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fgs^* - CLOSED SETS AND fgs^*-CONTINUOUS FUNCTIONS IN FUZZY TOPOLOGICAL SPACES

ANJANA BHATTACHARYYA

Abstract. In this paper, we introduce and study a new type of fuzzy generalized closed set and fuzzy generalized continuity in a fuzzy topological space. Also it is shown that fuzzy compactness and fuzzy normality remain invariant under this newly defined continuous function. Afterwards, a new type of fuzzy closure operator is introduced which is an idempotent operator which is distributive over union but not over intersection. Lastly, we introduce and characterize the notion of fgs^*-closed (resp., fgs^*-open) function which is weaker than that of fgs-closed (resp. fgs-open) function [8].

1. Introduction and Preliminaries

After the introduction of fuzzy generalized closed sets in [2], several types of fuzzy generalized forms of closed sets have been introduced and studied in [3, 4, 5, 6, 7, 8]. Here we introduce the class of fgs^*-closed sets which lies between the class of fuzzy semiclosed sets [1] and the class of fgs-closed sets [3].

Keywords: fs^*g-closed set, fgs^*-closed set, fs^*g-continuous function, fgs^*-continuous function, fgs^*-neighbourhood of a fuzzy point (fuzzy set), fgs^*-closed (open) function.
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In Section 2, we define a type of fuzzy space and in Section 3, we show that in this space fuzzy compactness and fuzzy normality remain invariant under fgs^*-continuous function.
Throughout this paper, by (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [10]. A fuzzy set [15] A in an fts X, denoted by $A \in I^X$, is defined to be a mapping from a non-empty set X into the closed interval $I = [0, 1]$. The support [15] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [15] of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [15] while AqB means A is quasi-coincident (q-coincident, for short) [12] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A\not{q}B$ respectively. A fuzzy set A in X is called a fuzzy neighbourhood (nbd, for short) [12] of a fuzzy point (resp., fuzzy set) x_t (resp., B) if there exists a fuzzy open set G in X such that $x_t \in G \leq A$ (resp., $B \leq G \leq A$). For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [10] and fuzzy interior [10] respectively. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy α-open [9]) if $A = intclA$ (resp., $A \leq clintA$, $A \leq intclintA$). The complement of a fuzzy semiopen (resp., fuzzy α-open) set is called fuzzy semiclosed [1] (resp., fuzzy α-closed [9]). The intersection of all fuzzy semiclosed (resp., fuzzy α-closed) sets containing a fuzzy set A in X is called fuzzy semiclosure (resp., fuzzy α-closure) of A, to be denoted by $sclA$ [1] (resp., αclA [9]). The union of all fuzzy semiopen sets contained in a fuzzy set A is called fuzzy semi interior of A, denoted by $sintA$ [1]. The collection of all fuzzy semiopen sets in X is denoted by $FSO(X)$ and that of fuzzy semiclosed sets in X is denoted by $FSC(X)$. A fuzzy set A is called fQ-set if $intclA = clintA$.

2. fgs^*-Closed Sets : Some Properties

We first recall some definitions from [2, 3, 5] for ease of reference.

Definition 2.1 [2]. A fuzzy set A in an fts (X, τ) is called fuzzy generalized closed (fg-closed, for short) [2] if $clA \leq U$ whenever $A \leq U \in \tau$.

The complement of an fg-closed set is called an fg-open set.

Definition 2.2. A fuzzy set A in an fts (X, τ) is called
(i) fuzzy semi generalized closed (fsg-closed, for short) [3] if $sclA \leq U$ whenever $A \leq U \in FSO(X)$,
(ii) fuzzy generalized semiclosed (fgs-closed, for short) [3] if $sclA \leq U$ whenever $A \leq U \in \tau$,
(iii) fuzzy α-generalized closed ($f\alpha g$-closed, for short) [3] if $\alpha clA \leq U$ whenever $A \leq U \in \tau$,
(iv) fuzzy strongly generalized closed (fs^*g-closed, for short) [5] if $clA \leq U$ whenever $A \leq U$ *and* U is fg-open in X.

