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Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 26(2016), No. 2, 83-94

NOTE ON THE ITERATES OF $q-$ AND $(p,q)-$BERNSTEIN OPERATORS

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Abstract. In this note we are concerned with the limit behavior of the iterates of q and (p,q)-Bernstein operators. The convergence of the iterates of q-Bernstein operators was proved by H. Oruç and N. Tuncer in 2002 and by X. Xiang, Q. He and W. Yang in 2007, using the q-differences, Stirling polynomials and matrix techniques and by S. Ostrovska in 2003, with the aids of eigenvalues. We try to prove the convergence from some different points of view: using contraction principle (Weakly Picard Operators Theory). The theory of (weakly) Picard operator is very useful to study some properties of the solutions of those equations for which the method of successive approximations works.

Work presented at "Gheorghe Vrănceanu" Conference on Mathematics and Informatics, December 6, 2016, "Vasile Alecsandri" University of Bacău

1. Introduction

In 1912, using the theory of probability, Bernstein defined the polynomials called nowadays **Bernstein polynomials**, in need to proof the famous Weierstrass Approximation Theorem. Later, it was found that this polynomials possess many remarkable properties. Due to this fact, these polynomials and their generalizations have been intensively studied.

Keywords and phrases: Iterates, q - Bernstein operator, $(p,q)-$ Bernstein operator, contraction principles, weakly Picard operators.
(2010) Mathematics Subject Classification: 41A10, 41A36, 47H10

In 1987, A. Lupaş [12] introduced a q-analogue of the Bernstein operators and in 1997 another generalization of these operators based on q-integers was introduced by G.M. Phillips [17]. For $q = 1$, these polynomials are just the classical ones, but for $q \neq 1$, there are considerable differences between them. Many properties of q-Bernstein operators were studied by numerous mathematicians, starting with [17]. Convergence properties for iterates of both q-Bernstein polynomials and their Boolean sum have been studied in 2002 by H. Oruç and N. Tuncer [15] and in 2007 by X. Xiang, Q. He and W. Yang [21]. The convergence of $B_n(f, q; x)$ as $q \to \infty$ and the convergence of the iterates $B_n^{j_n}(f, q; x)$ as both $n \to \infty$ and $j_n \to \infty$ have been investigated by S. Ostrovska in 2003, in [16]. Recently, in 2015, the pointwise convergence and properties of the q-derivatives of q-Bernstein operators have been studied by T. Acar and A. Aral [2].

The definition and elementary properties of q-Bernstein polynomials are given as follows, see, e.g., [12], [3], [13] and [17]. For any fixed real number $q > 0$, the q-integer $[k]_q$, for $k \in \mathbb{N}$ is defined as

$$[k]_q = \begin{cases} \left(1 - q^k\right)/(1 - q), & q \neq 1 \\ k, & q = 1. \end{cases}$$

Set $[0]_q = 0$. The q-factorial $[k]_q!$ and q-binomial coefficients $\left[\begin{array}{c} n \\ k \end{array} \right]_q$ are defined as follows

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ... \\ 1, & k = 0 \end{cases}$$

and

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \frac{[n]_q!}{[k]_q![n-k]_q!}, \text{ for } k \in \{0, 1, \ldots, n\}.$$

The q-analogue of $(x - a)^n$ is the polynomial

$$(x - a)_q^n = \begin{cases} 1, & n = 0 \\ (x - a)(x - qa)...(x - q^{n-1}a), & n \geq 1. \end{cases}$$

In 1997, G.M. Phillips [17] defined the q-Bernstein operators by:

$$B_n(f, q; x) = \sum_{k=0}^{n} b_{n,k}(x; q) f\left(\frac{[k]_q}{[n]_q}\right), \; x \in [0, 1], \tag{1}$$

where $b_{n,k}(x;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-k-1}(1-q^j x)$ and an empty product is equal to 1.

For any function f we define $\Delta_q^0 f_i = f_i$ for $i = 0, 1, \cdots, n-1$ and, recursively, $\Delta_q^{k+1} f_i = \Delta_q^k f_{i+1} - q^k \Delta_q^k f_i$ for $k \geq 0$, where f_i denotes $f\left(\frac{[i]}{[m]}\right)$.

