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A CHARACTERIZATION OF ORLICZ-POINCARÉ INEQUALITY ON METRIC MEASURE SPACES

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Abstract. In this note we characterize a weak Orlicz-Poincaré inequality through the Hölder continuity of locally integrable functions possessing upper gradients in the corresponding Orlicz space, under some growth assumptions on the Young function. Our results are proved in the setting of doubling metric measure spaces.

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1. Introduction

Poincaré inequality is an essential tool in the extensions to metric measure spaces of the theory of Sobolev spaces [11], [22], [9], of quasiconformal theory [16], [15], of nonlinear potential theory [3], in the study of differentiability of Lipschitz functions on metric measure spaces [4].

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Orlicz-Poincaré inequalities have been introduced by Tuominen [23] in the setting of metric measure space. Orlicz-Poincaré inequalities are closely related to $(1, p)-$Poincaré inequalities with $1 \leq p < \infty$. An Orlicz-Poincaré defined via a Young function Φ, called a $(1, \Phi)-$Poincaré inequality, is a $(1, p)-$Poincaré inequality if $\Phi(t) = t^p$. Moreover, every pair of functions (u, g) satisfying a $(1, \Phi)-$Orlicz-Poincaré with Φ doubling also satisfies a $(1, p)-$Poincaré inequality, provided that $\log_2 C_\Phi \leq p < \infty$, where C_Φ denotes the doubling constant of Φ. A generalization of Orlicz-Poincaré inequalities in which the Orlicz space is replaced by a Banach function space has been introduced in [20].

Orlicz-Poincaré inequalities have self-improving properties as proved by Heikkinen [12] , that generalized results proved by Cianchi in the Euclidean setting and by Hajłasz and Koskela [11] for $(1, p)-$Poincaré inequalities on metric spaces. Heikkinen also gave a characterization of Orlicz-Sobolev spaces by means of generalized Orlicz-Poincaré inequalities [13]. J. Björn characterized Orlicz–Poincaré inequalities on doubling metric measure spaces by means of pointwise inequalities involving maximal functions of the gradient [2]. Orlicz-Poincaré inequalities on metric measure spaces have applications to nonlinear potential theory [19]. Geometric consequences of $(1, p)-$Poincaré inequalities, even in the case $p = \infty$ have been investigated in [6], [7], [8].

Very recently, E. Durand-Cartagena, J. Jaramillo and N. Shanmugalingam [8] established the following result, providing two characterizations of $p-$Poincaré inequality in a complete Ahlfors $Q-$regular space with $p > Q$, one in terms of the locally $\left(1 - \frac{Q}{p}\right)-$Hölder continuity of the functions in $N^{1,p}(X)$ and the other in terms of the $p-$modulus of quasiconvex curves connecting pairs of points in the space.

Theorem 1. *Let X be a complete Ahlfors $Q-$regular space and $p > Q$. Then the following conditions are equivalent:*

(1) X supports a weak $p-$Poincaré inequality.

(2) There are constants $C > 0$, $\tau \geq 1$ such that every $u \in N^{1,p}(X)$ is $\left(1 - \frac{Q}{p}\right)-$Hölder continuous and for all $x, y \in X$ we have

$$|u(x) - u(y)| \leq C \left\| g_u \right\|_{L^p(B(x, \tau d(x,y)))} d(x, y)^{1 - \frac{Q}{p}},$$

where g_u is the minimal $p-$ weak upper gradient of u.

(3) There is a constant $C \geq 1$ such that, for every pair of distinct points $x, y \in X$,

$$Mod_p \left(\Gamma\left(\{x\}, \{y\}, C\right)\right) \geq \frac{1}{Cd\left(x, y\right)^{p-Q}},$$

where $\Gamma\left(\{x\}, \{y\}, C\right)$ denotes the family of $C-$quasiconvex curves connecting x to y.

In this note we give a characterization of a weak Orlicz-Poincaré inequality in a doubling metric measure space through the Hölder continuity of locally integrable functions possesing upper gradients in the corresponding Orlicz space, under some growth assumptions on the Young function.

2. Preliminaries

Let (X, d, μ) be a metric measure space. We will denote by $B\left(x, r\right)$ the open ball centered at $x \in X$ of radius $r > 0$. If $B = B\left(x, r\right)$ is a ball given with center and radius and $\sigma > 0$, we denote by σB the ball $B\left(x, \sigma r\right)$ with the same center and the radius multiplied by σ. Note that the pair center-radius of a ball in a metric space may be not unique.

The measure μ is called doubling if there exists a constant $C_\mu \geq 1$ so that $\mu\left(B\left(x, 2r\right)\right) \leq C_\mu \mu\left(B\left(x, r\right)\right)$ for all $x \in X$ and all $r > 0$. Doubling measures play an important role in harmonic analysis, through the notion of space of homogeneous type [5]. The following lower bound estimate holds for a doubling measure

$$\frac{\mu\left(B\left(x, r\right)\right)}{\mu\left(B\left(x_0, r_0\right)\right)} \geq C_b \left(\frac{r}{r_0}\right)^Q,$$

where $Q = \log_2 C_\mu$, for every ball $B\left(x_0, r_0\right)$, whenever $x \in B\left(x_0, r_0\right)$ and $0 < r \leq r_0$. If μ satisfies the above lower bound estimate for some $Q > 0$, then the constant Q is called a homogeneous dimension of the metric measure space (X, d, μ).

