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STRICT FIXED POINTS FOR MULTIFUNCTIONS SATISFYING A GENERALIZED IMPLICIT GREGUŠ TYPE CONDITION IN SYMMETRIC SPACES AND APPLICATIONS

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Abstract. In this paper a general fixed point theorem of generalized Greguš type in symmetric spaces for hybrid pair satisfying a new type of implicit relation using generalized altering distance and we obtain simultaneous results for contractive and extensive mappings. As application, new results for hybrid pairs satisfying contractive and expansive conditions of integral type are obtained.

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1. Introduction

Let f and g be self mappings of a metric space (X, d). Jungck [13] defined f and g to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of f and g if $fx = gx$. We denote by $C(f, g)$ the set of all coincidence points of f and g.

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Definition 1.1 ([14])**.** f and g are said to be weakly compatible if $fgu = gfu$ for all $u \in C(f,g)$.

Definition 1.2 ([3])**.** f and g are said to be occasionally weakly compatible (owc) if $fgu = gfu$ for some $u \in C(f,g)$.

Remark 1.3. *If f and g are weakly compatible and $C(f,g) \neq \emptyset$, then f and g are owc, but the converse is not true (see Example* [3]*).*

Some fixed point theorems for owc mappings are proved in [1], [2], [15] and in other papers.

It has been observed that in [11] that some of defining properties of metric space are not used in the proof of certain metric theorems. Hicks and Rhoades [11] established some common fixed point theorems in symmetric space and proved that every general probabilistic structures admit a compatible symmetric.

Definition 1.4. Let X be a nonempty set. A symmetric on X is a nonnegative real valued function d on $X \times X$ such that

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$.

A symmetric space is a nonempty set X with a symmetric d and is denoted by (X,d).

The study of fixed points for mappings satisfying an implicit relation is initiated in [20], [21] and other papers. Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, probabilistic metric spaces, in two or three metric spaces for single valued functions, hybrid pairs of mappings and set valued functions. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, intuitionistic metric spaces and in G - metric spaces.

With this method the proofs of some fixed point theorems are more simple. Also, the method allow the study of local and global properties of fixed point structures.

The study of fixed point for hybrid pairs of mappings satisfying implicit relation is initiated in [22] and [23].

2. Preliminaries

Let (X, d) be a symmetric space. We denote by $B(X)$ the family of all nonempty bounded subsets of X.

As in [22] we define the functions $D(A, B)$ and $\delta(A, B)$ as follows

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\},\ A, B \in B(X),$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},\ ,\ A, B \in B(X).$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B consists also of a single point b, we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of δ that

$$\delta(A, B) = \delta(B, A),$$
$$\delta(A, B) = 0 \text{ if and only if } A = B = \{a\}.$$

Definition 2.1. 1) A point $x \in X$ is said to be a coincidence point of $f : X \to X$ and $F : X \to B(X)$ if $fx \in Fx$. We denote $C(f, F)$ the set of all coincidence points of f and F.

2) A point $x \in X$ is said to be a strict coincidence point of f and F if $\{fx\} = Fx$. The set of all strict coincidence points of f and F is denoted by $SC(f, F)$ and $\{z\} = \{fx\} = Fx$ is said a point of strict coincidence of f and F.

3) A point $x \in X$ is a fixed point of $F : X \to B(X)$ if $x \in Fx$.

4) A point $x \in X$ is a strict fixed point of F if $\{x\} = Fx$.

Definition 2.2. The hybrid pair (f, F), where $f : X \to X$ and $F : X \to B(X)$, is said to be occasionally weakly compatible if there exists $x \in SC(f, F)$ such that $fFx = Ffx$.

Some strict fixed point theorem for hybrid pairs of mappings are recently proved in [25].

Greguš [10] proved the following:

Theorem 2.3 ([10])**.** *Let C be a nonempty closed convex subset of a Banach space X and let T be a mapping of C into itself satisfying the inequality*

$$\|Tx - Ty\| \le a\,\|x - y\| + b\,\|x - Tx\| + c\,\|y - Ty\| \tag{2.1}$$

for all $x, y \in X$, where $a > 0$, $b, c \ge 0$ and $a + b + c = 1$. Then T has an unique fixed point.

