(1) *upper* (τ, m)-*continuous* at $x \in X$ if for each $V \in m_Y$ containing $F(x)$, there exists $U \in \tau$ containing x such that $F(U) \subset V$,
(2) *lower* (τ, m)-*continuous* at $x \in X$ if for each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \tau$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,
(3) *upper/lower* (τ, m)-*continuous* if F has this property at each $x \in X$.

Theorem 4.1. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*
(1) F is upper (τ, m)-continuous at $x \in X$;
(2) $x \in \mathrm{Int}(F^+(V))$ for each m-open set V of Y containing $F(x)$;
(3) $x \in F^-(\mathrm{mCl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^-(B))$;
(4) $x \in \mathrm{Int}(F^+(B))$ for every subset B of Y such that $x \in F^+(\mathrm{mInt}(B))$.

Proof. (1) $\Rightarrow$ (2): Let V be any m-open set V of Y containing $F(x)$. There exists an open set U containing x such that $F(U) \subset V$. Then $x \in U \subset F^+(V)$. Since U is open, we have $x \in \mathrm{Int}(F^+(V))$.

(2) $\Rightarrow$ (3): Suppose that B is any subset of Y such that $x \in \mathrm{Cl}(F^-(B))$. Since m_Y has property $\mathcal{B}$, by Lemma 3.3 $\mathrm{mCl}(B)$ is m_Y-closed. Suppose that $x \notin F^-(\mathrm{mCl}(B))$. Then $x \in X - F^-(\mathrm{mCl}(B)) = F^+(Y - \mathrm{mCl}(B))$. This implies that $F(x) \subset Y - \mathrm{mCl}(B)$. Since $Y - \mathrm{mCl}(B)$ is an m_Y-open set, by (2) we have $x \in \mathrm{Int}(F^+(Y - \mathrm{mCl}(B))) = \mathrm{Int}(X - F^-(\mathrm{mCl}(B))) = X - \mathrm{Cl}(F^-(\mathrm{mCl}(B))) \subset X - \mathrm{Cl}(F^-(B))$. Hence $x \notin \mathrm{Cl}(F^-(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y such that $x \in F^+(\mathrm{mInt}(B))$. Suppose that $x \notin \mathrm{Int}(F^+(B))$. Then $x \in X - \mathrm{Int}(F^+(B)) = \mathrm{Cl}(X - F^+(B)) = \mathrm{Cl}(F^-(Y - B))$. By (3) we have $x \in F^-(\mathrm{mCl}(Y - B)) = F^-(Y - \mathrm{mInt}(B)) = X - F^+(\mathrm{mInt}(B))$. Hence $x \notin F^+(\mathrm{mInt}(B))$.

(4) $\Rightarrow$ (1): Let V be any m-open set of Y containing $F(x)$. Since m_Y has property $\mathcal{B}$, $V = \mathrm{mInt}(V)$ and $x \in F^+(V) = F^+(\mathrm{mInt}(V))$. Then, by (4), $x \in \mathrm{Int}(F^+(V))$. Therefore, there exists $U \in \tau$ containing x such that $U \subset F^+(V)$. Hence $F(U) \subset V$. This shows that F is upper (τ, m)-continuous at $x \in X$.

Theorem 4.2. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) F is lower (τ, m)-continuous at $x \in X$;

(2) $x \in \mathrm{Int}(F^-(V))$ for each m-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) $x \in F^+(\mathrm{mCl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^+(B))$;

(4) $x \in \mathrm{Int}(F^-(B))$ for every subset B of Y such that $x \in F^-(\mathrm{mInt}(B))$;

(5) $x \in F^+(\mathrm{mCl}(F(A)))$ for every subset A of X such that $x \in \mathrm{Cl}(A)$.

Proof. We prove only implications (3) $\Rightarrow$ (5) $\Rightarrow$ (4), being the proofs of the other are similar as in Theorem 4.1.

(3) $\Rightarrow$ (5): Let A be any subset of X such that $x \in \mathrm{Cl}(A)$. Since $\mathrm{Cl}(A) \subset \mathrm{Cl}(F^+(F(A))$, by (5) we have $x \in F^+(\mathrm{mCl}(F(A)))$.

(5) $\Rightarrow$ (4): Let B be any subset of Y such that $x \in F^-(\mathrm{mInt}(B))$. Suppose that $x \notin \mathrm{Int}(F^-(B))$. Then $x \in X - \mathrm{Int}(F^-(B)) = \mathrm{Cl}(X - F^-(B)) = \mathrm{Cl}(F^+(Y - B))$. Then by (5), $x \in F^+(\mathrm{mCl}(F(F^+(Y - B))) \subset F^+(\mathrm{mCl}(Y - B)) = F^+(Y - \mathrm{mInt}(B)) = X - F^-(\mathrm{mInt}(B))$. Hence $x \notin F^-(\mathrm{mInt}(B))$.

Corollary 4.1. *For a function $f : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) f is (τ, m)-continuous at $x \in X$;

(2) $x \in \mathrm{Int}(f^{-1}(V))$ for each m-open set V of Y containing $f(x)$;

(3) $x \in f^{-1}(\mathrm{mCl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(f^{-1}(B))$;

(4) $x \in \mathrm{Int}(f^{-1}(B))$ for every subset B of Y such that $x \in f^{-1}(\mathrm{mInt}(B))$;

(5) $x \in f^{-1}(\mathrm{mCl}(f(A)))$ for every subset A of X such that $x \in \mathrm{Cl}(A)$.

Theorem 4.3. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) F is upper (τ, m)-continuous;
(2) $F^+(V)$ is open in X for every $V \in m_Y$;
(3) $F^-(K)$ for closed in X every m_Y-closed set K;
(4) $\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{mCl}(B))$ for every subset B of Y;
(5) $F^+(\mathrm{mInt}(B)) \subset \mathrm{Int}(F^+(B))$ for every subset B of Y.

Proof. This follows from Theorem 3.1 of [28].

Theorem 4.4. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) F is lower (τ, m)-continuous;
(2) $F^-(V)$ is open in X for every $V \in m_Y$;
(3) $F^+(K)$ is closed in X for every m_Y-closed set K;
(4) $\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{mCl}(B))$ for every subset B of Y;
(5) $F(\mathrm{Cl}(A)) \subset \mathrm{mCl}(F(A))$ for every subset A of X;
(6) $F^-(\mathrm{mInt}(B)) \subset \mathrm{Int}(F^-(B))$ for every subset B of Y.

Proof. This follows from Theorem 3.2 of [28].

Remark 4.1. By Theorem 4.4, we obtain Theorem 3.1 for the single valued functions.

For a multifunction $F : (X, \tau) \to (Y, m_Y)$, we define $D^+_{\tau m}(F)$ and $D^-_{\tau m}(F)$ as follows:

$D^+_{\tau m}(F) = \{x \in X : F$ is not upper (τ, m)-continuous at $x \in X\}$,
$D^-_{\tau m}(F) = \{x \in X : F$ is not lower (τ, m)-continuous at $x \in X\}$.

Theorem 4.5. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following equalities hold:*

$$\begin{aligned} D^+_{\tau m}(F) &= \bigcup_{G \in m_Y} \{F^+(G) - \mathrm{Int}(F^+(G))\} \\ &= \bigcup_{B \in P(Y)} \{F^+(\mathrm{mInt}(B)) - \mathrm{Int}(F^+(B))\} \\ &= \bigcup_{B \in P(Y)} \{\mathrm{Cl}(F^-(B)) - F^-(\mathrm{mCl}(B))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\mathrm{Cl}(F^-(H)) - F^-(H)\}, \text{ where} \end{aligned}$$

$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all m-closed sets of (Y, m_Y).