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.3. A fuzzy set A in an fts (X, τ) is called fuzzy generalized strongly closed (fgs^*-closed, for short) if $sclA \leq U$ whenever $A \leq U$ where U is fg-open in X.

The complement of an fgs^*-closed set is called an fgs^*-open set.

Remark 2.4 (i). Every fuzzy semiclosed set is fgs^*-closed set, but not conversely, as it can be seen from Example 2.5.
(ii). Every fgs^*-closed set is fgs-closed, but not conversely, as it can be seen from Example 2.6.

Example 2.5. There exists an fgs^*-closed set which is not fuzzy semiclosed.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, A(b) = 0.55, B(a) = 0.5, B(b) = 0.6$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, B, U\}$ where $U \geq A$ and that of $FSC(X) = \{0_X, 1_X, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus A$. Consider the fuzzy set D defined by $D(a) = 0.6, D(b) = 0.5$. Clearly D is not fuzzy semiopen. Again any non-zero fuzzy set $V \leq A$ is not fg-closed. So other than 1_X, $1_X \setminus V \geq 1_X \setminus A$ is not fg-open. So 1_X is the only fg-open set in X such that $D < 1_X$ and so $sclD \leq 1_X$, therefore D is fgs^*-closed.

Example 2.6. There exists an fgs-closed set which is not fgs^*-closed.
Consider Example 2.5 and consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.8$. Now $C < 1_X (\in \tau)$ only and so

$sclC \leq 1_X$, therefore C is fgs-closed. Now $1_X \setminus C < B \in \tau$ and $cl(1_X \setminus C) = 1_X \setminus B < B$, hence $1_X \setminus C$ is fg-closed , i.e., C is fg-open in (X, τ). Now $C \leq C$, but $sclC = 1_X \not\leq C$, therefore C is not fgs^*-closed.

Remark 2.7. Every fs^*g-closed set is fgs^*-closed, but not conversely, as follows from the next example.

Example 2.8. There exists an fgs^*-closed set which is not fs^*g-closed.
Consider Example 2.5. Consider the fuzzy set E defined by $E(a) = E(b) = 0.4$. Clearly E is fg-open and so $E \leq E, sclE = E \leq E$, hence E is fgs^*-closed. But $clE = 1_X \setminus B \not\leq E$ implies E is not fs^*g-closed.

Remark 2.9 (i). fsg-closedness and fgs^*-closedness are independent notions, as follows from Example 2.10 and Example 2.11.
(ii) $f\alpha g$-closedness and fgs^*-closedness are independent notions, as follows from Example 2.12 and Example 2.13.

Example 2.10. There exists an fsg-closed set which is not fgs^*-closed.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Then $FSO(X) = \{0_X, 1_X, U\}$ where $A \leq U \leq 1_X \setminus A$ *and* $FSC(X) = \{0_X, 1_X, 1_X \setminus U\}$ where $A \leq 1_X \setminus U \leq 1_X \setminus A$. The collection of all fg-closed sets is $\{0_X, 1_X, V\}$ where $V > A$ and that of fg-open sets is $\{0_X, 1_X, 1_X \setminus V\}$, where $1_X \setminus V < 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then $B \leq B$ where B is fg-open in X and $sclB = A \not\leq B$. Hence B is not fgs^*-closed in X. But $B \leq A$ where A is fuzzy semiopen in X, hence $sclB = A \leq A$. Therefore B is fsg-closed in X.

Example 2.11. There exists an fgs^*-closed set which is not fsg-closed.
Consider Example 2.5. Here D is fgs^*-closed. Now $D < U$, where U being fuzzy semiopen in X defined by $U(a) = 0.6, U(b) = 0.55$. Then $sclD = 1_X \not\leq U$. Therefore, D is not fsg-closed.

Example 2.12. There exists an $f\alpha g$-closed set which is not fgs^*-closed.

Consider Example 2.6. Here C is not fgs^*-closed. But 1_X is the only fuzzy open set in X such that $C < 1_X$ and so $\alpha clC \leq 1_X$. Hence C is $f\alpha g$-closed.