Some authors have generalized Theorem 2.3 in [5], [6], [7], [8], [19], [24] and in other papers.

Quite recently, the following theorem is proved in [18].

Theorem 2.4 ([18]). *Let f and S be occasionally weakly compatible self - mappings of a metric space (X,d) satisfying*

$$d(fx,f^2x)\neq\max\{d(Sx,Sf(x)),d(fx,Sx),\\ d(f^2x,Sfx),d(fx,Sfx),d(Sx,f^2x)\} \tag{2.2}$$

where $fx\neq f^2x$. Then f and S have a common fixed point.

An altering distance is a mapping $\psi:[0,\infty)\to[0,\infty)$ such that
(ψ_1) : ψ is increasing and continuous,
(ψ_2) : $\psi(t)=0$ if and only if $t=0$.

Fixed point theorems involving altering distances have been proved in [16], [27], [30], [31] and in other papers.

Some fixed point theorems using a weakly form of altering distances are recently proved in [25] and [28].

We introduce a generalized form of altering distance.

Definition 2.5. A generalized altering distance is a mapping ψ : $[0,\infty)\to[0,\infty)$ such that $\psi(t)=0$ if and only if $t=0$.

3. Generalized implicit Greguš type functions

In the following we denote by $\mathfrak{F}_G$ the family of all functions F : $\mathbb{R}_+^6\to\mathbb{R}$ satisfying

$$F(t,t,0,0,t,t)=0,\forall t>0$$

and named this family, generalized implicit Greguš type functions.

Example 3.1. $\phi(t_1,...,t_6)=t_1-\max\{t_2,t_3,t_4,t_5,t_6\}$.

Example 3.2. $\phi(t_1,...,t_6)=t_1^p-at_2^p-(1-a)\max\{t_3^p,t_4^p,(t_3t_5)^{\frac{p}{2}},(t_5t_6)^{\frac{p}{2}}\}$, *where* $0<a<1$ *and* $p\geq 1$.

Example 3.3. $\phi(t_1,...,t_6)=t_1-at_2-b\max\{t_3,t_4\}-c\max\{t_2,t_5,t_6\}$, *where* a, b, $c\geq 0$ *and* $a+c=1$.

Example 3.4. $\phi(t_1,...,t_6)=t_1-\alpha\max\{t_2,t_3,t_4\}-(1-\alpha)(at_5+bt_6)$, *where* $0<\alpha<1$, $a,b\geq 0$ *and* $a+b=1$.

Example 3.5. $\phi(t_1,...,t_6)=t_1-at_2-b\dfrac{t_5+t_6}{1+t_3+t_4}$, *where* a, $b\geq 0$ *and* $a+2b=1$.

Example 3.6. $\phi(t_1,...,t_6)=t_1-\max\left\{t_2,\frac{1}{2}(t_3+t_4),\frac{1}{2}(t_5+t_6)\right\}$.

Example 3.7. $\phi(t_1,...,t_6)=t_1-at_2-b(t_3+t_4)-c\min\{t_5,t_6\}$, *where* a, b, $c\geq 0$ *and* $a+c=1$.

Example 3.8. $\phi(t_1, ..., t_6) = t_1(1 + \alpha t_2) - \alpha(t_3t_4 + t_5t_6) - at_2 - (1 - a)\max\{t_3, t_4,$
$(t_5t_6)^{\frac{1}{2}}, (t_3t_6)^{\frac{1}{2}}\}$, *where* $\alpha > 0$ *and* $0 < a \leq 1$.

Example 3.9. $\phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $0 < c \leq 1$, $b \geq 0$ *and* $a + b = 1$.

Example 3.10. $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, *where* $a, b, c \geq 0$ *and* $a + 2c = 1$.