Proof. We shall show only the first equality and the last equality since the proofs of other are similar to the first.

Let $x \in D^+_{\tau m}(F)$. Then, by Theorem 4.1, there exists $V \in m_Y$ containing $F(x)$ such that $x \notin \mathrm{Int}(F^+(V))$. Therefore, we have

$$x \in F^+(V) - \mathrm{Int}(F^+(V)) \subset \bigcup_{G \in m_Y} \{F^+(G) - \mathrm{Int}(F^+(G))\}.$$

Conversely, let $x \in \bigcup_{G\in m_Y}\{F^+(G) - \mathrm{Int}(F^+(G))\}$. Then there exists $V \in m_Y$ such that $x \in F^+(V) - \mathrm{Int}(F^+(V))$. By Theorem 4.1, $x \in D^+_{\tau m}(F)$.

We prove the last equality.

$$\bigcup_{H\in\mathcal{F}} \{\mathrm{Cl}(F^-(H)) - F^-(H)\} \subset \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^-(B)) - F^-(\mathrm{mCl}(B))\} = D^+_{\tau m}(F).$$

Conversely, since m_Y has property $\mathcal{B}$, by Lemmas 3.1 and 3.3 we have

$$D^+_{\tau m}(F) = \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^-(B)) - F^-(\mathrm{mCl}(B))\} \subset \bigcup_{H\in\mathcal{F}} \{\mathrm{Cl}(F^-(H)) - F^-(H)\}.$$

Theorem 4.6. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following equalities hold:*

$$\begin{aligned} D^-_{\tau m}(F) &= \bigcup_{G\in m_Y}\{F^-(G) - \mathrm{Int}(F^-(G))\} \\ &= \bigcup_{B\in P(Y)}\{F^-(\mathrm{mInt}(B)) - \mathrm{Int}(F^-(B))\} \\ &= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^+(B)) - F^+(\mathrm{mCl}(B))\} \\ &= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - F^+(\mathrm{mCl}(F(A)))\} \\ &= \bigcup_{H\in\mathcal{F}} \{\mathrm{Cl}(F^+(H)) - F^+(H)\}, \end{aligned}$$ *where*

$P(X)$ is the family of all subsets of X,
$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all m-closed sets of (Y, m_Y).

Proof. The proof is similar to that of Theorem 4.5.

Let (X, τ) be a topological space and (Y, m_Y) an m-space. For a function $f : (X, \tau) \to (Y, m_Y)$, we define $D_{\tau m}(f)$ as follows:

$$D_{\tau m}(f) = \{x \in X : f \text{ is not } (\tau, m)\text{-continuous at } x\}.$$

Corollary 4.2. *For a function $f : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following equalities hold:*

$$\begin{aligned} D_{\tau m}(f) &= \bigcup_{G\in m_Y}\{f^{-1}(G) - \mathrm{Int}(f^{-1}(G))\} \\ &= \bigcup_{B\in P(Y)}\{f^{-1}(\mathrm{mInt}(B)) - \mathrm{Int}(f^{-1}(B))\} \\ &= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(f^{-1}(B)) - f^{-1}(\mathrm{mCl}(B))\} \\ &= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - f^{-1}(\mathrm{mCl}(f(A)))\} \\ &= \bigcup_{H\in\mathcal{F}} \{\mathrm{Cl}(f^{-1}(H)) - f^{-1}(H)\}, \end{aligned}$$ *where*

$P(X)$ is the family of all subsets of X,
$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all m-closed sets of (Y, m_Y).

Theorem 4.7. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, $D^+_{\tau m}(F)$ (resp. $D^-_{\tau m}(F)$) is identical with the union of the frontiers of the upper* *(resp. lower) inverse images of m-open sets of Y containing (resp. meeting) F(x).*

Proof. We shall prove the first case since the proof of the second is similar.
Let $x \in D^+_{\tau m}(F)$. Then, there exists an m-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every open set U containing x. By Lemma 3.2, we have $x \in \mathrm{mCl}(X - F^+(V))$. On the other hand, since $x \in F^+(V) \subset \mathrm{Cl}(F^+(V))$ and hence $x \in \mathrm{Fr}(F^+(V))$.

Conversely, suppose that F is upper (τ, m)-continuous at $x \in X$. Then, for any m-open set V of Y containing $F(x)$, there exists $U \in \tau$ containing x such that $F(U) \subset V$; hence $x \in U \subset F^+(V)$. Therefore, we have $x \in U \subset \mathrm{Int}(F^+(V))$. This is contrary to the fact that $x \in \mathrm{Fr}(F^+(V))$.

Corollary 4.3. *For a function $f : (X, \tau) \to (Y, m_Y)$, $D_{\tau m}(f)$ is identical with the union of the frontiers of the inverse images of m-open sets of Y containing f(x).*

Definition 4.2. Let S be a subset of an m-space (Y, m_Y). A point $y \in Y$ is called an *m_θ-adherent* point of S if $\mathrm{mCl}(V) \cap S \neq \emptyset$ for every m_Y-open set V containing y.

The set of all m_θ-adherent points of S is called the *m_θ-closure* of S and is denoted by $\mathrm{mCl}_\theta(S)$. If $S = \mathrm{mCl}_\theta(S)$, then S is said to be *m_θ-closed*. The complement of an m_θ-closed set is said to be *m_θ-open*.

Remark 4.2. Let S be a subset of a topological space (X, τ) and $m_X = \tau$ (resp. $\mathrm{SO}(X), \mathrm{PO}(X)$), then $\mathrm{mCl}_\theta(S) = \mathrm{Cl}_\theta(S)$ [41] (resp. $\mathrm{sCl}_\theta(S)$ [6], $\mathrm{pCl}_\theta(S)$ [29]).

Lemma 4.1. (Popa and Noiri [35]) *Let S be a subset of an m-space (Y, m_Y). If m_Y satisfies property $\mathcal{B}$, then* $\mathrm{mCl}_\theta(S)$ *is m_Y-closed for every subset S of Y.*

Definition 4.3. An m-space (Y, m_Y) is said to be *m-regular* [35] if for each m_Y-closed set F and each $y \notin F$, there exist disjoint m_Y-open sets U and V such that $y \in U$ and $F \subset V$.

Remark 4.3. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$). Then m-regularity coincides with regularity (resp. semi-regularity [7], pre-regularity [29]).

Lemma 4.2. (Popa and Noiri [35]) *Let (Y, m_Y) be an m-regular space. Then every m-open set is m_θ-open.*

Theorem 4.8. *Let (Y, m_Y) be an m-regular space, where m_Y has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*

(1) F is upper (τ, m)-continuous;
(2) $F^-(\mathrm{mCl}_\theta(B))$ is closed in X for every subset B of Y;
(3) $F^-(K)$ is closed in X for every m_θ-closed set K of Y;
(4) $F^+(V)$ is open in X for every m_θ-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then by Lemma 4.1, $\mathrm{mCl}_\theta(B)$ is m-closed in Y and by Theorem 4.3, $F^-(\mathrm{mCl}_\theta(B))$ is closed in X.

(2) $\Rightarrow$ (3): Let K be a m_θ-closed set of Y. Then $K = \mathrm{mCl}_\theta(K)$ and by (2) $F^-(K)$ is closed in X.