Example 2.13. There exists an fgs^*-closed set which is not $f\alpha g$-closed.
Consider Example 2.10 and the fuzzy set A. Then A is fg-open in X with $A \leq A$ and $sclA = A \leq A$ implies A is fgs^*-closed set. But $A \leq A \in \tau, \alpha clA = 1_X \setminus A \nleq A$ and so A is not $f\alpha g$-closed. Infact, the collection of all fuzzy α-open sets in X is $\{0_X, 1_X, A\}$ and that of fuzzy α-closed sets is $\{0_X, 1_X, 1_X \setminus A\}$.

Remark 2.14 (i). Since for any two fuzzy sets A, B in an fts (X, τ), $scl(A \bigvee B) = sclA \bigvee sclB$, it is clear that union of any two fgs^*-closed sets is fgs^*-closed. So intersection of any two fgs^*-open sets is fgs^*-open.
(ii) Intersection of two fgs^*-closed sets need not be fgs^*-closed as it can be seen from the following example.

Example 2.15. Consider Example 2.10. Let a fuzzy set C be defined by $C(a) = 0.4, C(b) = 0.7$. Consider two fuzzy sets A and C. It is shown in Example 2.13 that A is fgs^*-closed. Now 1_X is the only fg-open set in X such that $C < 1_X$ and so $sclC < 1_X$ implies C is fgs^*-closed in X. But $D = A \bigwedge C$ defined by $D(a) = D(b) = 0.4$ is not fgs^*-closed in X. Indeed, D is fg-open set in X with $D \leq D$, but $sclD = A \nleq D$ and hence D is not fgs^*-closed.

Lemma 2.16. *Let $B \in I^X$ be an fgs^*-closed set in X. Then there does not exist an fg-closed set V such that $V \not{q} B$ and $V q sclB$.*

Proof. Assume that V is an fg-closed set such that $V \not{q} B$. Then $B \leq 1_X \setminus V$ where $1_X \setminus V$ is fg-open set in X. As B is fgs^*-closed, $sclB \leq 1_X \setminus V$, hence $V \not{q} sclB$.

Lemma 2.17. *Let (X, τ) be an fts. If $B \in I^X$ is fg-open and fgs^*-closed in X, then B is fuzzy semiclosed.*

Proof. $B \leq B$ implies $sclB \leq B$ (as B is fgs^*-closed). which shows that B is fuzzy semiclosed.

Corollary 2.18. *If $B \in I^X$* is *fuzzy open and fgs^*-closed, then B is fuzzy semiclosed.*

Lemma 2.19. *Let $B \in I^X$ in an fts (X, τ). Then the following statements are equivalent:*
(i) B is fuzzy regular open,
(ii) B is fuzzy open and fgs^*-closed.

Proof (i) $\Rightarrow$ (ii). B is fuzzy regular open implies B is fuzzy open. Let H be an fg-open set in X with $B \leq H$. Then $B \bigvee intclB = B \leq H$, hence $sclB \leq H$. Therefore B is fgs^*-closed.

(ii) $\Rightarrow$ (i). Let $B \in \tau$ and fgs^*-closed in X. Then B is fg-open in X with $B \leq B$. By hypothesis, $sclB \leq B$ and so $B \bigvee intclB \leq B$ implies $intclB \leq B$. Again as B is fuzzy open, $B \leq intclB$, hence $B = intclB$. Therefore B is fuzzy regular open in X.

Theorem 2.20. *Let $B \in I^X$ in an fts (X, τ). Then the following statements are equivalent :*
(i) $B \in \tau$ *and* $B \in \tau^c$,
(ii) $B \in \tau$, B is an fQ-set and *an* fgs^*-closed set.

Proof (i) $\Rightarrow$ (ii). By (i), $B = intB, B = clB$ and as a result $B = intclB = clintB$, therefore B is an fQ-set. Let H be an fg-open set in X such that $B \leq H$. Then $clB = B \leq H$, therefore $sclB \leq clB \leq H$. It follows that B is fgs^*-closed.