A useful special case of doubling metric measures spaces is that of Ahlfors regular spaces. Recall that the metric measure space (X, d, μ) is said to be $Q-$Ahlfors regular, where $Q > 0$, if there is a constant $C_A \geq 1$ so that

$$\frac{1}{C_A} r^Q \leq \mu(B(x, r)) \leq C_A r^Q \tag{1}$$

for every ball $B(x,r)$ in X with $0 < r < diam\,(X)$. Note that for an $Q-$Ahlfors regular space a homogeneous dimension is Q.

Given a locally integrable function u on X and $E \subset X$ a measurable set of positive finite measure, we denote the integral mean of u on E by u_E, i. e. $u_E = \frac{1}{\mu(E)} \int\limits_E u\, d\mu$.

In first order calculus on metric spaces, the notion of upper gradient, introduced by Heinonen and Koskela in [16], is a substitute for the length of the gradient of a smooth function. A Borel measurable function $g : X \to [0, \infty]$ is an *upper gradient* of a real-valued function u on X if

$$|u(\gamma\,(a)) - u(\gamma\,(b))| \leq \int\limits_\gamma g\, ds \tag{2}$$

for every non-constant rectifiable curve $\gamma : [a, b] \to X$. We will recall later the more general notion of weak upper gradient on a metric measure space, with respect to a Young function.

In applications of Orlicz spaces, some growth conditions for the corresponding Young functions are very useful. A Young function $\Phi : [0, \infty) \to [0, \infty]$ is said to be doubling or to satisfy a Δ_2-condition if there is a constant $C \geq 1$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t \geq 0$; in this case we will denote $C_\Phi = \sup\limits_{t>0} \frac{\Phi(2t)}{\Phi(t)}$ and will call C_Φ the doubling constant of Φ. A doubling Young function is real-valued, strictly increasing and continuous. The inverse of a strictly increasing Young function $\Phi : [0, \infty) \to [0, \infty)$ is an increasing concave function which is subadditive, hence is doubling with a doubling constant $1 \leq C_{\Phi^{-1}} \leq 2$ [23, Lemma 2.9]. By [23, Lemma 2.7], iterating the Δ_2-condition $\Phi(2t) \leq C_\Phi \Phi(t)$ for a doubling Young function Φ we get

$$\Phi\,(\lambda t) \leq C_\Phi \lambda^{\log_2 C_\Phi} \Phi\,(t) \text{ for all } \lambda \geq 1,\, t \geq 0. \tag{3}$$

We collect some estimates on the growth of the inverse of a doubling Young function, see [23, Lemma 2.7] and [23, (5.7)].

Lemma 2. *Let $\Phi : [0, \infty) \to [0, \infty)$ be a doubling Young function, with a doubling constant C_Φ. Then the inverse $\Phi^{-1} : [0, \infty) \to [0, \infty)$ satisfies for all $t \geq 0$ the inequalities*

$$\Phi^{-1}\,(\lambda t) \leq 2\lambda^{\log_2 C_{\Phi^{-1}}} \Phi^{-1}\,(t) \;\; \textit{if } \lambda \geq 1 \tag{4}$$

and

$$\Phi^{-1}\,(\lambda t) \leq 2\lambda^{\frac{1}{\log_2 C_\Phi}} \Phi^{-1}\,(t) \;\; \textit{if } 0 \leq \lambda \leq 1. \tag{5}$$

Proof. The first inequality follows from condition (3) applied to Φ^{-1}, since $C_{\Phi^{-1}} \leq 2$.

Let $t \geq 0$ and $0 \leq \lambda \leq 1$. If $t = 0$ or $\lambda = 0$ inequality (5) obviously holds. Let $t > 0$ and $0 < \lambda \leq 1$. Inequality (3) shows that for all $0 < a \leq b$ we have $\Phi(b) \leq C_\Phi \left(\frac{b}{a}\right)^{\log_2 C_\Phi} \Phi(a)$, hence

$$\frac{a}{b} \leq C_\Phi^{\frac{1}{\log_2 C_\Phi}} \left(\frac{\Phi(a)}{\Phi(b)}\right)^{\frac{1}{\log_2 C_\Phi}}.$$

For $a = \Phi^{-1}(\lambda t)$ and $b = \Phi^{-1}(t)$ the latter inequality implies (5), since $C_\Phi^{\frac{1}{\log_2 C_\Phi}} = 2$. ∎

Corollary 3. *If* $\log_2 C_\Phi \cdot \log_2 C_{\Phi/^{-1}} \leq 1$, *then*

$$\Phi^{-1}(\lambda t) \leq C \min\left(\lambda^{\frac{1}{\log_2 C_\Phi}}, \lambda^{\log_2 C_{\Phi^{-1}}}\right) \Phi^{-1}(t) \tag{6}$$

for all $\lambda \geq 0$ *and* $t \geq 0$.

Remark 1. *In the classical case where* $\Phi(t) = t^p$ *with* $p \geq 1$ *one has* $\log_2 C_\Phi = p$ *and* $\log_2 C_{\Phi^{-1}} = \frac{1}{p}$, *hence* $\log_2 C_\Phi \cdot \log_2 C_{\Phi/^{-1}} = 1$.