(3) $\Rightarrow$ (4): Let V be an m_θ-open set of Y. Then $Y - V$ is m_θ-closed and hence $F^-(Y - V) = X - F^+(V)$ is closed in X. Hence $F^+(V)$ is open in X.

(4) $\Rightarrow$ (1): Let V be any m-open set of Y. Since (Y, m_Y) is m-regular, by Lemma 4.2 V is m_θ-open. By (4), $F^+(V)$ is open in X and by Theorem 4.3 F is upper (τ, m)-continuous.

Theorem 4.9. *Let (Y, m_Y) be an m-regular space, where m_Y has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*

(1) F is lower (τ, m)-continuous;
(2) $F^+(\mathrm{mCl}_\theta(B))$ is closed in X for every subset B of Y;
(3) $F^+(K)$ is closed in X for every m_θ-closed set K of Y;
(4) $F^-(V)$ is open in X for every m_θ-open set V of Y.

Proof. The proof is similar to that of Theorem 4.8.

Let (Y, m_Y) be an m-space. By $\mathrm{mCl}(F) : (X, \tau) \to (Y, m_Y)$, we denote a multifunction defined by $\mathrm{mCl}(F)(x) = \mathrm{mCl}(F(x))$ for each $x \in X$.

Lemma 4.3. *Let $F : (X, \tau) \to (Y, m_Y)$ be a multifunction. Then $(\mathrm{mCl}(F))^-(V) = F^-(V)$ for every m-open set V of Y.*

Proof. Let V be any m-open set of Y and $x \in (\mathrm{mCl}(F))^-(V)$. Then $\mathrm{mCl}(F(x)) \cap V \neq \emptyset$ and there exists $y \in Y$ and $y \in \mathrm{mCl}(F(x))$. Since V is m-open, by Lemma 3.2 $V \cap F(x) \neq \emptyset$ and hence $x \in F^-(V)$. This shows that $(\mathrm{mCl}(F))^-(V) \subset F^-(V)$.

Conversely, let $x \in F^-(V)$. Then we have $\emptyset \neq F(x) \cap V \subset \mathrm{mCl}(F(x)) \cap V$ and hence $x \in (\mathrm{mCl}(F))^-(V)$. This shows that $F^-(V) \subset \mathrm{mCl}(F(x))^-(V)$.

Theorem 4.10. *A multifunction $F : (X, \tau) \to (Y, m_Y)$ is lower (τ, m)-continuous if and only if* $\mathrm{mCl}(F) : (X, \tau) \to (Y, m_Y)$ *is lower (τ, m)-continuous.*

Proof. Let F be lower (τ, m)-continuous and $V \in m_Y$. Then, by Theorem 4.4 $F^-(V)$ is open in X. By Lemma 4.3, $(\mathrm{mCl}(F))^-(V) = F^-(V)$ is open and by Theorem 4.4 $\mathrm{mCl}(F)$ is lower (τ, m)-continuous. Similarly, if $\mathrm{mCl}(F)$ is lower (τ, m)-continuous, F is lower (τ, m)-continuous.

Definition 4.4. An m-space (Y, m_Y) is said to be *m-compact* [28] if every cover of Y by m-open sets of Y has a finite subcover. A subset B of (Y, m_Y) is said to be *m-compact* if every cover of B by m-open sets of Y has a finite subcover.

Theorem 4.11. *If $F : (X, \tau) \to (Y, m_Y)$ is an upper (τ, m)-continuous surjective multifunction such that F(x) is m-compact for each $x \in X$ and (X, τ) is compact, then (Y, m_Y) is m-compact.*

Proof. The proof follows from Corollary 3.1 of [28].

5. Minimal structures in bitopological spaces

In this section, we recall four types of generaliztions of open sets in bitopological spaces. Every family belonging to these types is an m-space having property $\mathcal{B}$.

A. $(i, j)m_X$-open sets.

Definition 5.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) (i, j)-*semi-open* [20] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, i, $j = 1$, 2,

(2) (i, j)-*preopen* [11] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, i, $j = 1$, 2,

(3) (i, j)-α-*open* [12] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, i, $j = 1$, 2,

(4) (i, j)-*semi-preopen* (briefly (i, j)-*sp-open*) [14] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j, i, j = 1, 2$.

The family of all (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-*sp*-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

Remark 5.1. Let (X, τ_1, τ_2) be a bitoppological space and A a subset of X. Then $(i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, and $(i, j)\mathrm{SPO}(X)$

are all minimal structures on X. If $(i,j)m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$), then

(1) $(i,j)\mathrm{mCl}(A) = (i,j)$-$\mathrm{sCl}(A)$ (resp. (i,j)-$\mathrm{pCl}(A)$, (i,j)-$\alpha\mathrm{Cl}(A)$, (i,j)-$\mathrm{spCl}(A)$),

(2) $(i,j)\mathrm{mInt}(A) = (i,j)$-$\mathrm{sInt}(A)$ (resp. (i,j)-$\mathrm{pInt}(A)$, (i,j)-$\alpha\mathrm{Int}(A)$, (i,j)-$\mathrm{spInt}(A)$).

Remark 5.2. Let (X, τ_1, τ_2) be a bitopological space.

(1) Let $(i,j)m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\alpha(X)$). Then by Lemma 3.1 we obtain the results established in Theorem 13 of [20] (resp. Theorem 3.6 of [25]).

(2) Let $(i,j)m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$). Then by Lemma 3.2 we obtain the result established in Theorem 15 of [19] (resp. Theorem 3.5 of [14], Theorem 3.5 of [25], Theorem 3.5 of [14]).

Remark 5.3. Let (X, τ_1, τ_2) be a bitopological space.

(1) $(i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$) is an minimal structure on X satisfying $\mathcal{B}$ by Theorem 2 of [20] (resp. Theorem 4.2 of [13] or Theorem 3.2 of [14], Theorem 3.2 of [25], Theorem 3.2 of [14]).

(2) Let $(i,j)m_X = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$). Then by Lemma 3.3 we obtain the result established in Theorem 1.13 of [19] (resp. Theorem 3.5 of [14], Theorem 3.6 of [25], Theorem 3.6 of [14]).

B. Quasi m-open sets.

Definition 5.2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *quasi open* [9], [21] or *$\tau_1\tau_2$-open* [37], [38] if $A = B \cup C$, where $B \in \tau_1$ and $C \in \tau_2$,

(2) *quasi semi-open* [15] if $A = B \cup C$, where $B \in \mathrm{SO}(X, \tau_1)$ and $C \in \mathrm{SO}(X, \tau_2)$,

(3) *quasi preopen* [30] if $A = B \cup C$, where $B \in \mathrm{PO}(X, \tau_1)$ and $C \in \mathrm{PO}(X, \tau_2)$,

(4) *quasi semipreopen* [40] if $A = B \cup C$, where $B \in \mathrm{SPO}(X, \tau_1)$ and $C \in \mathrm{SPO}(X, \tau_2)$,

(5) *quasi α-open* [39] if $A = B \cup C$, where $B \in \alpha(X, \tau_1)$ and $C \in \alpha(X, \tau_2)$.

The family of all quasi open (resp. quasi semi-open, quasi preopen, quasi semipreopen, quasi α-open) sets of a bitopological space (X, τ_1, τ_2) is denoted by QO(X), $\tau_1\tau_2(X)$ or (1,2)O(X) (resp. QSO(X), QPO(X), QSPO(X), Qα(X)).