It is established by induction that

$$\Delta_q^k f_i = \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r}.$$

The q-Bernstein polynomial may be written in the q-difference form

$$B_n(f,q;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \Delta_q^k f_0 x^k. \tag{2}$$

We may deduce from (2) that $B_n(f,q;x)$ reproduces linear polynomials, that is,

$$B_n(ax+b,q;x) = ax+b, \quad a,b \in \mathbb{R}.$$

Also, we can see that these operators verify for the test function $e_j(x) = x^j, \quad j = 0, 1, ...,$ that

$$B_n(e_0,q;x) = 1, \quad B_n(e_1,q;x) = x, \quad B_n(e_2,q;x) = x^2 + \frac{x(1-x)}{[n]_q}.$$

2. Preliminaries

Fixed point theory is a fascinating subject and became, in the last decades, not only a field with a great development, but also a strong tool for solving different kind of problems from various fields of pure and applied mathematics. An important result of the metric fixed point theory is the Banach-Caccioppoli Contraction Principle. Today we have many generalizations of this result.

One of the most important concepts of fixed point theory is the Picard operator. In 1993 in [19], Ioan A. Rus introduced and developed these theory of Picard and weakly Picard operators, which is now one of the most strong tools of the Fixed point theory with several applications to operatorial equations and inclusions. The theory of (weakly) Picard operators is very useful to study some properties of the solutions of those equations for which the method of successive approximations works.

We know that a self mapping T of a nonempty set X is Picard operator if T has a unique fixed point and the sequence of successive approximation for any initial point converges to this unique fixed point. Therefore, a lot of different types of contractions are Picard operators. (See for more details the article of V. Berinde from 2004, [6]). On the other hand, in [19] I.A. Rus introduced the concept of *weakly Picard operator*: a self mapping T of a nonempty set X is a weakly Picard operator if the set of fixed points of T is nonempty and the sequence of successive approximation for any initial point converges to a fixed point of T. There exist many classes of weakly Picard operators in the literature, but one of the most interesting of them is the class of almost contractions. In [6], V. Berinde showed that the concept of almost contraction is more general than several different types of contractions in the literature. Recently, an interesting generalization in Kasahara spaces was given by A.D. Filip and A. Petruşel in [9].

In the following, we recall some basic features from fixed point theory, see e.g. [5] and [19].

Definition 2.1. *Let (X, d) be a metric space. The operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence of iterates $(A^m(x))_{m\geq 1}$ converges for all $x \in X$ and the limit is a fixed point of A.*

We denote as usually $A^0 = I_X,\ A^{m+1} = A \circ A^m,\ m \in \mathbb{N}$.

Let $\mathcal{F}_A := \{x \in X | A(x) = x\}$ be the set of fixed points of A. If the operator A is a WPO and $\mathcal{F}_A$ has exactly one element, then A *is* called a Picard operator (PO).

Moreover we have the following characterization of the WPOs.

Theorem 2.1. *Let (X, d) be a metric space. The operator $A : X \to X$ is a weakly Picard operator (WPO) if and only if there exists a partition $\{X_\lambda : \lambda \in \Lambda\}$ such that for every $\lambda \in \Lambda$ one has*

(i) $X_\lambda \in I(A)$,

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a PO,*

where $I(A) := \{\emptyset \neq Y \subset X | A(Y) \subset Y\}$ represents the family of all non-empty subsets invariant under A.

Further on, if A is a WPO we consider $A^\infty \in X^X$ defined by

$$A^\infty(x) := \lim_{m\to\infty} A^m(x),\ x \in X.$$

Clearly, we have $A^\infty(X) = \mathcal{F}_A$. Also, if A is WPO, then the identities $\mathcal{F}_A^m = \mathcal{F}_A \neq \emptyset,\ m \in \mathbb{N}$, hold true.