Let $t > 0$. We have

$$2t = \Phi\left(\Phi^{-1}(2t)\right) \leq \Phi\left(C_{\Phi^{-1}}\Phi^{-1}(t)\right) \leq C_\Phi \left(C_{\Phi^{-1}}\right)^{\log_2 C_\Phi} t,$$

We applied (3) to get the second inequality. Then $2 \leq C_\Phi \left(C_{\Phi^{-1}}\right)^{\log_2 C_\Phi}$, which implies $\log_2 C_\Phi \left(1 + \log_2 C_{\Phi^{-1}}\right) \geq 1$.

It would be interesting to find more examples of Young functions satisfying $\log_2 C_\Phi \cdot \log_2 C_{\Phi/^{-1}} \leq 1$, besides the classical case $\Phi(t) = Ct^p$ with $1 \leq p < \infty$.

Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a complete, $\sigma-$finite measure and let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all real-valued measurable functions u in X such that $\int\limits_X \Phi(\lambda |u|) d\mu < \infty$ for some $\lambda > 0$. We identify any two functions that agree $\mu-$a.e. $L^\Phi(X)$ is a vector space and $\|u\|_{L^\Phi(X)} = \inf \left\{ k > 0 : \int\limits_X \Phi(\frac{|u|}{k}) d\mu \leq 1 \right\}$ defines a norm on $L^\Phi(X)$, called the Luxemburg norm.

Let Γ be a family of curves in the metric measure space (X, d, μ). An admissible metric for Γ is a nonnegative Borel function ρ on X such that $\int\limits_{\gamma} \rho ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. The Φ-modulus of the family Γ is

$$M_{\Phi}(\Gamma) = \inf \|\rho\|_{L^{\Phi}(X)},$$

where the infimum is taken over all admissible metrics for Γ. Note that for $\Phi(t) = t^p$ with $1 \leq p < \infty$ the Φ-modulus does not coincide with the usual $p-$modulus Mod_p, since $M_{\Phi}(\Gamma) = (Mod_p(\Gamma))^{1/p}$, but $p-$modulus and Φ-modulus are both outer measures on the set of curves in X.

A Borel nonnegative function g on X is a $\Phi-$weak upper gradient of real-valued function u on X if the inequality (2) from the definition of an upper gradient holds for all non-constant rectifiable curve $\gamma : [a, b] \to X$ except for a curve family of zero $\Phi-$modulus.

Tuominen [23] introduced the Orlicz-Sobolev space $N^{1,\Phi}(X)$. The collection $\widetilde{N}^{1,\Phi}(X)$ of all real-valued functions $u \in L^{\Phi}(X)$ possesing $\Phi-$weak upper gradients in $L^{\Phi}(X)$ is a vector space. The application defined on $\widetilde{N}^{1,\Phi}(X)$ by $\|u\|_{1,\Phi} = \|u\|_{L^{\Phi}(X)} + \inf \|g\|_{L^{\Phi}(X)}$, where the infimum is taken over all $\Phi-$weak upper gradients $g \in L^{\Phi}(X)$ of u is a seminorm. Setting $u \sim v$ if $\|u - v\|_{1,\Phi} = 0$, for $u, v \in \widetilde{N}^{1,\Phi}(X)$, we obtain an equivalence relation on $\widetilde{N}^{1,\Phi}(X)$. The set of equivalence classes of functions $u \in \widetilde{N}^{1,\Phi}(X)$ is a vector space denoted by $N^{1,\Phi}(X)$ and its norm as a quotient space is defined by $\|u\|_{N^{1,\Phi}(X)} = \|u\|_{1,\Phi}$.

If $\Phi(t) = t^p$ with $1 \leq p < \infty$, then $N^{1,\Phi}(X) = N^{1,p}(X)$ is the Newtonian space introduced by Shanmugalingam [22].

The $(1, \Phi)-$Poincaré inequality introduced in [23] is a generalization of the well-known $(1, p)-$Poincaré inequality [16], [11]. Let Φ be a strictly increasing real-valued Young function. It is said that a function $u \in L^1_{loc}(X)$ and a non-negative measurable function g on X satisfy a weak $(1, \Phi)-$Poincaré inequality if there exist constants $C_P > 0$ and $\tau \geq 1$ such that

$$\frac{1}{\mu(B)} \int\limits_{B} |u - u_B| \, d\mu \leq C_P r \Phi^{-1} \left(\frac{1}{\mu(\tau B)} \int\limits_{\tau B} \Phi(g) \, d\mu \right). \tag{7}$$

Moreover, it is said that the metric measure space (X, d, μ) supports a weak $(1, \Phi)-$Poincaré inequality if the inequality (7) holds for each pair (u, g) with $u \in L^1_{loc}(X)$ and g an upper gradient of u, with constants C_P, τ not depending on (u, g).

For $\Phi(t) = t^p$, $1 \le p < \infty$, the $(1,\Phi)-$Poincaré inequality is the $(1,p)-$Poincaré inequality

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \le C_P r\left(\frac{1}{\mu(\tau B)}\int_{\tau B} g^p d\mu\right)^{\frac{1}{p}}. \tag{8}$$

Assume that (X,d,μ) supports a weak $(1,\Phi)-$Poincaré inequality (7) and that Φ satisfies a Δ_2-condition. Then replacing C_P by $C_P C_\Phi$ in (7) we obtain an inequality that holds for each pair (u,g_0) with $u \in L^1_{loc}(X)$ and g_0 a $\Phi-$weak upper gradient of u, as we show in the following.