Example 3.11. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4, \frac{2t_4+t_5}{2}, \frac{2t_4+t_6}{2}, \frac{t_5+t_6}{2}\}$, *where* a, $b \geq 0$ *and* $a + b = 1$.

Example 3.12. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\max\{2t_4 + t_5, 2t_4 + t_6, t_3 + t_5 + t_6\}$, *where* a, $b \geq 0$ *and* $a + 2b = 1$.

The following theorem is proved in [24].

Theorem 3.13. *Let* (X, d) *be a symmetric space and* $\phi \in \mathfrak{F}_G$. *Suppose that* f, g, S *and* T *are self mappings of* X *such that each of the pairs* $\{f, S\}$ *and* $\{g, T\}$ *is owc and the inequality*

$$\phi(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) < 0$$

holds for all $x, y \in X$ *and least one of* $d(Sx, Ty)$, $d(fx, Sx)$, $d(gy, Ty)$, $d(fx, Ty)$, $d(gy, Sx)$ *is positive. Then* f, g, S *and* T *have an unique common fixed point.*

In this paper a general fixed point theorem of generalized Greguš type is proved similarly with Theorem 2.4 for hybrid pairs of mappings satisfying a new type of implicit relation using generalized altering distance and we obtain simultaneous results for hybrid pairs satisfying contractive / extensive conditions. As applications, we determine new results for hybrid pair satisfying contractive / extensive conditions of integral type.

4. Main results

Lemma 4.1 ([[25]]). *Let* (X, d) *be a symmetric space,* $f : X \to X$ *and* $F : X \to B(X)$ *strict occasionally weakly compatible self mappings. If* f *and* F *have an unique point of strict coincidence* $\{z\} = \{fx\} = Fx$, *then* z *is the unique common fixed point of* f *and* F *which is the unique strict fixed point for* F.

Theorem 4.2. *Let (X, d) be a symmetric space, $f, g : X \to X$ and $F, G : X \to B(X)$ satisfying one of the relations*

$$\phi(\psi(\delta(Fx, Gy)), \psi(d(fx, gy)), \psi(\delta(fx, Fx)), \psi(\delta(gy, Gy)), \psi(D(fx, Gy)), \psi(D(gy, Fx))) \neq 0 \tag{4.1}$$

$$\phi(\psi(\delta(Fx, Gy)), \psi(d(fx, gy)), \psi(D(fx, Fx)), \psi(D(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))) \neq 0 \tag{4.2}$$

$$\phi(\psi(\delta(Fx, Gy)), \psi(d(fx, gy)), \psi(D(fx, Fx)), \psi(D(gy, Gy)), \psi(D(fx, Gy)), \psi(D(gy, Fx))) \neq 0 \tag{4.3}$$

$$\phi(\psi(\delta(Fx, Gy)), \psi(d(fx, gy)), \psi(\delta(fx, Fx)), \psi(\delta(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))) \neq 0 \tag{4.4}$$

for all $x, y \in X$ with $fx \neq gy$, where $\phi \in \mathfrak{F}_G$ and ψ is a generalized altering distance. Suppose that there exists $x, y \in X$ such that $\{u\} = \{fx\} = Fx$ and $\{v\} = \{gy\} = Gy$. Then, u is the unique point of strict coincidence of f and F and v is the unique point of strict coincidence of g and G. Moreover, $u = v$.

Proof. We prove this theorem only in case (4.1). For cases (4.2, 4.3, 4.4) the proof is similar.

First we prove that $fx = gy$. Suppose $fx \neq gy$. Then by (4.1) we obtain

$$\phi(\psi(d(fx, gy)), \psi(d(fx, gy)), 0, 0, \psi(d(fx, gy)), \psi(d(fx, gy))) \neq 0,$$

which is a contradiction of $\phi \in \mathfrak{F}_G$. Hence, $fx = gy$ and $\{u\} = \{fx\} = \{gy\} = Fx = Gy = v$. Suppose that there exist $z \in X$ such that $\{w\} = \{fz\} = Fz$ and $w \neq v = u$. Then by (4.1) we obtain

$$\phi(\psi(d(fz, gy)), \psi(d(fz, gy)), 0, 0, \psi(d(fz, gy)), \psi(d(fz, gy))) \neq 0,$$

a contradiction. Hence, $u = v$ is the unique point of strict coincidence of f and F. Similarly, $u = v$ is the unique strict point of coincidence of g and G. □