3. Iterates of q-Bernstein operator

In the last decades the iterated of positive and linear operators in various classes were intensively investigated. The study of the convergence of iterates of Bernstein operators has connections, for example, to probability theory, spectral theory, matrix theory, Korovkin-type theorems, quantitative results about the approximation of functions by positive linear operators, fixed point theorems and the theory of semigroups of operators. We emphasize here the importance of the work of some mathematicians, such as U. Abel, H.H. Gonska, M. Ivan and I. Raşa [1], [10], [18]. The direction of studying the convergence of iterates of linear operators using the fixed point theory was introduced by O. Agratini and I.A. Rus [4], [5] and [20], based on theory of weakly Picard operators development by I.A. Rus, in [19]. For Bernstein operators, the study of iterative properties was developed, for example, in [7] and [8].

We want to extend the study of the iterate of Bernstein operators using q-calculus. Our aim is to study the convergence for iterates of q-Bernstein operators, using contraction principle (weakly Picard operators theory).

In case of classical Bernstein polynomials, the iterates converge to linear end point interpolation on $[0,1]$. R.P. Kelisky and T.J. Rivlin (see [11]) considered this problem both when M is dependent of the degree of $B_n f$ and when M is dependent on n. We have the following well known result:

Theorem 3.1. *(Kelisky & Rivlin,* [11]*) If $n \in \mathbb{N}^*$ is fixed, then for all $f \in C[0,1]$,*

$$\lim_{M\to\infty} B_n^M(f;x) = f(0) + (f(1) - f(0))\,x, \quad x \in C[0,1].$$

In [20] I. A. Rus proved the result from above, using contraction principle.

The iterates of the q-Bernstein polynomial are defined by H. Oruç and N. Tuncer in 2002 in [15] as

$$B_n^{M+1}(f,q;x) = B_n\left(B_n^M(f,q;x);x\right) \quad M = 1,2,...,$$

and $B_n^1(f,q;x) = B_n(f,q;x)$.

In the same paper, [15] the authors proved the convergence of the iterates of the q-Bernstein operators, on the general case when $q > 0$, using the q-differences, Stirling polynomials and matrix techniques.

In 2003, in paper [16], S. Ostrovska proved also the convergence of the iterates of the q-Bernstein operators by using eigenvalues.

In the following, we study the convergence, from a different point of view, namely, by using the contraction principle.

Theorem 3.2. *Let $B_n(f,q;x)$ be the q-Bernstein operator defined in (1),* $0<q<1$.

Then the iterates sequence $\left(B_n^M\right)_{M\geq 1}$ verifies, for all $f \in C[0,1]$ and all $x \in [0,1]$,

$$\lim_{M\to\infty} B_n^M(f,q;x) = f(0) + (f(1)-f(0))\,x. \tag{3}$$

Proof. First we define $X_{\alpha,\beta} := \{f \in C[0,1] \mid f(0)=\alpha, f(1)=\beta\}$, $\alpha,\beta \in \mathbb{R}$. Clearly every $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$ and $\{X_{\alpha,\beta},\ (\alpha,\beta) \in \mathbb{R}\times\mathbb{R}\}$ is a partition of this space.

It follows directly from the definition that q-Bernstein polynomials possess the end-point interpolation property [16], i.e.

$$B_n(f,q;0) = f(0),\ B_n(f,q;1) = f(1).$$

for all $q>0$ (not only for $0<q<1$) and all $n = 1,2,....$

These relations ensure that for all $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ and $n \in \mathbb{N}$, $X_{\alpha,\beta}$ is an invariant subset of $f \mapsto B_n(f,q;\cdot)$.

Further on, we prove that the restriction of $f \mapsto B_n(f,q;\cdot)$ *to* $X_{\alpha,\beta}$ is a contraction for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

Let $u_n := min_{x\in[0,1]}\,(b_{n,0}(x;q) - b_{n,n}(x;q))$, i.e.

$$u_n := min_{x\in[0,1]}\left(x^n + (1-x)_q^n\right).$$

Then $0 < u_n \leq 1$.