There exist a sequence of upper gradients $(g_n)_{n\ge1}$ of u such that $\lim\limits_{n\to\infty}\|g_n-g_0\|_{L^\Phi(X)} = 0$, which implies $\lim\limits_{n\to\infty}\int_{\tau B}\Phi(|g_n-g_0|)\,d\mu = 0$ [23, Lemma 4.3]. For each $n \ge 1$ the pair $(u,g)=(u,g_n)$ satisfies (7). Since $\Phi(g_n) \le \frac{C_\Phi}{2}\Phi(g_0) + \frac{C_\Phi}{2}\Phi(|g_n-g_0|)$ and $\Phi^{-1}\left(\frac{C_\Phi}{2}u+\frac{C_\Phi}{2}v\right) \le C_\Phi\Phi^{-1}(u) + C_\Phi\Phi^{-1}(v)$ for all $u,v \ge 0$, we get

$$\begin{aligned}\Phi^{-1}\left(\frac{1}{\mu(\tau B)}\int_{\tau B}\Phi(g_n)\,d\mu\right) &\le C_\Phi\Phi^{-1}\left(\frac{1}{\mu(\tau B)}\int_{\tau B}\Phi(g_0)\,d\mu\right)\\ &\quad + C_\Phi\Phi^{-1}\left(\frac{1}{\mu(\tau B)}\int_{\tau B}\Phi(|g_n-g_0|)\,d\mu\right).\end{aligned}$$

It follows that for all $n \ge 1$

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu &\le C_P C_\Phi r\Phi^{-1}\left(\frac{1}{\mu(\tau B)}\int_{\tau B}\Phi(g_0)\,d\mu\right)\\ &\quad + C_P C_\Phi\Phi^{-1}\left(\frac{1}{\mu(\tau B)}\int_{\tau B}\Phi(|g_n-g_0|)\,d\mu\right).\end{aligned}$$

Letting n tend to infinity we see that (u,g_0) satisfies the analogous (7) with $C_P C_\Phi$ instead C_P.

3. The interplay between Orlicz-Poincaré inequality and Hölder continuity

Let Φ be a Young function satisfying a Δ_2-condition. If $g \in L^\Phi(X)$, then for every measurable set $A \subset X$ we have $\int_A \Phi(g)\,d\mu < \infty$.

We denote $I_{\Phi}(g,A) := \Phi^{-1}\left(\int\limits_A \Phi(g)\,d\mu\right)$, using the convention that $I_{\Phi}(g,A) = \infty$ if $\int\limits_A \Phi(g)\,d\mu = \infty$. If in addition $\lim\limits_{t\to\infty} \frac{\Phi(t)}{t} = \infty$, then $L^{\Phi}(X) \subset L^1_{loc}(X)$ [21]. Note that in general $I_{\Phi}(g,A)$ and $\|g\|_{L^{\Phi}(A)}$ are not comparable.

In what follows, we denote $p := \log_2 C_{\Phi}$ and $q := \log_2 C_{\Phi/^{-1}}$.

Theorem 4. *Let (X,d,μ) be a metric space with a doubling measure μ and a homogeneous dimension $Q > 1$. Let Φ be an Young function satisfying the Δ_2- condition, such that $p > Q$ and $q < 1/Q$. Then there exists some constant $C > 0$ such that for every pair (u,g) satisfying a weak $\Phi-$Poincaré inequality (7), where $u \in L^1_{loc}(X)$ and g is a nonnegative measurable function, we have*

(9)
$$|u(x) - u(y)| \le C \max\left(d(x,y)^{1-\frac{Q}{p}}, d(x,y)^{1-Qq}\right) I_{\Phi}(g, B(x, 2\tau d(x,y)))$$

for all distinct Lebesgue points $x,y \in X$ of u. The constant C depends only on p,q, on the constants C_{μ}, Q, C_A associated with μ and on the constants τ, C_{PI} associated with the weak $\Phi-$Poincaré inequality (7).

Proof. Let $u \in L^1_{loc}(X)$. Let $x,y \in X$ be distinct Lebesgue points of u. We apply a telescoping argument similar to that from the proof of [22, Theorem 4.1]. Denoting $B_0 = B(x, 2d(x,y))$ and $B_1 = B(y, d(x,y))$, define $B_{(-i-1)} = \frac{1}{2}B_{(-i)}$ and $B_{i+1} = \frac{1}{2}B_i$ for all integers $i \ge 1$. Then $B_{(-i)} = B(x, 2^{1-i}d(x,y))$ and $B_i = B(y, 2^{1-i}d(x,y))$ for all $i \ge 1$.