Theorem 4.3. *Let (X, d) be a symmetric space, $f, g : X \to X$ and $F, G : X \to B(X)$ satisfying one of the relations (4.1) - (4.4) for all $x, y \in X$ with $fx \neq gy$, where $\phi \in \mathfrak{F}_G$ and ψ is a generalized altering distance. If the pairs $\{f, F\}$ and $\{g, G\}$ are sowc, then, f, g, F and G have an unique common fixed point, which is a strict fixed point for F and G.*

Proof. Since $\{f, F\}$ and $\{g, G\}$ are sowc, there exists $x, y \in X$ such that $\{u\} = \{fx\} = Fx$ and $\{v\} = \{gy\} = Gy$. By Theorem 4.2, u is the unique point of strict coincidence of f and F and v is the unique point of strict coincidence of g and G and $u = v$. By Lemma 4.1, u is the unique common fixed point of f and F which is a strict fixed point for F. Similarly, $v = u$ is the unique common fixed point of g and G which is a strict point for g and G. Hence, u is the unique common fixed point of f, g, F and G, which is a strict fixed point for F and G. □

Remark 4.4. *1. If in relations (4.1) - (4.4) we have "$<$" or "$>$" instead of "$\neq$" we obtain new results for contractive or expansive hybrid pairs.*

2. By Examples 3.1 - 3.12 we obtain new particular results. For example, by Example 3.1 we obtain

Corollary 4.5. *Let (X, d) be a symmetric space, $f, g : X \to X$ and $F, G : X \to B(X)$ such that the pairs $\{f, F\}$ and $\{g, G\}$ are sowc satisfying one of the relations*

a)

$$\begin{aligned}\psi(\delta(Fx, Gy)) \neq\ & \max\{\psi(d(fx, gy)), \psi(\delta(fx, Fx)),\\ & \psi(\delta(gy, Gy)), \psi(D(fx, Gy)), \psi(D(gy, Fx))\},\end{aligned}$$

b)

$$\begin{aligned}\psi(\delta(Fx, Gy)) \neq\ & \max\{\psi(d(fx, gy)), \psi(D(fx, Fx)),\\ & \psi(D(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))\},\end{aligned}$$

c)

$$\begin{aligned}\psi(\delta(Fx, Gy)) \neq\ & \max\{\psi(d(fx, gy)), \psi(D(fx, Fx)),\\ & \psi(D(gy, Gy)), \psi(D(fx, Gy)), \psi(D(gy, Fx))\},\end{aligned}$$

d)

$$\begin{aligned}\psi(\delta(Fx, Gy)) \neq\ & \max\{\psi(d(fx, gy)), \psi(\delta(fx, Fx)),\\ & \psi(\delta(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))\}\end{aligned}$$

for all $x, y \in X$ with $fx \neq gy$, where ψ is a generalized altering distance. Then f, g, F and G have an unique common fixed point, which is a strict fixed point for F and G.

Remark 4.6. *1. If in relations (a) - (d) we have "$<$" or "$>$" instead of "$\neq$" we obtain new results for contractive or expansive hybrid pairs.*