Definition 5.3. Let (X, τ_1, τ_2) be a bitopological space and m^1_X (resp. m^2_X) an m-structure on the topological space (X, τ_1) (resp. (X, τ_2)). The family

$$qm_X = \{A \subset X : A = B \cup C, \text{ where } B \in m^1_X \text{ and } C \in m^2_X \}$$

is a minimal structure on X and hence is called a *quasi m-structure* on X. Each member of qm_X is said to be *quasi m_X-open* (or briefly quasi m-open). The complement of a quasi m_X-open set is said to be *quasi m_X-closed* (or briefly quasi m-closed).

Remark 5.4. Let (X, τ_1, τ_2) be a bitopological space.

(1) If m^1_X and m^2_X have property $\mathcal{B}$, then qm_X is an m-structure with property $\mathcal{B}$.

(2) If $m^1_X = \tau_1$ and $m^2_X = \tau_2$ (resp. SO(X, τ_1) and SO(X, τ_2), PO(X, τ_1) and PO(X, τ_2), SPO(X, τ_1) and SPO(X, τ_2), $\alpha(X, \tau_1)$ and $\alpha(X, \tau_2)$), then $qm_X =$ QO(X), $\tau_1\tau_2(X)$ or (1,2)O(X) (resp. QSO(X), QPO(X), QSPO(X), QαO(X)).

(3) Since SO(X, τ_i) (resp. PO(X, τ_i), SPO(X, τ_i) and $\alpha(X, \tau_i)$) has property $\mathcal{B}$ for $i = 1$, 2, QSO(X) (resp. QPO(X), QSPO(X) and QαO(X)) has property $\mathcal{B}$.

Definition 5.4. Let (X, τ_1, τ_2) be a bitopological space. For a subset A of X, the *quasi m_X-closure* of A and the *quasi m_X-interior* of A are defined as follows:

(1) qmCl$(A) = \cap\{F : A \subset F, X - F \in qm_X\}$,

(2) qmInt$(A) = \cup\{U : U \subset A, U \in qm_X\}$.

Remark 5.5. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. If $qm_X =$ QO(X) (resp. QSO(X), QPO(X), QSPO(X), QαO(X)), then we have

(1) qmCl(A) = qCl(A) [38] (resp. qsCl(A) [15], qpCl(A) [30], qspCl(A) [40], qαCl(A) [39]),

(2) qmInt(A) = qInt(A) (resp. qsInt(A), qpInt(A), qspInt(A), qαInt(A)).

The notations qCl(A) and qInt(A) are also denoted by $\tau_1\tau_2$Cl(X) (or (1,2)Cl(A)) and $\tau_1\tau_2$Int(X) (or (1,2)Int(A)), respectively.

C. $(1,2)^*$-m_X-open sets

Definition 5.5. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) $(1,2)^*$-*semi-open* [36] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(A))$,
(2) $(1,2)^*$-*preopen* [36] if $A \subset \tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A))$,
(3) $(1,2)^*$-α-*open* [36] if $A \subset \tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(A)))$,
(4) $(1,2)^*$-*semi-preopen* [36] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\tau_2\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A)))$.

The complement of a $(1,2)^*$-semi-open (resp. $(1,2)^*$-preopen, $(1,2)^*$-α-open, $(1,2)^*$-semi-preopen) set is said to be $(1,2)^*$-semi-closed (resp. $(1,2)^*$-preclosed, $(1,2)^*$-α-closed, $(1,2)^*$-semi-preclosed).

The family of all $(1,2)^*$-semi-open (resp. $(1,2)^*$-preopen, $(1,2)^*$-α-open, $(1,2)^*$-semi-preopen) sets is denoted by $(1,2)^*\mathrm{SO}(X)$ (resp. $(1,2)^*\mathrm{PO}(X)$, $(1,2)^*\alpha(X)$, $(1,2)^*\mathrm{SPO}(X)$).

Remark 5.6. Let (X,τ_1,τ_2) be a bitopological space and A a subset of X.

(1) The families $(1,2)^*\mathrm{SO}(X)$, $(1,2)^*\mathrm{PO}(X)$, $(1,2)^*\alpha(X)$, and $(1,2)^*\mathrm{SPO}(X)$ are all m-structures with property $\mathcal{B}$.

(2) By $(1,2)^*m_X$, we denote each member of the above five familes and call it an m-structure determinded by τ_1 and τ_2. Let $(1,2)^*m_X = \tau_{12}\mathrm{O}(X)$ (resp. $(1,2)^*\mathrm{SO}(X)$, $(1,2)^*\mathrm{PO}(X)$, $(1,2)^*\alpha(X)$, $(1,2)^*\mathrm{SPO}(X)$), then we have

(i) $(1,2)^*\mathrm{mCl}(A) = \tau_1\tau_2\mathrm{Cl}(A)$ (resp. $(1,2)^*\mathrm{sCl}(A)$, $(1,2)^*\mathrm{pCl}(A)$, $(1,2)^*\alpha\mathrm{Cl}(A)$, $(1,2)^*\mathrm{spCl}(A)$),

(ii) $(1,2)^*\mathrm{mInt}(A) = \tau_1\tau_2\mathrm{Int}(A)$ (resp. $(1,2)^*\mathrm{sInt}(A)$, $(1,2)^*\mathrm{pInt}(A)$, $(1,2)^*\alpha\mathrm{Int}(A)$, $(1,2)^*\mathrm{spInt}(A)$).

(3) Since each one of $(1,2)^*m_X$ has property $\mathcal{B}$, by Lemma 3.3 we have

(i) A is $(1,2)^*m_X$-closed if and only if $(1,2)^*\mathrm{mCl}(A) = A$,

(ii) A is $(1,2)^*m_X$-open if and only if $(1,2)^*\mathrm{mInt}(A) = A$

for $(1,2)^*m_X = \tau_{12}\mathrm{O}(X)$ (resp. $(1,2)^*\mathrm{SO}(X)$, $(1,2)^*\mathrm{PO}(X)$, $(1,2)^*\alpha(X)$, $(1,2)^*\mathrm{SPO}(X)$).

(4) By Lemma 3.2, we obtain the result established in Proposition 2.2(ii) of [38].

(5) By Lemma 3.1, we obtain the relations between $(1,2)^*\mathrm{mCl}(A)$ and $(1,2)^*\mathrm{mInt}(A)$.

D. $(1,2)m_X$-open sets.

Definition 5.6. A subset A of a bitopological space (X,τ_1,τ_2) is said to be

(1) *(1,2)-semi-open* [16] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(A))$,
(2) *(1,2)-preopen* [16] if $A \subset \tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A))$,
(3) *(1,2)-α-open* [16] if $A \subset \tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(A)))$,
(4) *(1,2)-semi-preopen* [17] if $A \subset \tau_1\tau_2\mathrm{Cl}(\tau_1\mathrm{Int}(\tau_1\tau_2\mathrm{Cl}(A)))$.

The complement of (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) set of X is said to be *(1,2)-semi-closed* (resp. *(1,2)-preclosed*, *(1,2)-α-closed*, *(1,2)-semi-preclosed*). The intersection of all (1,2)-semi-closed (resp. (1,2)-preclosed, (1,2)-α-closed, (1,2)-semi-preclosed) sets containing A is called the *(1,2)-semi-closure* (resp. *(1,2)-preclosure*, *(1,2)-α-closure*, *(1,2)-semi-preclosure*) of A and is denoted by (1,2)sCl(A) (resp. (1,2)pCl(A), (1,2)αCl(A), (1,2)spCl(A)). The union of (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) sets of X contained in A is called the *(1,2)-semi-interior* (resp. *(1,2)-preinterior*, *(1,2)-α-interior*, *(1,2)-semi-preinterior*) of A and is denoted by (1,2)sInt(A) (resp. (1,2)pInt(A), (1,2)αInt(A), (1,2)spInt(A)).