Let $f,g \in X_{\alpha,\beta}$. From the definition of q-Bernstein operators we can write

$$|B_n(f,q;x) - B_n(g,q;x)| = \left|\sum_{k=1}^{n-1} b_{n,k}(x;q)(f-g)\left(\frac{[k]_q}{[n]_q}\right)\right|$$

$$\leq \sum_{k=1}^{n-1} b_{n,k}(x;q)\,\|f-g\|_{[0,1]} = (1 - b_{n,0}(x;q) - b_{n,n}(x;q))\,\|f-g\|_{[0,1]}$$

$$= \left(1 - x^n - (1-x)_q^n\right)\|f-g\|_{[0,1]} \leq (1-u_n)\,\|f-g\|_{[0,1]}$$

and finally

$$\|B_n(f,q;\cdot) - B_n(g,q;\cdot)\|_{[0,1]} \leq (1-u_n)\,\|f-g\|_{[0,1]}\,.$$

The restriction of $f \mapsto B_n(f,q;\cdot)$ *to* $X_{\alpha,\beta}$ is a contraction.

On the other hand, $p*_{\alpha,\beta} := \alpha e_0 + (\beta-\alpha)e_1 \in X_{\alpha,\beta}$.

Since $B_n(e_0,q;x)=e_0$ and $B_n(e_1,q;x)=e_1$, it follows that $p*_{\alpha,\beta}$ is a fixed point of $B_n(f,q;\cdot)$. For any $f\in C([0,1])$ one has $f\in X_{f(0),f(1)}$ and, by using the contraction principle, we obtain the desired result (3).

In terms of WPOs, using (2), we can formulate the above theorem as follows

Theorem 3.3. *The q-Bernstein operator $f\mapsto B_n(f,q;\cdot)$ is WPO and*

$$B_n^{\infty}(f,q;x)=f(0)+(f(1)-f(0))\,x.$$

4. (p,q)-Bernstein operator

During the last two decades, the applications of q-calculus have emerged as a new area in the field of approximation theory. Now, a new extension of the q-calculus arises. Recently, M. Mursaleen, K.J. Ansari and A. Khan [14] used $(p,q)-$calculus in approximation theory and defined the (p,q)-analogue of Bernstein operators . They estimated uniform convergence of the operators, as well as the rate of convergence, and obtained Voronovskaya theorem as well.

Let us recall some definitions and notations regarding (p,q)-calculus, see, e.g., [14].

The(p,q)-integer of the number $n\in\mathbb{N}$ is defined as

$$[n]_{p,q}:=\left(p^n-q^n\right)/(p-q),\quad 0<q<p\leq 1.$$

The (p,q)-factorial $[k]_{p,q}!$ and the (p,q)-binomial coefficients $\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}$ are defined as follows

$$[k]_{p,q}!:=[k]_{p,q}[k-1]_{p,q}...[1]_{p,q},\quad k=1,2,...$$

and

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}:=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},\ \text{for } k\in\{0,1,\ldots,n\}.$$

Now by some simple calculation and induction on n, we obtain the (p,q)-binomial expansion as follows

$$(x+y)_{p,q}^n:=(x+y)(px+qy)(p^2x+q^2y)...(p^{n-1}x+q^{n-1}y),$$

$$(1-x)_{p,q}^n:=(1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x),$$

$$(ax+by)_{p,q}^n:=\sum_{k=0}^{k}p^{\dfrac{(n-k)(n-k-1)}{2}}q^{\dfrac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}a^{n-k}b^kx^{n-k}y^k.$$

Discovering that the (p,q)-Bernstein operators introduced in [14] do not reproduce the test function e_0, in a recent note (corrigendum), M. Mursaleen, K.J. Ansari and A. Khan introduced a revised form of the (p,q)-analogue of Bernstein operators as follows:

$$B_n(f,p,q;x)=\sum_{k=0}^{n} b_{n,k}(x;p,q) f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right),\ x\in[0,1], \tag{4}$$

where

$$b_{n,k}(x;p,q)=\frac{1}{p^{\frac{n(n-1)}{2}}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{j=0}^{n-k-1}(p^j-q^jx).$$

Note that for $p=1$, the (p,q)-Bernstein operators given by (4) turn out to be q-Bernstein operators (1).