Since x,y are Lebesgue points of u, we have $u(x) = \lim\limits_{i\to\infty} u_{B_{(-i)}}$ and $u(y) = \lim\limits_{i\to\infty} u_{B_i}$. Then

$$|u(x) - u(y)| \le \sum_{i=-\infty}^{\infty} \left|u_{B_i} - u_{B_{i+1}}\right|.$$

Recall the following estimate: if $B' \subset B \subset X$ are balls, then for every measure μ which is finite and positive on balls and each $u \in L^1_{loc}(X)$ we have

$$|u_{B'} - u_B| \le \frac{\mu(B)}{\mu(B')}\left(\frac{1}{\mu(B)}\int\limits_B u\,d\mu\right).$$

Note that $B_{\pm(i+1)} \subset B_{\pm i} = 2B_{\pm(i+1)}$ for all $i \ge 1$, while $B_1 \subset B_0 \subset 3B_1 \subset 2^2 B_1$. Since μ is doubling, it follows from the above inequality that for each $i \geq 1$, $\left|u_{B_{\pm(i+1)}} - u_{B\pm_i}\right| \leq C_\mu \left(\frac{1}{\mu(B_{\pm i})} \int\limits_{B\pm_i} u\, d\mu\right)$ and $|u_{B_1} - u_{B_0}| \leq (C_\mu)^2 \left(\frac{1}{\mu(B_0)} \int\limits_{B_0} u\, d\mu\right)$. Then

(10)
$$|u(x) - u(y)| \leq C_\mu \sum_{i=1}^{\infty} \frac{1}{\mu(B_{\pm i})} \int\limits_{B\pm_i} u\, d\mu + (C_\mu)^2 \left(\frac{1}{\mu(B_0)} \int\limits_{B_0} u\, d\mu\right).$$

Since the pair (u, g) satisfies the weak $\Phi-$Poincaré inequality (7), we have

(11)
$$\frac{1}{\mu(B_{\pm i})} \int\limits_{B\pm_i} u\, d\mu \leq C_{PI} 2^{1-i} d(x, y)\, \Phi^{-1} \left(\frac{1}{\mu(\tau B_{\pm i})} \int\limits_{\tau B\pm_i} \Phi(g)\, d\mu\right).$$

Our goal is to estimate the last factor from the right hand side of the above inequality $J_\Phi(g, \tau B\pm_i)$, where $J_\Phi(g, B) := \Phi^{-1}\left(\frac{1}{\mu(B)} \int\limits_B \Phi(g)\, d\mu\right)$, in terms of $I_\Phi(g, \tau B\pm_i)$. Note that $I_\Phi(g, \tau B\pm_i) \leq I_\Phi(g, \tau B_0)$, since $\Phi(g)$ is non-negative and Φ^{-1} is increasing. Let i be a nonnegative integer. By Lemma 2,

$$J_\Phi(g, \tau B\pm_i) \leq 2\mu(\tau B_{\pm i})^{-q} I_\Phi(g, \tau B\pm_i)$$

if $\mu(\tau B_{\pm i}) \leq 1$, while

$$J_\Phi(g, \tau B\pm_i) \leq 2\mu(\tau B_{\pm i})^{-1/p} I_\Phi(g, \tau B\pm_i)$$

if $\mu(\tau B_{\pm i}) \geq 1$.

Using the first inequality in (1) these inequalities yield

$$J_\Phi(g, \tau B\pm_i) \leq 2C_A^q \tau^{-qQ} \left(2^{1-i} d(x, y)\right)^{-qQ} I_\Phi(g, \tau B\pm_i)$$

if $\mu(\tau B_{\pm i}) \leq 1$, respectively

$$J_\Phi(g, \tau B\pm_i) \leq 2C_A^{1/p} \tau^{-Q/p} \left(2^{1-i} d(x, y)\right)^{-Q/p} I_\Phi(g, \tau B\pm_i)$$

if $\mu(\tau B_{\pm i}) \geq 1$. Then (11) implies

$$\frac{1}{\mu(B_{\pm i})} \int\limits_{B\pm_i} u\, d\mu \leq 2C_{PI} M(x, y)\, I_\Phi(g, \tau B\pm_i), \tag{12}$$

where we denoted

$M(x, y) = \max\left(C_A^q \tau^{-qQ} \left(2^{1-i} d(x, y)\right)^{1-qQ}, C_A^{1/p} \tau^{-Q/p} \left(2^{1-i} d(x, y)\right)^{1-Q/p}\right).$

But

$\max\left(C_A^q\tau^{-qQ}\left(2^{1-i}d\left(x,y\right)\right)^{1-qQ}, C_A^{1/p}\tau^{-Q/p}\left(2^{1-i}d\left(x,y\right)\right)^{1-Q/p}\right) \leq$

$C_1\max\left(2^{(1-i)(1-qQ)}, 2^{(1-i)(1-Q)/p}\right)\max\left(d\left(x,y\right)^{1-\frac{Q}{p}}, d\left(x,y\right)^{1-Qq}\right),$

where
$C_1 := \max\left(C_A^q\tau^{-qQ}, C_A^{1/p}\tau^{-Q/p}\right)$ and
$\max\left(2^{(1-i)(1-qQ)}, 2^{(1-i)(1-Q)/p}\right) = \max\left(2^{1-qQ}, 2^{1-Q/p}\right)$ for $i = 0$ and
$\max\left(2^{(1-i)(1-qQ)}, 2^{(1-i)(1-Q)/p}\right) = m^{(1-i)}$ for $i \geq 1$, where
$m := \min\left(2^{(1-qQ)}, 2^{(1-Q)/p}\right)$. Note that $m > 1$, due to our assumptions $p > Q$ and $q < 1/Q$, hence $\sum\limits_{i=1}^{\infty} m^{1-i} = m/\left(1-m\right)$.