The collection of all (1,2)-semi-open (resp. (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) sets of X is denoted by (1,2)SO(X) (resp. (1,2)PO(X), (1,2)αO(X), (1,2)SPO(X)).

Remark 5.7. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X.

(1) The families $\tau_1\tau_2$O(X), (1,2)SO(X), (1,2)PO(X), (1,2)αO(X) and (1,2)SPO(X) are all m-structures on X having property $\mathcal{B}$.

(2) By $(1,2)m_X$, we denote each one of the above families and call it an m-structure determined by the topologies τ_1 and τ_2 on X. If (1,2)m_X $=$ $\tau_1\tau_2$O(X) (resp. (1,2)SO(X), (1,2)PO(X), (1,2)αO(X), (1,2)SPO(X)), then we have

(i) (1,2)mCl(A) $=$ $\tau_1\tau_2$Cl(A) (resp. (1,2)sCl(A), (1,2)pCl(A), (1,2)αCl(A), (1,2)spCl(A)),

(ii) (1,2)mInt(A) $=$ $\tau_1\tau_2$Int(A) (resp. (1,2)sInt(A), (1,2)pInt(A), (1,2)αInt(A), (1,2)spInt(A)).

(3) Since the families $\tau_1\tau_2$O(X), (1,2)SO(X), (1,2)PO(X), (1,2)αO(X) and (1,2)SPO(X) have property $\mathcal{B}$, a subet A of X is $\tau_1\tau_2$-closed (resp. (1,2)-semi-closed, (1,2)-preclosed, (1,2)-α-closed, (1,2)-semi-preclosed) if and only if $A = \tau_1\tau_2$Cl(A) (resp. $A =$ (1,2)sCl(A), $A =$ (1,2)pCl(A), $A =$ (1,2)αCl(A), $A =$ (1,2)spCl(A)) and also A is $\tau_1\tau_2$-open (resp. (1,2)-semi-open, (1,2)-preopen, (1,2)-α-open, (1,2)-semi-preopen) if and only if $A = \tau_1\tau_2$Int(A) (resp. $A =$ (1,2)sInt(A), $A =$ (1,2)pInt(A), $A =$ (1,2)αInt(A), $A =$ (1,2)spInt(A)).

(4) By Lemma 3.2, we obtain the results established in Lemma 8(iii) of [4].

(5) By Lemma 3.1, we obtain the relations between (1,2)mCl(A) and (1,2)mInt(A).

Remark 5.8. It follows from **A**, **B**, **C** and **D** that if (X, τ_1, τ_2) is a bitopological space then some minimal structures on X determined by τ_1 and τ_2 are introduced. In the sequel, by $m(\tau_1, \tau_2)$ (simply m_{12}) we denote a minimal structure on X determined by τ_1 and τ_2, that is, $(i, j)m_X$, qm, $(1, 2)^*m_X$ or $(1, 2)m_X$.

6. (τ, m)-continuity for multifunctions

Definition 6.1. A multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *upper ultra continuous* at a point $x \in X$ [26] if for each (1,2)α-open set V containing $F(x)$, there exists an open set U containing x such that $F(U) \subset V$,

(2) *lower ultra continuous* at a point $x \in X$ [26] if for each (1,2)α-open set V such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) *upper/lower ultra continuous* if F has this property at each $x \in X$.

Hence, it turns out that $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is ultra upper/lower continuous at a point $x \in X$ if and only if $F : (X, \tau) \to (Y, (1, 2)\alpha(Y))$ is upper/lower (τ, m)-continuous at a point $x \in X$.

Definition 6.2. Let (X, τ) be a topological space, (Y, σ_1, σ_2) a bitopological space and $m_{12} = m(\sigma_1, \sigma_2)$ an minimal structure on Y determined by σ_1 and σ_2. A multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is said to be *upper/lower* (τ, m_{12})*-continuous* at a point $x \in X$ (resp. on X) if $F : (X, \tau) \to (Y, m_{12})$ is upper/lower (τ, m)-continuous at $x \in X$ (resp. on X).

Hence a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *upper* (τ, m_{12})*-continuous* at $x \in X$ if for each m_{12}-open set V containing $F(x)$, there exists an open set U containing x such that $F(U) \subset V$,

(2) *lower* (τ, m_{12})*-continuous* at $x \in X$ if for each m_{12}-open set V such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) *upper/lower* (τ, m_{12})*-continuous* if F has this property at each $x \in X$.

Remark 6.1. (1) If $m(\sigma_1, \sigma_2) = (1, 2)\alpha(Y)$, then we obtain the definitions of Definition 6.1

By Theorems 4.1-4.4, for the family $m(\sigma_1, \sigma_2) = (i, j)m_Y, qm, (1, 2)^*m_Y$ and $(1, 2)m_Y$, we obtain the following characterizations.

Theorem 6.1. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1, \sigma_2)$ on Y, the following properties are equivalent:*

(1) F is upper (τ, m_{12})-continuous at $x \in X$;

(2) $x \in \mathrm{Int}(F^+(V))$ for each m_{12}-open set V of Y containing F(x);

(3) $x \in F^-(\mathrm{m_{12}Cl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^-(B))$;

(4) $x \in \mathrm{Int}(F^+(B))$ for every subset B of Y such that $x \in F^+(\mathrm{m_{12}Int}(B))$.

Theorem 6.2. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1, \sigma_2)$ on Y, the following properties are equivalent:*

(1) F is lower (τ, m_{12})-continuous at $x \in X$;

(2) $x \in \mathrm{Int}(F^-(V))$ for each m_{12}-open set V of Y meeting F(x);

(3) $x \in F^+(\mathrm{m_{12}Cl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^+(B))$;

(4) $x \in \mathrm{Int}(F^-(B))$ for every subset B of Y such that $x \in F^-(\mathrm{m_{12}Int}(B))$;

(5) $x \in F^-(\mathrm{m_{12}Cl}(F(A)))$ for every subset A of Y such that $x \in \mathrm{Cl}(A)$.

Theorem 6.3. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1, \sigma_2)$ on Y, the following properties are equivalent:*

(1) F is upper (τ, m_{12})-continuous;

(2) $F^+(V)$ is open in X for every $V \in m_{12}$;

(3) $F^-(K)$ for closed in X every m_{12}-closed set K;

(4) $\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{m_{12}Cl}(B))$ for every subset B of Y;

(5) $F^+(\mathrm{m_{12}Int}(B)) \subset \mathrm{Int}(F^+(B))$ for every subset B of Y.

Theorem 6.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1, \sigma_2)$ on Y, the following properties are equivalent:*

(1) F is lower (τ, m_{12})-continuous;

(2) $F^-(V)$ is open in X for every $V \in m_{12}$;

(3) $F^+(K)$ is closed in X for every m_{12}-closed set K;

(4) $\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{m_{12}Cl}(B))$ for every subset B of Y;

(5) $F(\mathrm{Cl}(A)) \subset \mathrm{m_{12}Cl}(F(A))$ for every subset A of X;

(6) $F^-(\mathrm{m_{12}Int}(B)) \subset \mathrm{Int}(F^-(B))$ for every subset B of Y.

If $m(\sigma_1, \sigma_2) = (1, 2)\alpha(Y)$, then we obtain the following four corollaries.