(ii) $\Rightarrow$ (i). By Lemma 2.19 (ii)$\Rightarrow$ (i), B is fuzzy regular open. Again B is an fQ-set implies $intclB = clintB$, hence $B = clB$, i.e., $B \in \tau^c$.

3. fgs^*-Continuity : Some Results

We first recall some definitions for ease of reference.

Definition 3.1. A fuzzy function $f : (X, \tau) \to (Y, \tau_1)$ is called
(i) fuzzy continuous [13] if $f^{-1}(V) \in \tau^c$ for every $V \in \tau_1^c$,
(ii) fgs-continuous [3] if $f^{-1}(V)$ is fgs-closed in X for every $V \in \tau_1^c$,
(iii) fs^*g-continuous [5] if $f^{-1}(V) \in \tau^c$ for every fg-closed set V in Y,
(iv) fuzzy open function [14] if $f(F)$ is fuzzy open in Y for every fuzzy open set F in X .

Definition 3.2. A fuzzy function $f : (X, \tau) \to (Y, \tau_1)$ is called fgs^*-continuous if $f^{-1}(V)$ is fgs^*-closed in X for every $V \in \tau_1^c$.

Theorem 3.3. *Every fuzzy continuous function is fgs^*-continuous.*

Proof. The proof follows from the fact that every fuzzy closed set is fgs^*-closed.

The converse of Theorem 3.3. is not true, as it can be seen from the following example.

Example 3.4. fgs^*-continuity does not imply fuzzy continuity
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.4$, $B(a) = 0.5, B(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Here $1_X \setminus B \in \tau_1^c, i^{-1}(1_X \setminus B) = 1_X \setminus B$ which is fgs^*-closed (as shown in Example 2.13) in (X, τ), but not fuzzy closed in (X, τ). Hence i is fgs^*-continuous but not fuzzy continuous.

Remark 3.5. By Remark 2.4(ii), every fgs^*-continuous function is fgs-continuous. But the converse *is not* true, as it can be seen from the following example.

Example 3.6. fgs-continuity does not imply fgs^*-continuity
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.4$, $A(b) = 0.55$, $B(a) = 0.5$, $B(b) = 0.6$, $C(a) = 0.5$, $C(b) = 0.2$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Then $1_X \setminus C \in \tau_1^c, i^{-1}(1_X \setminus C) = 1_X \setminus C$. Now 1_X is the only fuzzy open set in (X, τ) with $1_X \setminus C < 1_X$ and so $1_X \setminus C$ is fgs-closed in (X, τ), therefore i is an fgs-continuous function. But $1_X \setminus C$ is fg-open in (X, τ) and so $1_X \setminus C \leq 1_X \setminus C$ implies $scl(1_X \setminus C) = 1_X \not\leq 1_X \setminus C$. It follows that the function i is not fgs^*-continuous.

Remark 3.7. The inverse image of some fgs^*-closed set under fgs^*-continuous function may not be fgs^*-closed as it can be seen from the following example.

Example 3.8. Consider Example 3.4. Let us consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.3$. Now the collection of all fg-open as well as $FSO(X, \tau_1)$ is $\{0_X, 1_X, W\}$ where $W \geq B$ and

$FSC(X, \tau_1) = \{0_X, 1_X, 1_X \setminus W\}$ where $1_X \setminus W \leq 1_X \setminus B$. Now $C < B$ where B is fg-open in (X, τ_1) and $sclC = C < B$ implies C is fgs^*-closed in (X, τ_1). But $i^{-1}(C) = C < C$ where C is fg-open in (X, τ) and $sclC = A \nleq C$. It follows that C is not fgs^*-closed in (X, τ).

Remark 3.9. Since a fgs^*-closed set is not necessarily fuzzy closed always, the composition of two fgs^*-continuous functions need not be fgs^*-continuous.

To achieve this result we have to define some new type of fuzzy space.

Definition 3.10. An fts (X, τ) is called fT_{gs^*}-space if every fgs^*-closed set is fuzzy closed.