From (10), (12) and $I_\Phi\left(g,\tau B\pm_i\right) \leq I_\Phi\left(g,\tau B_0\right)$ for $i \geq 0$ we get (9) with

$$C = 2C_\mu C_{PI}C_1\left(\frac{2m}{1-m} + C_\mu\max\left(2^{(1-qQ)}, 2^{(1-Q)/p}\right)\right).$$

∎

Corollary 5. *Consider the assumptions of the above theorem. Suppose that that X supports a weak $\Phi-$Poincaré inequality (7). If $pq \leq 1$, then there exists some constant $C > 0$ such that for every $u \in L^1_{loc}\left(X\right)$ with a $\Phi-$ weak upper gradient g we have*

$$|u(x) - u(y)| \leq C\min\left(d\left(x,y\right)^{1-\frac{Q}{p}}, d\left(x,y\right)^{1-Qq}\right)I_\Phi\left(g, B\left(x, 2\tau d\left(x,y\right)\right)\right)$$

for all distinct Lebesgue points $x, y \in X$ of u. Here
$C = C\left(p,q,C_\mu,Q,C_A,\tau,C_{PI}\right)$.

Proof. Let $u \in L^1_{loc}\left(X\right)$ and g be a $\Phi-$ weak upper gradient of u. Since X supports a weak $\Phi-$Poincaré inequality, the pair (u,g) satisfies a weak $\Phi-$Poincaré inequality (7).

We slightly modify the proof of Theorem 4. Taking into account Corollary 3, we may replace inequality (12) by the stronger inequality

$$\frac{1}{\mu\left(B_{\pm i}\right)}\int\limits_{B\pm_i} u\,d\mu \leq 2C_{PI}m\left(x,y\right)I_\Phi\left(g,\tau B\pm_i\right), \tag{13}$$

where
$m(x,y) := \min\left(C_A^q\tau^{-qQ}\left(2^{1-i}d\left(x,y\right)\right)^{1-qQ}, C_A^{1/p}\tau^{-Q/p}\left(2^{1-i}d\left(x,y\right)\right)^{1-Q/p}\right)$.

From (10), (13) and $I_\Phi\left(g,\tau B\pm_i\right) \leq I_\Phi\left(g,\tau B_0\right)$ for $i \geq 0$ we get

$$|u(x) - u(y)| \leq C_2 d\left(x,y\right)^{1-Qq} I_\Phi\left(g,\tau B_0\right)$$

and

$$|u(x)-u(y)| \leq C_3 d(x,y)^{1-\frac{Q}{p}} I_\Phi(g,\tau B_0),$$

where we denoted $C_2 := 2C_\mu C_{PI} C_A^q \tau^{-qQ} 2^{1-qQ}\left(\frac{2}{2^{1-qQ}-1}+C_\mu\right)$ and

$$C_3 := 2C_\mu C_{PI} C_A^{1/p} \tau^{-Q/p} 2^{1-Q/p}\left(\frac{2}{2^{1-Q/p}-1}+C_\mu\right).$$

Setting $C=\max(C_2,C_3)$ the claim (5) follows. ∎

Corollary 6. *Consider the assumptions of the above theorem. Suppose that that X supports a weak $\Phi-$Poincaré inequality (7), where $\lim\limits_{t\to\infty}\frac{\Phi(t)}{t}=\infty$. Then for every $u\in N^{1,\Phi}(X)$ there exist a constant $C(u)>0$ and a continuous representative $\widetilde{u}$ of u satisfying*

$$|\widetilde{u}(x)-\widetilde{u}(y)| \leq C(u)\max\left(d(x,y)^{1-\frac{Q}{p}}, d(x,y)^{1-Qq}\right)$$

for all $x,y\in X$. Moreover, if $pq\leq 1$, then every $u\in N^{1,\Phi}(X)$ has a representative $\widetilde{u}$ which is $(1-Qq)-$Hölder continuous and $(1-Q/p)-$ Hölder continuous.

Remark 2. *The above Corollary generalizes implication* (1) $\Rightarrow$ (2) *from* [8, Theorem 5.1].

Remark 3. *By* [18, Theorem 1], *if the pair (u,g) with $u\in L^1_{loc}(X)$ and $g\in L^\Phi_{loc}(X)$ satisfies a weak $\Phi-$Poincaré inequality, then u has a representative that is* locally *$(1-Qq)-$Hölder continuous, provided that $q<1/Q$. There it was not assumed that $p>Q$, but the claim is weaker than that of Theorem 4.*

We are looking for a generalization of the implication (2) $\Rightarrow$ (1) from [8, Theorem 5.1].