Corollary 6.1. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m(\sigma_1, \sigma_2) = (1, 2)\alpha(Y)$ on Y, the following properties are equivalent:*

(1) F is upper $(\tau, (1, 2)\alpha(Y))$-continuous at $x \in X$;

(2) $x \in \mathrm{Int}(F^+(V))$ for each $V \in (1, 2)\alpha(Y)$ containing $F(x)$;

(3) $x \in F^-((1, 2)\alpha(\mathrm{Y})\mathrm{Cl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^-(B))$;

(4) $x \in \mathrm{Int}(F^+(B))$ for every subset B of Y such that $x \in F^+((1, 2)\alpha(\mathrm{Y})\mathrm{Int}(B))$.

Corollary 6.2. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m(\sigma_1, \sigma_2) = (1, 2)\alpha(Y)$ on Y, the following properties are equivalent:*

(1) F is lower $(\tau, (1, 2)\alpha(Y))$-continuous at $x \in X$;

(2) $x \in \mathrm{Int}(F^-(V))$ for each $(1, 2)\alpha(Y)$-open set V of Y meeting $F(x)$;

(3) $x \in F^+((1, 2)\alpha(\mathrm{Y})\mathrm{Cl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^+(B))$;

(4) $x \in \mathrm{Int}(F^-(B))$ for every subset B of Y such that $x \in F^-((1, 2)\alpha(\mathrm{Y})\mathrm{Int}(B))$;

(5) $x \in F^-((1, 2)\alpha(\mathrm{Y})\mathrm{Cl}(F(A)))$ for every subset A of Y such that $x \in \mathrm{Cl}(A)$.

Corollary 6.3. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m(\sigma_1, \sigma_2) = (1, 2)\alpha(Y)$ on Y, the following properties are equivalent:*

(1) F is upper $(\tau, (1, 2)\alpha(Y))$-continuous;

(2) $F^+(V)$ is open in X for every $V \in (1, 2)\alpha(Y)$;

(3) $F^-(K)$ for closed in X every $(1, 2)\alpha(Y)$-closed set K;

(4) $\mathrm{Cl}(F^-(B)) \subset F^-((1, 2)\alpha(\mathrm{Y})\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^+((1, 2)\alpha(\mathrm{Y})\mathrm{Int}(B)) \subset \mathrm{Int}(F^+(B))$ for every subset B of Y.

Corollary 6.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m(\sigma_1, \sigma_2) = (1, 2)\alpha(Y)$ on Y, the following properties are equivalent:*

(1) F is lower $(\tau, (1, 2)\alpha(Y))$-continuous;

(2) $F^-(V)$ is open in X for every $V \in (1, 2)\alpha(Y)$;

(3) $F^+(K)$ is closed in X for every $(1, 2)\alpha(Y)$-closed set K;

(4) $\mathrm{Cl}(F^+(B)) \subset F^+((1, 2)\alpha(\mathrm{Y})\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F(\mathrm{Cl}(A)) \subset (1, 2)\alpha(\mathrm{Y})\mathrm{Cl}(F(A))$ for every subset A of X;

(6) $F^-((1, 2)\alpha(\mathrm{Y})\mathrm{Int}(B)) \subset \mathrm{Int}(F^-(B))$ for every subset B of Y.

Definition 6.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j) K-continuous* [8] if for each (i, j)-semi-open set V of Y containing $f(x)$, there exists a τ_i-open set U of X containing x such that $f(U) \subset V$.

Hence a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(i, j)K$-continuous if and only if a function $f : (X, \tau_i) \to (Y, (i, j)\mathrm{SO}(Y))$ is (τ, m)-continuous.

Definition 6.4. Let (Y, σ_1, σ_2) be a bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure on Y determined by σ_1 and σ_2. A function $f : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is said to be (τ, m)*-continuous* if a function $f : (X, \tau) \to (Y, m(\sigma_1, \sigma_2))$ is (τ, m)-continuous.

Remark 6.2. (1) If $m(\sigma_1, \sigma_2) = (i, j)\mathrm{SO}(Y)$, then we obtain Definition 6.3.

(2) By Theorem 6.4, we obtain the following theorem for single valued functions.

Theorem 6.5. *For a function $f : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1, \sigma_2)$ on Y, the following properties are equivalent:*

(1) f is (τ, m_{12})-continuous;
(2) $f^{-1}(V)$ is open in X for every $V \in m_{12}$;
(3) $f^{-1}(K)$ is closed in X for every m_{12}-closed set K;
(4) $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{m_{12}Cl}(B))$ for every subset B of Y;
(5) $f(\mathrm{Cl}(A)) \subset \mathrm{m_{12}Cl}(f(A))$ for every subset A of X;
(6) $f^{-1}(\mathrm{m_{12}Int}(B)) \subset \mathrm{Int}(f^{-1}(B))$ for every subset B of Y.

Remark 6.3. If $m(\sigma_1, \sigma_2) = (i, j)\mathrm{SO}(Y)$, then by Theorem 6.5 we obtain the result established in Theorem 1 of [8].

For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m(\sigma_1, \sigma_2)$ on Y determined by σ_1 and σ_2, we define $D^+_{\tau m_{12}}(F)$ and $D^-_{\tau m_{12}}(F)$ as follows:

$D^+_{\tau m_{12}}(F) = \{x \in X : F$ is not upper $(\tau, m(\sigma_1, \sigma_2)$-continuous at $x \in X\}$,

$D^-_{\tau m_{12}}(F) = \{x \in X : F$ is not lower $(\tau, m(\sigma_1, \sigma_2))$-continuous at $x \in X\}$.

The following theorems follow immediately from Theorems 4.5 and 4.6.

Theorem 6.6. *For a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1, \sigma_2)$ on Y determined by σ_1 and σ_2, the* *following properties hold:*

$$\begin{aligned} D^+_{\tau m_{12}}(F) &= \bigcup_{G\in m(\sigma_1,\sigma_2)}\{F^+(G) - \mathrm{Int}(F^+(G))\} \\ &= \bigcup_{B\in P(Y)}\{F^+(\mathrm{m}_{12}\mathrm{Int}(B)) - \mathrm{Int}(F^+(B))\} \\ &= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^-(B)) - F^-(\mathrm{m}_{12}\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in \mathcal{F}}\ \{\mathrm{Cl}(F^-(H)) - F^-(H)\}, \text{ where} \end{aligned}$$

$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all $m(\sigma_1,\sigma_2)$-closed sets of Y.

Theorem 6.7. *For a multifunction $F : (X,\tau) \to (Y,\sigma_1,\sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1,\sigma_2)$ on Y determined by σ_1 and σ_2, the following properties hold:*

$$\begin{aligned} D^-_{\tau m_{12}}(F) &= \bigcup_{G\in m(\sigma_1,\sigma_2)}\{F^-(G) - \mathrm{Int}(F^-(G))\} \\ &= \bigcup_{B\in P(Y)}\{F^-(\mathrm{m}_{12}\mathrm{Int}(B)) - \mathrm{Int}(F^-(B))\} \\ &= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^+(B)) - F^+(\mathrm{m}_{12}\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - F^+(\mathrm{m}_{12}\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{H\in \mathcal{F}}\ \{\mathrm{Cl}(F^+(H)) - F^+(H)\}, \text{ where} \end{aligned}$$

$P(X)$ is the family of all subsets of X,
$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all $m(\sigma_1,\sigma_2)$-closed sets of Y.

For a function $f : (X,\tau) \to (Y,\sigma_1,\sigma_2)$ and a minimal structure $m_{12} = m(\sigma_1,\sigma_2)$ on Y determined by σ_1 and σ_2, by Corollary 4.2 we obtain the following corollary.