Theorem 3.11. *Let $f : (X, \tau) \to (Y, \tau_1)$ and $g : (Y, \tau_1) \to (Z, \tau_2)$ be two fgs^*-continuous functions where (Y, τ_1) is fT_{gs^*}-space. Then the* function *$g \circ f : (X, \tau) \to (Z, \tau_2)$ is fgs^*-continuous.*

Proof. The proof is obvious.

Theorem 3.12. *Every fs^*g-continuous function is fgs^*-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \tau_1)$ be fs^*g-continuous and $V \in \tau_1^c$. Then V is fg-closed in Y. As f is fs^*g-continuous, $f^{-1}(V) \in \tau^c$ implies $f^{-1}(V)$ is fgs^*-closed in X. Hence the proof.

But the converse may not be true as it can be seen from the following example.

Example 3.13. fgs^*-continuity does not imply fs^*g-continuity
Consider Example 3.4. Here the fuzzy set F defined by $F(a) = 0.5, F(b) = 0.7$ is fg-closed in (X, τ_1). Now $i^{-1}(F) = F \notin \tau^c$, therefore i is not fs^*g-continuous. But here i is fgs^*-continuous (as shown in Example 3.4).

Remark 3.14. In an fT_{gs^*}-space, fgs^*-closed set is fsg-closed, fg-closed.

4. gs^*-Closure Operator and fgs^*-Closed (Open) Functions : Some Properties

In this section a new type of closure (interior) operator, viz. gs^*-closure (resp., gs^*-interior) operator is introduced and studied. It is shown that this operator is an idempotent operator. Also a new type of fuzzy closed (resp., fuzzy open) function is introduced and characterized. It is shown that this class of fuzzy closed functions is strictly larger than the class of fgs-closed functions.

Definition 4.1. The intersection of all fgs^*-closed sets containing a fuzzy set A in an fts (X, τ) is called gs^*-closure of A to be denoted by gs^*clA, i.e., $gs^*cl(A) = \bigwedge\{F : A \leq F$ and F is fgs^*-closed in $X\}$.

Now we recall a definition from [8] for ease of reference.

Definition 4.2 [8]. The intersection of all fgs-closed sets containing a fuzzy set A in an fts (X, τ) is called gs-closure of A and will be denoted by $gscl(A)$, i.e., $gscl(A) = \bigwedge\{F : A \leq F$ and F is fgs-closed in $X\}$.

Remark 4.3. Since every fgs^*-closed set is fgs-closed, it is clear that $A \leq gscl(A) \leq gs^*cl(A) \leq clA$.

Remark 4.4. If a fuzzy set A is fgs^*-closed, then $A = gs^*cl(A)$. But $gs^*cl(A)$ may not be fgs^*-closed since intersection of two fgs^*-closed sets may not be so, as it is seen in Example 2.15.

Theorem 4.5. *Let (X, τ) be an fts and $A \in I^X$. Then for a fuzzy point x_t in X, $x_t \in gs^*cl(A)$* if and only if for *every fgs^*-open set U, x_tqU implies UqA.*

Proof. Let $x_t \in gs^*cl(A)$ and *let* U be any fgs^*-open set in X with x_tqU. Then $x_t \in F$ for every fgs^*-closed set F containing A. Now $U(x) + t > 1$ implies $x_t \notin 1_X \setminus U$, but $1_X \setminus U$ is an fgs^*-closed set in X and so $A \not\leq 1_X \setminus U$ shows that there exists $y \in X$ such that $A(y) > 1 - U(y)$, hence AqU.

Conversely, assume that for every fgs^*-open set U, x_tqU implies UqA. We have to prove that $x_t \in F$, for *every* fgs^*-closed *set* F in X containing A. Let F be *an* fgs^*-closed set in X containing A. *Assume by contrary that* $x_t \notin F$. Then $F(x) < t$ implies $1 - F(x) > 1 - t$, i.e., $x_t q(1_X \setminus F)$, but $1_X \setminus F$ is fgs^*-open in X and so by hypothesis, $Aq(1_X \setminus F)$. Then there exists $y \in X$ such that $A(y) + 1 - F(y) > 1$, hence $A(y) > F(y)$, a contradiction.