2. By case "<" instead of "≠" and (a) we obtain a generalization of Corollary 1 [28].

If $\psi(t) = t$ by Theorem 4.3 we obtain

Theorem 4.7. *Let (X, d) be a symmetric space, $f, g : X \to X$ and $F, G : X \to B(X)$ satisfying one of the relations*

$$\begin{aligned} &\phi(\delta(Fx, Gy), d(fx, gy), \delta(fx, Fx), \\ &\delta(gy, Gy), D(fx, Gy), D(gy, Fx)) \neq 0 \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\phi(\delta(Fx, Gy), d(fx, gy), D(fx, Fx), \\ &D(gy, Gy), \delta(fx, Gy), \delta(gy, Fx)) \neq 0 \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\phi(\delta(Fx, Gy), d(fx, gy), D(fx, Fx), \\ &D(gy, Gy), D(fx, Gy), D(gy, Fx)) \neq 0 \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\phi(\delta(Fx, Gy), d(fx, gy), \delta(fx, Fx), \\ &\delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx)) \neq 0 \end{aligned} \tag{4.8}$$

for all $x, y \in X$ with $fx \neq gy$, where $\phi \in \mathfrak{F}_G$. If the pairs $\{f, F\}$ and $\{g, G\}$ are sowc, then, f, g, F and G have an unique common fixed point, which is a strict fixed point for F and G.

Remark 4.8. *1. If in relations (4.5) - (4.8) we have "<" or ">" instead of "≠" we obtain new results for contractive or expansive hybrid pairs.*

2. By Examples 3.1 - 3.12 we obtain new particular results.

If f, g, F and G are single valued functions, by Theorem 4.3 we obtain

Theorem 4.9. *Let f, g, F, G be self mappings of a symmetric space (X, d) such that $\{f, F\}$ and $\{g, G\}$ are owc. If*

$$\begin{aligned} &\phi(\psi(d(Fx, Gy)), \psi(d(fx, gy)), \psi(d(fx, Fx)), \\ &\psi(d(gy, Gy)), \psi(d(fx, Gy)), \psi(d(gy, Fx))) \neq 0 \end{aligned} \tag{4.9}$$

for all $x, y \in X$ with $fx \neq gy$, where $\psi(t)$ is a generalized altering distance and $\phi \in \mathfrak{F}_G$, then, f, g, F and G have an unique common fixed point.

If $\psi(t) = t$ by Theorem 4.9 we obtain

Theorem 4.10. *Let f, g, F, G be self mappings of a symmetric space (X, d) such that $\{f, F\}$ and $\{g, G\}$ are owc. If*

$$\phi(d(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) \neq 0 \tag{4.10}$$

for all $x, y \in X$ with $fx \neq gy$ and $\phi \in \mathfrak{F}_G$, then, f, g, F and G have an unique common fixed point.

Corollary 4.11. *Let (X, d) be a symmetric space. Suppose that each of the pairs $\{f, F\}$ and $\{g, G\}$ is owc. If*

$$d(Fx, Gy) \neq \max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\} \tag{4.11}$$

for all $x, y \in X$ with $fx \neq gy$, then f, g, F and G have an unique common fixed point.

Proof. The proof it follows by Theorem 4.10 and Example 3.1. □

Remark 4.12. *1. If in (4.11) we have "$<$" instead of "$\neq$" we obtain Theorem 1* [15].

2. If in (4.11) we have "$>$" instead of "$\neq$" we obtain a new result for extensive mappings which is different by the results from recent paper [12].

Corollary 4.13. *Let (X, d) be a symmetric space, $0 < a < 1$ and $p > 1$. Let f, g, F and G be self mappings of X such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. Suppose that*

$$d^p(Fx, Gy) \neq ad^p(fx, gy) + (1 - a)M(x, y), \tag{4.12}$$

for all $x, y \in X$ with $fx \neq gy$, where

$$M(x, y) = \max\{d^p(fx, Fx), d^p(gy, Gy), d^{\frac{p}{2}}(fx, Fx) \cdot d^{\frac{p}{2}}(fx, Gy), d^{\frac{p}{2}}(fx, Gy) \cdot d^{\frac{p}{2}}(gy, Fx)\}.$$

Then f, g, F and G have an unique common fixed point.

Proof. The proof it follows by Theorem 4.10 and Example 3.2. □

Remark 4.14. *1. If in (4.12) we have "$<$" instead of "$\neq$" we obtain Theorem 2* [15].

2. If in (4.12) we have "$>$" instead of "$\neq$" we obtain a new result which is different by the results from [12].

5. Applications

In [4], Branciari established the following result.