Corollary 6.5. *For a function $f : (X,\tau) \to (Y,\sigma_1,\sigma_2)$ and a minimal structure $m(\sigma_1,\sigma_2)$ on Y determined by σ_1 and σ_2, the following properties hold:*

$$\begin{aligned} D_{\tau m_{12}}(f) &= \bigcup_{G\in m(\sigma_1,\sigma_2)}\{f^{-1}(G) - \mathrm{Int}(f^{-1}(G))\} \\ &= \bigcup_{B\in P(Y)}\{f^{-1}(\mathrm{m}_{12}\mathrm{Int}(B)) - \mathrm{Int}(f^{-1}(B))\} \\ &= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(f^{-1}(B)) - f^{-1}(\mathrm{m}_{12}\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - f^{-1}(\mathrm{m}_{12}\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{H\in \mathcal{F}}\ \{\mathrm{Cl}(f^{-1}(H)) - f^{-1}(H)\}, \text{ where} \end{aligned}$$

$P(X)$ is the family of all subsets of X,
$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all $m(\sigma_1,\sigma_2)$-closed sets of Y.

Let $m(\sigma_1,\sigma_2) = (i,j)\mathrm{SO}(Y)$, then by Corollary 6.5 we obtain the following corollary.

Corollary 6.6. *For a function $f : (X,\tau) \to (Y,\sigma_1,\sigma_2)$ and a minimal structure $m(\sigma_1,\sigma_2) = (i,j)SO(Y)$, the following properties hold:*

$$D_{\tau m_{12}}(f) = \bigcup_{G\in (i,j)\mathrm{SO}(Y)}\{f^{-1}(G) - \mathrm{Int}(f^{-1}(G))\}$$

$= \bigcup_{B\in P(Y)}\{f^{-1}((i,j)\mathrm{sInt}(B)) - \mathrm{Int}(f^{-1}(B))\}$
$= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(f^{-1}(B)) - f^{-1}((i,j)\mathrm{sCl}(B))\}$
$= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - f^{-1}((i,j)\mathrm{sCl}(F(A)))\}$
$= \bigcup_{H\in\mathcal{F}}\{\mathrm{Cl}(f^{-1}(H)) - f^{-1}(H)\}$, *where*
$P(X)$ *is the family of all subsets of* X,
$P(Y)$ *is the family of all subsets of* Y,
$\mathcal{F}$ *is the family of all* (i,j)*-semi-closed sets of* Y.

Theorem 6.8. *Let* (Y,σ_1,σ_2) *be a bitopological space and* $m(\sigma_1,\sigma_2)$ *a minimal structure of* Y *determined by* σ_1 *and* σ_2. *The set of all points at which a multifunction* $F:(X,\tau)\to(Y,\sigma_1,\sigma_2)$ *is not upper/lower* $(\tau,m(\sigma_1,\sigma_2))$*-continuous is identical with the union of the frontiers of the upper/lower inverse images of* $m(\sigma_1,\sigma_2)$*-open sets of* Y *containing/meeting F(x).*

Proof. The proof follows from Theorem 4.7.

If $m(\sigma_1,\sigma_2)=(1,2)\alpha(Y)$, then we obtain the following corollary.

Theorem 6.9. *Let* (Y,σ_1,σ_2) *be a bitopological space and* $m(\sigma_1,\sigma_2)=(1,2)\alpha(Y)$. *The set of all points at which a multifunction* $F:(X,\tau)\to(Y,\sigma_1,\sigma_2)$ *is not upper/lower ultra-continuous is identical with the union of the frontiers of the upper/lower inverse images of* $(1,2)\alpha(Y)$*-open sets of* Y *containing/meeting F(x).*

For a single valued function $f:(X,\tau)\to(Y,\sigma_1,\sigma_2)$, we obtain the following theorem.

Theorem 6.10. *Let* (Y,σ_1,σ_2) *be a bitopological space and* $m(\sigma_1,\sigma_2)=(1,2)\alpha(Y)$. *The set of all points at which a function* $f:(X,\tau)\to(Y,\sigma_1,\sigma_2)$ *is not ultra-continuous is identical with the union of the frontiers of the inverse images of* $(1,2)\alpha(Y)$*-open sets of* Y *containing f(x).*

Let (Y,σ_1,σ_2) be a bitopological space and $m(\sigma_1,\sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2.

Definition 6.5. Let S be a subset of Y. A point $y\in Y$ is called an $m_\theta(\sigma_1,\sigma_2)$*-adherent point* of S if y is m_θ-adherent in $(Y,m(\sigma_1,\sigma_2))$. The set of all $m_\theta(\sigma_1,\sigma_2)$-adherent points of S is called the $m_\theta(\sigma_1,\sigma_2)$-closure of S and is denoted by $m(\sigma_1,\sigma_2)\mathrm{Cl}_\theta(S)$.

If $S=m(\sigma_1,\sigma_2)\mathrm{Cl}_\theta(S)$, then S is said to be $m_\theta(\sigma_1,\sigma_2)$-closed. The complement of an $m_\theta(\sigma_1,\sigma_2)$-closed set is said to be $m_\theta(\sigma_1,\sigma_2)$-open.

Definition 6.6. A bitopological space (Y, σ_1, σ_2) is said to be $m(\sigma_1, \sigma_2)$-*regular* if the space $(Y, m(\sigma_1, \sigma_2))$ is m-regular.

Remark 6.4. If $m(\sigma_1, \sigma_2) = (i, j)\mathrm{PO}(Y)$, then by Definitiion 6.5 we obtain the definition of (i, j)-pre-regular spaces due to [4].

By Definitions 6.5 and 6.6 and Theorems 4.8 and 4.9, we obtain the following theorems.

Theorem 6.11. *Let (Y, σ_1, σ_2) be an $m(\sigma_1, \sigma_2)$-regular bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2. Then, for a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*
(1) F is upper $(\tau, m(\sigma_1, \sigma_2))$-continuous;
(2) $F^-(m_\theta(\sigma_1, \sigma_2)\mathrm{Cl}(B))$ is closed in X for every subset B of Y;
(3) $F^-(K)$ is closed in X for every $m_\theta(\sigma_1, \sigma_2)$-closed set K of Y;
(4) $F^+(V)$ is open in X for every $m_\theta(\sigma_1, \sigma_2)$-open set V of Y.

Theorem 6.12. *Let (Y, σ_1, σ_2) be an $m(\sigma_1, \sigma_2)$-regular bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2. Then, for a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*
(1) F is lower $(\tau, m(\sigma_1, \sigma_2))$-continuous;
(2) $F^+(m_\theta(\sigma_1, \sigma_2)\mathrm{Cl}(B))$ is closed in X for every subset B of Y;
(3) $F^+(K)$ is closed in X for every $m_\theta(\sigma_1, \sigma_2)$-closed set K of Y;
(4) $F^-(V)$ is open in X for every $m_\theta(\sigma_1, \sigma_2)$-open set V of Y.

Let (Y, σ_1, σ_2) be a bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2. Then, for a multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$, we denote by $m(\sigma_1, \sigma_2)\mathrm{Cl}(F) : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ a multifunction defined by $(m(\sigma_1, \sigma_2)\mathrm{Cl}(F))(x) = m(\sigma_1, \sigma_2)\mathrm{Cl}(F(x))$ for each $x \in X$. Then, by Theorem 4.10 we obtain the following theorem.