Theorem 4.6. *Let (X, τ) be an fts and $A, B \in I^X$. Then the following statements are true :*
(i) $gs^*cl(0_X) = 0_X$,
(ii) $gs^*cl(1_X) = 1_X$,
(iii) If $A \leq B$, then $gs^*cl(A) \leq gs^*cl(B)$,
(iv) $gs^*cl(A \bigvee B) = gs^*cl(A) \bigvee gs^*cl(B)$,
(v) $gs^*cl(A \bigwedge B) \leq gs^*cl(A) \bigwedge gs^*cl(B)$
(vi) $gs^*cl(gs^*cl(A)) = gs^*cl(A)$.

Proof. The proofs of (i), (ii) and (iii) are obvious.
(iv) $gs^*cl(A) \bigvee gs^*cl(B) \leq gs^*cl(A \bigvee B)$ follows from (iii).

To prove the converse, let $x_t \in gs^*cl(A \bigvee B)$. Then by Theorem 4.5, for any fgs^*-open set U in X with $x_t qU$, we have $Uq(A \bigvee B)$. Then there exists $y \in X$ such that $U(y) + max\{A(y), B(y)\} > 1$ which shows that either $U(y) + A(y) > 1$ or $U(y) + B(y) > 1$. Then either UqA or UqB, therefore either $x_t \in gs^*cl(A)$ or $x_t \in gs^*cl(B)$, hence $x_t \in gs^*cl(A) \bigvee gs^*cl(B)$.
(v) Follows from (iii).
(vi) From definition, $A \leq gs^*cl(A)$ implies $gs^*cl(A) \leq gs^*cl(gs^*cl(A))$ by (iii).

To prove the converse, let $x_t \in gs^*cl(gs^*cl(A)) = gs^*cl(B)$ where $B = gs^*cl(A)$. Let U be any fgs^*-open set in X with $x_t qU$. Then UqB, therefore there exists $y \in X$ such that $U(y) + B(y) > 1$. Let $B(y) = t$. Then $y_t \in B$ and $y_t qU$. Since $y_t \in gs^*cl(A)$, UqA by Theorem 4.5, which implies $x_t \in gs^*cl(A)$ and hence $gs^*cl(gs^*cl(A)) \leq gs^*cl(A)$.

In Theorem 4.6 (v), equality does not hold in general, as it can be seen in Example 4.7.

Example 4.7. Consider Example 2.15. Here A and C being fgs^*-closed sets in X, $A = gs^*cl(A), C = gs^*cl(C)$ and so $gs^*cl(A) \bigwedge gs^*cl(C) = A \bigwedge C = D \neq gs^*cl(D) = gs^*cl(A \bigwedge C)$ as D is not fgs^*-closed in X.

Definition 4.8. A function $f : X \to Y$ is said to be fgs^*-closed (resp., fgs^*-open) if $f(F)$ is fgs^*-closed (resp., fgs^*-open) in Y for every fuzzy closed (resp., fuzzy open) set F in X.

Remark 4.9. It is clear from definition that every fgs^*-closed (resp., fgs^*-open) function is fgs-closed (resp., fgs-open) function. The converse does not hold, as it can be seen from the following example.

Example 4.10. There exists an fgs-closed function which is not fgs^*-closed.
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Now $FSO(X, \tau_2) = \{0_X, 1_X, U\}$ where $A \leq U \leq 1_X \setminus A$ and $FSC(X, \tau_2) = \{0_X, 1_X, 1_X \setminus U\}$ where $A \leq 1_X \setminus U \leq 1_X \setminus A$. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_1^c, i(1_X \setminus B) = 1_X \setminus B$ ($= D$, say) is not fgs^*-closed in (X, τ_2) as shown in Example 2.15. So i is not fgs^*-closed function. We claim that i is fgs-closed function. Indeed, $1_X \setminus B < A \in \tau_2$, hence $scl_{\tau_2}(1_X \setminus B) = A \leq A$, which implies that $1_X \setminus B$ is fgs-closed in (X, τ_2) and so i is fgs-closed function.