Theorem 5.1. *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt, \tag{5.1}$$

where $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$ such that for $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then f has an unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Remark 5.2. *Theorem 5.1 has been generalized in several papers.*

Let (X,d) be a symmetric space and $\psi(t) = \int_0^t h(x)dx$, where $h : [0,\infty) \to [0,\infty)$ as in Theorem 5.1.

Lemma 5.3. *ψ is a generalized altering distance.*

Proof. The proof it follows by the definition of ψ. □

Remark 5.4. *The contractive condition of integral type in a metric space (X,d), used in fixed point theory can be written as usual contractive condition in a symmetric space* [17], [26].

The following theorem is proved in [24].

Theorem 5.5. *Let (X,d) be a metric space and $h : [0,\infty) \to [0,\infty)$ be a function as in Theorem 5.1. Suppose that A, B, S and T are self mappings of X such that the pairs $\{A,S\}$ and $\{B,T\}$ are sowc. If $F \in \mathfrak{F}_G$ and*

$$\begin{aligned} &F(\textstyle\int_0^{d(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt, \\ &\textstyle\int_0^{d(Ty,By)} h(t)dt, \int_0^{d(Ax,Ty)} h(t)dt, \int_0^{d(By,Sx)} h(t)dt) < 0 \end{aligned} \tag{5.2}$$

for all $x, y \in X$ and least one of the distances $d(Sx,Ty)$, $d(Sx,Ax)$, $d(Ty,By)$, $d(Ax,Ty)$, $d(By,Sx)$ is positive. Then A, B, S and T have an unique common fixed point.

Theorem 5.6. *Let f, g be self mappings of a symmetric space (X,d) and F, G be maps of X into $B(X)$ such that the pairs $\{f,F\}$ and $\{g,G\}$ are sowc satisfying one of the relations*

$$\begin{aligned} &\phi(\textstyle\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{\delta(fx,Fx)} h(t)dt, \\ &\textstyle\int_0^{\delta(gy,Gy)} h(t)dt, \int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(gy,Fx)} h(t)dt) \neq 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned}\phi(&\textstyle\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt,\\ &\textstyle\int_0^{D(gy,Gy)} h(t)dt, \int_0^{\delta(fx,Gy)} h(t)dt, \int_0^{\delta(gy,Fx)} h(t)dt) \neq 0,\end{aligned} \tag{5.4}$$

$$\begin{aligned}\phi(&\textstyle\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt,\\ &\textstyle\int_0^{D(gy,Gy)} h(t)dt, \int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(gy,Fx)} h(t)dt) \neq 0,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\phi(&\textstyle\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{\delta(fx,Fx)} h(t)dt,\\ &\textstyle\int_0^{\delta(gy,Gy)} h(t)dt, \int_0^{\delta(fx,Gy)} h(t)dt, \int_0^{\delta(gy,Fx)} h(t)dt) \neq 0,\end{aligned} \tag{5.6}$$

for all $x, y \in X$, with $fx \neq gy$, $\phi \in \mathfrak{F}_G$ and $h(t)$ is as in Theorem 5.1.

Then f, g, F and G have an unique common fixed point, which is a strict fixed point for F and G.

Proof. We prove this theorem in case (5.3), because the cases (5.4), (5.5), (5.6) are similar proved.

As in Lemma 5.3 we have

$$\begin{aligned}
\psi(\delta(Fx,Gy)) &= \int_0^{\delta(Fx,Gy)} h(t)dt, \psi(d(fx,gy)) = \int_0^{d(fx,gy)} h(t)dt,\\
\psi(\delta(fx,Fx)) &= \int_0^{\delta(fx,Fx)} h(t)dt, \psi(\delta(gy,Gy)) = \int_0^{\delta(gy,Gy)} h(t)dt,\\
\psi(D(fx,Gy)) &= \int_0^{D(fx,Gy)} h(t)dt, \psi(D(gy,Fx)) = \int_0^{D(gy,Fx)} h(t)dt.
\end{aligned}$$