We have the following basic result

$$B_n(e_0,p,q;x)=1,\quad B_n(e_1,p,q;x)=x,\quad B_n(e_2,p,q;x)=\frac{p^{n-1}}{[n]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2.$$

5. Iterates of (p,q)-Bernstein operator

Now we extend the study of the iterate of Bernstein operators in the framework of (p,q)-calculus.

The iterates of the (p,q)-Bernstein polynomial are defined as

$$B_n^{M+1}(f,p,q;x)=B_n\left(B_n^M(f,p,q;x);x\right)\quad M=1,2,...,$$

and $B_n^1(f,p,q;x)=B_n(f,p,q;x)$.

Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of the (p,q)-Bernstein operators.

Theorem 5.1. *Let $B_n(f,p,q;x)$ be the (p,q)-Bernstein operator defined in (4), where $0<q<p\leq 1$.*

the given (p,q)-Bernstein operator is a weakly Picard operator and its iterates sequence $\left(B_n^M\right)_{M\geq 1}$ verifies

$$\lim_{M\to\infty} B_n^M(f,p,q;x)=f(0)+(f(1)-f(0))\,x. \tag{5}$$

Proof. The proof follows the same steps as in Theorem 3.2.

Let $X_{\alpha,\beta} := \{f \in C[0,1] |\ f(0) = \alpha, f(1) = \beta\},\ \alpha, \beta \in \mathbb{R}$.

(p,q)-Bernstein polynomials possess the end-point interpolation property

$$B_n(f,p,q;0) = f(0),\ B_n(f,p,q;1) = f(1).$$

Then $X_{\alpha,\beta}$ is an invariant subset of $f \mapsto B_n(f,p,q;\cdot)$, for all $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ and $n \in \mathbb{N}$.

We prove that the restriction of $f \mapsto B_n(f,p,q;\cdot)$ to $X_{\alpha,\beta}$ is a contraction for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

Let $\alpha, \beta \in \mathbb{R}$. Let $f, g \in X_{\alpha,\beta}$. From definition of (p,q) -Bernstein operator we can write

$$|B_n(f,p,q;x) - B_n(g,p,q;x)| = \left| \sum_{k=1}^{n-1} b_{n,k}(x;p,q)(f-g)\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right) \right|$$

$$\leq \sum_{k=1}^{n-1} b_{n,k}(x;p,q) \left\| f-g \right\|_{[0,1]} = (1 - b_{n,0}(x;p,q) - b_{n,n}(x;p,q)) \left\| f-g \right\|_{[0,1]}.$$

Let $v_n := min_{x\in[0,1]} \left(b_{n,0}(x;p,q) + b_{n,n}(x;p,q)\right)$, i.e.

$$v_n := min_{x\in[0,1]} \left(x^n + (1-x)^n_{p,q}\right).$$

Then $0 < v_n \leq 1$.

Therefore,

$$\begin{aligned} |B_n(f,p,q;x) - B_n(g,p,q;x)| &\leq \left(1 - x^n - (1-x)^n_{p,q}\right) \left\| f-g \right\|_{[0,1]} \\ &\leq (1 - v_n) \left\| f-g \right\|_{[0,1]} \end{aligned}$$

for any $f, g \in X_{\alpha,\beta}$, that is the restriction of $f \mapsto B_n(f,p,q;\cdot)$ to $X_{\alpha,\beta}$ is a contraction.

On the other hand, $p*_{\alpha,\beta} := \alpha e_0 + (\beta - \alpha)e_1 \in X_{\alpha,\beta}$, where $e_0(x) = 1$ and $e_1(x) = x$ for all $x \in [0,1]$. Since $B_n(e_0,p,q;x) = e_0$ and $B_n(e_1,p,q;x) = e_1$, it follows that $p*_{\alpha,\beta}$ is a fixed point of $B_n(f,p,q;\cdot)$.

By Theorem 2.1, the (p,q) Bernstein operator is a WPO and using the contraction principle, we obtain the claim (5).

6. Acknowledgement

The author wish to thank the referee for useful comments and suggestions.

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