Lemma 7. *Let (X,d,μ) be a metric space with a doubling measure μ satisfying $\mu(B(x,r))\leq C_A r^Q$ for every ball $B(x,r)$ in X with $0<r<diam(X)$, where C_A, $Q>0$ are constants. Let Φ be an Young function satisfying the Δ_2-condition. Let $u\in L^1_{loc}(X)$ and g be a non-negative measurable function. Assume that there exist some constants $\alpha>0$, $\lambda\geq 1$ and $C>0$ such that the following inequality holds, for almost all $x,y\in X$ with $x\neq y$:*

$$|u(x)-u(y)| \leq C d(x,y)^\alpha I_\Phi(g, B(x,\lambda d(x,y))).$$

Then for every ball $B=B(a,r)\subset X$ we have

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \leq C_1 r^{\alpha+Qq}\Phi^{-1}\left(\frac{1}{\mu(5\lambda B)}\int_{5\lambda B}\Phi(g)\,d\mu\right)$$

if $\mu(5\lambda B) \geq 1$ *and*

$$(14) \qquad \frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C_2 r^{\alpha + \frac{Q}{p}} \Phi^{-1} \left(\frac{1}{\mu(5\lambda B)} \int_{5\lambda B} \Phi(g) \, d\mu \right)$$

if $\mu(5\lambda B) \leq 1$. *Here* $C_1 = 2^{\alpha+1} C (C_A)^q$ *and* $C_2 = 2^{\alpha+1} C (C_A)^{1/p}$.

If in addition $pq \leq 1$, *then (14) holds for every ball* $B = B(a,r) \subset X$.

Proof. Let $u \in L^1_{loc}(X)$ with a $\Phi-$ weak upper gradient g. Let $B = B(a,r) \subset X$ be a ball. Since
$|u(x) - u_B| = \left| \frac{1}{\mu(B)} \int_B (u(x) - u(y)) \, d\mu(y) \right| \leq \frac{1}{\mu(B)} \int_B |u(x) - u(y)| \, d\mu(x)$
for every $x \in X$, we have

$$(15) \quad \frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq \left(\frac{1}{\mu(B)} \right)^2 \int_B \int_B |u(x) - u(y)| \, d\mu(y) \, d\mu(x).$$

There exists a null set $E \subset X$ such that

$$|u(x) - u(y)| \leq C d(x,y)^\alpha I_\Phi(g, B(x, \lambda d(x,y)))$$

for all $x, y \in X \backslash E$ with $x \neq y$.

If $x, y \in B$, then
$B(x, \lambda d(x,y)) \subset (4\lambda + 1) B \subset 5\lambda B$, hence
$I_\Phi(g, B(x, \lambda d(x,y))) \leq I_\Phi(g, 5\lambda B)$. Note that
$|u(x) - u(y)| \leq 2^\alpha C r^\alpha I_\Phi(g, B(x, \lambda d(x,y)))$ for $x, y \in B \setminus E$.
It follows that
$|u(x) - u(y)| \leq 2^\alpha C r^\alpha I_\Phi(g, 5\lambda B)$ for all $x, y \in B \setminus E$, hence

$$(16) \quad \left(\frac{1}{\mu(B)} \right)^2 \int_B \int_B |u(x) - u(y)| \, d\mu(y) \, d\mu(x) \leq 2^\alpha C r^\alpha I_\Phi(g, 5\lambda B).$$

By (15) and (16) we get

$$(17) \qquad \frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq 2^\alpha C r^\alpha I_\Phi(g, 5\lambda B).$$

But
$I_\Phi (g, 5\lambda B) = \Phi^{-1} \left(\int\limits_{5\lambda B} \Phi (g)\, d\mu \right) \leq$
$\leq 2 (\mu (5\lambda B))^s \Phi^{-1} \left(\frac{1}{\mu(5\lambda B)} \int\limits_{5\lambda B} \Phi (g)\, d\mu \right),$
where $s = q$ if $\mu (5\lambda B) \geq 1$ and $s = \frac{1}{p}$ if $\mu (5\lambda B) \leq 1$.
Using the Ahlfors regularity of μ we obtain

$$I_\Phi (g, 5\lambda B) \leq 2 (C_A)^s r^{Qs} \Phi^{-1} \left(\frac{1}{\mu (5\lambda B)} \int\limits_{5\lambda B} \Phi (g)\, d\mu \right), \tag{18}$$

where $s = q$ if $\mu (5\lambda B) \geq 1$ and $s = \frac{1}{p}$ if $\mu (5\lambda B) \leq 1$. If in addition $pq \leq 1$, then for $\mu (5\lambda B) \geq 1$ we have $(\mu (5\lambda B))^q \leq (\mu (5\lambda B))^{1/p}$, therefore inequality (18) with $s = \frac{1}{p}$ holds for every ball $B = B (a, r) \subset X$.

Now (17) and (18) imply the claim. ∎

The following Proposition is a straightforward consequence of Lemma 7.

Proposition 8. *Let (X, d, μ) be a metric space with a doubling measure μ satisfying $\mu (B (x, r)) \leq C_A r^Q$ for every ball $B(x, r)$ in X with $0 < r < diam (X)$, where C_A, $Q > 0$ are constants. Let Φ be an Young function satisfying the Δ_2- condition, such that $p > Q$ and $pq \leq 1$. Assume that there exist some constants $\lambda \geq 1$ and $C > 0$ such that for every $u \in L^1_{loc} (X)$ with a $\Phi-$ weak upper gradient g the following inequality holds, for almost all $x, y \in X$ with $x \neq y$:*

$$|u(x) - u(y)| \leq Cd (x, y)^{1-\frac{Q}{p}} I_\Phi (g, B (x, \lambda d (x, y))).$$

Then X supports a weak $\Phi-$Poincaré inequality.