Then by (5.3) we have

$$\begin{aligned}&\phi(\psi(\delta(Fx,Gy)), \psi(d(fx,gy)), \psi(\delta(fx,Fx)),\\ &\psi(\delta(gy,Gy)), \psi(D(fx,Gy)), \psi(D(gy,Fx))) \neq 0,\end{aligned}$$

which is relation (4.1). □

Because in Lemma 5.3 $\psi(t) = \int_0^t h(x)dx$ is a generalized altering distance, the conditions of Theorems 4.7 are satisfied, from Theorem 5.6 and Example 3.1 we obtain

Corollary 5.7. *Let f, g be self mappings of a symmetric space (X, d) and F, G be maps of X into $B(X)$ such that the pairs $\{f, F\}$ and $\{g, G\}$ are sowc satisfying one of the relations:*

a)

$$\begin{aligned}\int_0^{\delta(Fx,Gy)} h(t)dt &\neq \max\{\int_0^{d(fx,gy)} h(t)dt, \int_0^{\delta(fx,Fx)} h(t)dt,\\ &\int_0^{\delta(gy,Gy)} h(t)dt, \int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(gy,Fx)} h(t)dt\},\end{aligned}$$

b)

$$\begin{aligned}\int_0^{\delta(Fx,Gy)} h(t)dt &\neq \max\{\int_0^{d(fx,gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt,\\ &\int_0^{D(gy,Gy)} h(t)dt, \int_0^{\delta(fx,Gy)} h(t)dt, \int_0^{\delta(gy,Fx)} h(t)dt\},\end{aligned}$$

c)

$$\begin{aligned}\int_0^{\delta(Fx,Gy)} h(t)dt &\neq \max\{\int_0^{d(fx,gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt,\\ &\int_0^{D(gy,Gy)} h(t)dt, \int_0^{D(fx,Gy)} h(t)dt, \int_0^{D(gy,Fx)} h(t)dt\},\end{aligned}$$

d)

$$\begin{aligned}\int_0^{\delta(Fx,Gy)} h(t)dt &\neq \max\{\int_0^{d(fx,gy)} h(t)dt, \int_0^{\delta(fx,Fx)} h(t)dt,\\ &\int_0^{\delta(gy,Gy)} h(t)dt, \int_0^{\delta(fx,Gy)} h(t)dt, \int_0^{\delta(gy,Fx)} h(t)dt\}\end{aligned}$$

for all $x, y \in X$ with $fx \neq gy$, where $h(t)$ is as in Theorem 5.1. Then f, g, F and G have an unique fixed point, which is a strict fixed point for F and G.

Remark 5.8. *1) By Examples 3.1 - 3.12 we obtain new results.*
2) By Theorem 5.5 and Corollary 5.7 and "$<$" or "$>$" instead of "$\neq$" we obtain new results.

In f, g, F and G are single valued mappings, by Theorem 5.6 we obtain

Theorem 5.9. *Let f, g, F and G be self mappings of a symmetric space (X, d) such that $\{f, F\}$ and $\{g, G\}$ are owc satisfying the inequality:*

$$\begin{aligned}&\phi(\textstyle\int_0^{d(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{d(fx,Fx)} h(t)dt,\\ &\textstyle\int_0^{d(gy,Gy)} h(t)dt, \int_0^{d(fx,Gy)} h(t)dt, \int_0^{d(gy,Fx)} h(t)dt) \neq 0\end{aligned} \tag{5.7}$$

for all $x, y \in X$ with $fx \neq gy$, where $\phi \in \mathfrak{F}_G$ and $h(t)$ is as in Theorem 5.1. Then f, g, F and G have an unique common fixed point.

Remark 5.10. *1) If in (5.7) we have "$<$" instead of "$\neq$" we obtain a generalization of Theorem 4.1* [24].

2) By Theorem 5.9, Examples 3.1 - 3.12 and "$<$" or "$>$" instead of "$\neq$" we obtain new particular results.

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