Theorem 6.13. *Let (Y, σ_1, σ_2) be a bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2. A multifunction $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is lower $(\tau, m(\sigma_1, \sigma_2))$-continuous if and only if $m(\sigma_1, \sigma_2)\mathrm{Cl}(F) : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is lower $(\tau, m(\sigma_1, \sigma_2))$-continuous.*

Definition 6.7. Let (Y, σ_1, σ_2) be a bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2. The space (Y, σ_1, σ_2) is said to be $m(\sigma_1, \sigma_2)$-*compact* if $(Y, m(\sigma_1, \sigma_2))$ is m-compact.

By Theorem 4.11, we obtain the following theorem.

Theorem 6.14. *Let (Y, σ_1, σ_2) be a bitopological space and $m(\sigma_1, \sigma_2)$ a minimal structure of Y determined by σ_1 and σ_2. If $F : (X, \tau) \to (Y, \sigma_1, \sigma_2)$ is an upper $(\tau, m(\sigma_1, \sigma_2))$-continuous surjective multifunction such that F(x) is $m(\sigma_1, \sigma_2)$-compact for each $x \in X$ and (X, τ) is compact, then (Y, σ_1, σ_2) is $m(\sigma_1, \sigma_2)$-compact.*

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, *β-closure and β-interior*, J. Fac. Ed. Ain Shams Univ. **10** (1986), 235–245.

[3] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[4] S. Athisaya Panmani, R. Raja Rajeswari, M. Lellis Thivagar and E. Ekici, *On some characterizations of vividly and blurly (1, 2)-β-irresolute mappings*, Filomat **21** (2007), 85–98.

[5] S. G. Crossley and S. K. Hildebrand, *Semi-closure*, Texas J. Sci. **22** (1971), 99–112.

[6] G. Di Maio and T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math. **18** (1987), 226–233.

[7] C. Dorsett, *Semi-regular spaces*, Soochow J. Math. **8** (1982), 45–53.

[8] K. K. Dube, R. K. Tiwar and O. S. Panwar, *A note on pairwise K-continuous mappings*, J. Indian Acad. Math. **12** (1990), 156–162.

[9] P. C. Dutta, *Contributions of the Theory of Bitopological Spaces*, Ph. D. Thesis, Pilan (India), 1971.

[10] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **27(75)** (1983), 311–315.

[11] M. Jelić, *A decomposition of pairwise continuity*, J. Inst. Math. Comput. Sci. Math. Ser. **3** (1990), 25–29.

[12] M. Jelić, *Feebly p-continuous mappings*, Suppl. Rend. Circ. Mat. Palermo (2) **24** (1999), 387–395.

[13] A. Kar and P. Battacharyya, *Bitopological preopen, precontinuity and preopen mappings*, Indian J. Math. **34** (1992), 295–309.

[14] F. H. Khedr, S. M. Al-Areefi and T. Noiri, *Precontinuity and semi-precontinuity in bitopological spaces*, Indian J. Pure Appl. Math. **23** (1992),625–633.

[15] Y. J. Lee and J. J. Lee, *Quasi semi-open sets*, Ulsan Inst. Techn. Report **13** (1982), 171–173.

[16] M. Lellis Thivagar, *Generalizations on pairwise α-continuous functions*, Pure Appl. Math. Sci. **28** (1991), 53–63.

[17] M. Lellis Thivagar, R. Raja Rajeswari and E. Ekici, *On extension of semi-preopen sets in bitopological spaces*, Kochi J. Math. **3** (2008), 55–60.

[18] N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly **70** (1963), 36–41.

[19] T. Lipski, *Quasi continuous multivalued maps in bitopological spaces*, Slupsskie Prace Mat. Przyrodnicze, Slupsk **7** (1988), 3–31.

[20] S. N. Maheshwari and R. Prasad, *Semi-open sets and semi-continuous functions in bitopological spaces*, Math. Notae **26** (1977/78), 29–37.

[21] S. N. Maheshwari, P. C. Jain and G. I. Chae, *On quasi open sets*, Ulsan Inst. Tech. Report **11** (1980), 291–292.

[22] H. Maki, K. C. Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets*, Pure Appl. Math. Sci. **49** (1999), 17–29.

[23] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings,* Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[24] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar. **41** (1983), 213–218.

[25] A. Nasef and T. Noiri, *Feebly open sets and feebly continuity in bitopological spaces*, Ann. Univ. Timişoara, Ser. Mat. Infor. **36** (1998), 79–88.

[26] G. B. Navalagi, M. Lellis Thivagar and R. Raja Rajeswari, *On ultra function in bitopological spaces*, Indian J. Math. Comput. Sci. Inform. Techn. **1**(1) (2008), 69–74.

[27] O. Njåstad, *On some classes of nearly open sets,* Pacific J. Math. **15** (1965), 961–970.

[28] T. Noiri and V. Popa, *On upper and lower M-continuous multifunctions*, Filomat **14** (2000), 73–86.

[29] M. C. Pal and P. Bhattacharyya, *Feeble and strong forms of preirresolute functions*, Bull. Malaysian Math. Soc. (2) **19** (1996), 63–75.

[30] V. Popa, *Quasi-preopen sets and quasi almost continuity in bitopological spaces*, Stud. Cerc. Bacău (1981), 180–184.

[31] V. Popa and T. Noiri, *On M-continuous functions,* Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. (2) **18** (**23**) (2000), 31–41.

[32] V. Popa and T. Noiri, *A unified theory of S-continuity for multifunctions,* J. Math. Fac. Sci. Univ. Istanbul **59** (2000), 31–41.

[33] V. Popa and T. Noiri, *On the definitions of some generalized forms of continuity under minimal conditions*, Mem. Fac. Sci. Kochi Univ. Ser. Math. **22** (2001), 31–41.

[34] V. Popa and T. Noiri, *On weakly (τ, m)-continuous functions,* Rend Circ. Mat. Palermo (2) **51** (2002), 295–316.

[35] V. Popa and T. Noiri, *A unified theorey of weak continuity for functions,* Rend Circ. Mat. Palermo (2) **51** (2002), 439–464.

[36] O. Ravi and M. Lellis Thivagar, *On strong forms of $(1,2)^*$-quotient mappings in bitopological spaces*, Internat. J. Math. Game Theory Algebra **14** (2004), 481–492.

[37] O. Ravi and M. Lellis Thivagar, *A bitopological $(1,2)^*$-semigeneralized continuous maps*, Bull. Malays. Math. Sci. Soc. (2) **29** (2006), 1–9.

[38] M. S. Sarsak, *On quasi-continuous functions,* J. Indian Acad. Math. **27**(2006), 407–414.

[39] S. S. Thakur and P. Paik, *Quasi α-sets*, J. Indian Acad. Math. **7** (1985), 91–95.
[40] S. S. Thakur and P. Vermay, *Quasi semi-preopen sets*, Vikram Math. J. **11** (1991), 56–61.
[41] N. V. Veličko, *H-closed topological spaces,* Amer. Math. Soc. Transl. (2) **78** (1968), 103–118.

Takashi NOIRI:
2949-1 Shiokita-cho, Hinagu,
Yatsushiro-shi, Kumamoto-ken,
869-5142 JAPAN
e-mail: t.noiri@nifty.com

Valeriu POPA:
Department of Mathematics, Informatics and Education Sciences,
Faculty of Sciences,
"Vasile Alecsandri" University of Bacau ,
157 Calea Marasesti, 600115 Bacau , ROMANIA
e-mail: vpopa@ub.ro