Proposition 8 partly generalizes the implication (2) ⇒ (1) from [8, Theorem 5.1]. To obtain a full generalization we would need a result guaranteeing that a space supporting a Orlicz-Poincaré inequality for functions $u \in N^{1,\Phi} (X)$ also supports a Orlicz-Poincaré inequality for functions $u \in L^1_{loc} (X)$, that would be a counterpart of a result of Keith [17] where $\Phi (t) = t^p$, $1 \leq p < \infty$.

We conclude with the following result, combining Corollary 5 and Proposition 8.

Theorem 9. *Let (X, d, μ) be a $Q-$Ahlfor regular metric measure space with $Q > 1$. Let Φ be an Young function satisfying the Δ_2- condition, such that $p > Q$ and $pq \leq 1$. Then the following conditions are equivalent:*

(1) X supports a weak $\Phi-$Poincaré inequality.

(2) There exist some constants $\lambda \geq 1$ and $C > 0$ such that for every $u \in L^1_{loc}(X)$ with a $\Phi-$ weak upper gradient g the following inequality holds, for almost all $x, y \in X$ with $x \neq y$:

$$|u(x) - u(y)| \leq Cd(x,y)^{1-\frac{Q}{p}} \Phi^{-1}\left(\int_{B(x,\lambda d(x,y))} \Phi(g)\, d\mu \right).$$

References

[1] N. Aïssaoui, **Another extension of Orlicz-Sobolev spaces to metric spaces**, Abstr. Appl. Anal. 2004, no.1, 1-26.

[2] J. Björn, **Orlicz-Poincaré inequalities, maximal functions and A_Φ-conditions**, Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), 31-48.

[3] A. Björn, J. Björn, **Nonlinear potential theory on metric spaces**, EMS Tracts in Mathematics 17, European Mathematical Society, Zürich, 2011, 415 pp.

[4] J. Cheeger, **Differentiability of Lipschitz functions on metric measure spaces**, Geom. Funct. Anal. 9 (1999), 428-517.

[5] R. R. Coifman and G. Weiss, **Analyse Harmonique Non-Commutative Sur Certains Espaces Homogènes**, vol. 242 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1971.

[6] E. Durand-Cartagena, J. A. Jaramillo, N. Shanmugalingam: **The ∞-Poincaré inequality in metric measure spaces.** Michigan Math. Journal 61 No.1 (2012), 63-85.

[7] E. Durand-Cartagena, N. Shanmugalingam, A. Williams: **p-Poincaré inequality versus ∞-Poincaré inequality: some counterexamples,** Math. Z. 271 (2012), no. 1-2, 447-467.

[8] E. Durand-Cartagena, J. Jaramillo and N. Shanmugalingam, **Geometric characterizations of the $p-$Poincaré inequality in the metric setting**, Publ. Mat. 60 (2016), 81-111.

[9] P. Hajłasz, **Sobolev spaces on metric-measure spaces**, Contemporary Math., 338 (2003), 173-218.

[10] P. Hajłasz and P. Koskela, **Sobolev meets Poincaré**, C. R. Acad. Sci. Paris Ser. I Math. 320 (1995), 1211–1215.

[11] P. Hajłasz and P. Koskela, **Sobolev met Poincaré,** Memoirs Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.

[12] T. Heikkinen, **Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in metric measure spaces,** Indiana University Mathematics Journal 59(3) (2010), 957–986.

[13] T. Heikkinen, **Characterizations of Orlicz-Sobolev spaces by means of generalized Orlicz-Poincaré inequalities**, Journal of Function Spaces and Applications, Volume 2012, Article ID 426067, 15 pages, doi:10.1155/2012/426067

[14] T. Heikkinen, P. Koskela and H. Tuominen, **Sobolev-type spaces from generalized Poincaré inequalities,** Studia Math 181 (1), 2007, 1-16

[15] J. Heinonen, **Lectures on Analysis on Metric Measure Spaces**, Springer- Verlag, New York, 2001.

[16] J. Heinonen and P. Koskela, **Quasiconformal maps on metric spaces with controlled geometry**, Acta Math. 181 (1998), 1–61.

[17] S. Keith, **Modulus and the Poincaré inequality on metric measure spaces**. Math. Z. 245(2003), 255{292.

[18] M. Mocanu, **Maximal operators and Orlicz-Sobolev functions on metric measure spaces,** Proc. Sixth Congress of Romanian Math., Bucharest, vol. 1 (2009), pp. 169–178.

[19] M. Mocanu, **A Poincaré inequality for Orlicz-Sobolev functions with zero boundary values on metric spaces**, Complex Analysis and Operator Theory, 5 (3) (2011), 799-810.

[20] M. Mocanu, **Poincaré inequalities based on Banach function spaces on metric measure spaces**, Scientific Studies and Research. Series Mathematics and Informatics, 23 (1) (2013), 63-75

[21] M. M. Rao and Z. D. Ren, **Theory of Orlicz Spaces**, Monographs and Textbooks in Pure and Applied Mathematics 146, Marcel Dekker, New York, 1991.

[22] N. Shanmugalingam, **Newtonian spaces: an extension of Sobolev spaces to metric measure spaces**, Rev. Mat. Iberoamericana 16 (2000), no.2, 243-279.

[23] H. Tuominen, **Orlicz-Sobolev spaces on metric measure spaces**, Ann. Acad. Sci. Fenn., Diss.135 (2004) 86 pp

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