Theorem 4.11. *If $f : X \to Y$ is an fgs^*-closed function, then $gs^*cl(f(A)) \leq f(clA)$ for all $A \in I^X$.*

Proof. Let $A \in I^X$. Then clA is fuzzy closed in X. As f is fgs^*-closed function, $f(clA)$ is fgs^*-closed in Y. Now $f(A) \leq f(clA)$. So $gs^*cl(f(A)) \leq gs^*cl(f(clA)) = f(clA)$.

Definition 4.12. The union of all fgs^*-open sets contained in a fuzzy set A in an fts X is called $gs^*int(A)$.

Remark 4.13. It is clear from definitions that for a fuzzy set A in an fts (X, τ), $intA \leq gs^*intA \leq gsintA \leq A$.

Lemma 4.14. *For a fuzzy set A in an fts (X, τ), the following statements are equivalent:*
(i) $gs^*cl(1_X \setminus A) = 1_X \setminus gs^*int(A)$
(ii) $gs^*int(1_X \setminus A) = 1_X \setminus gs^*cl(A)$.

Proof (i). Let $x_t \in gs^*cl(1_X \setminus A)$. If possible, let $x_t \notin 1_X \setminus gs^*int(A)$. Then $1 - (gs^*int(A))(x) < t$, i.e. $[gs^*int(A)](x) + t > 1$, hence $gs^*int(A)qx_t$. Then there exists at least one fgs^*-open set $F \leq A$ with x_tqF, which shows that x_tqA. As $x_t \in gs^*cl(1_X \setminus A)$ and $Fq(1_X \setminus A)$, it follows that $Aq(1_X \setminus A)$, a contradiction. Hence

$$gs^*cl(1_X \setminus A) \leq 1_X \setminus gs^*int(A) \tag{1}$$

Conversely, let $x_t \in 1_X \setminus gs^*int(A)$. Then

$$1 - [gs^*int(A)](x) \geq t \;\Rightarrow x_t \not{q}(gs^*int(A)) \Rightarrow x_t \not{q} F \tag{2}$$

where F is any fgs^*-open set contained in A .
Let U be any fgs^*-closed set in X such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$. Now $1_X \setminus U$ is fgs^*-open set in X contained in A. By (2), $x_t \not{q}(1_X \setminus U)$ implies $x_t \in U$. Therefore $x_t \in gs^*cl(1_X \setminus A)$ and so

$$1_X \setminus gs^*int(A) \leq gs^*cl(1_X \setminus A). \tag{3}$$

Combining (1) and (3), (i) follows.

(ii) Putting $1_X \setminus A$ for A in (i), we get $gs^*cl(A) = 1_X \setminus gs^*int(1_X \setminus A)$, hence $gs^*int(1_X \setminus A) = 1_X \setminus gs^*cl(A)$.

Theorem 4.15. *For a bijective function $f : X \to Y$, the following statements are equivalent:*
(i) f is fgs^*-open,
(ii) $f(intA) \leq gs^*int(f(A))$, for all $A \in I^X$,
(iii) For each fuzzy point x_t in X and each fuzzy open set U in X containing x_t, there exists an fgs^*-open set V containing $f(x_t)$ such that $V \leq f(U)$.

Proof (i) $\Rightarrow$ (ii). Let $A \in I^X$. Then $intA$ is fuzzy open in X. By (i), $f(intA)$ is fgs^*-open in Y. Since $f(intA) \leq f(A)$ and $gs^*int(f(A))$ is the union of all fgs^*-open sets contained in $f(A)$, we have $f(intA) \leq gs^*int(f(A))$.

(ii) $\Rightarrow$ (i). Let U be a fuzzy open set in X. Then $f(U) = f(intU) \leq gs^*int(f(U))$ (by (ii)), hence $f(U)$ is fgs^*-open in Y.

(ii) $\Rightarrow$ (iii). Let x_t be a fuzzy point in X and let U be a fuzzy open set in X such that $x_t \in U$. Then $f(x_t) \in f(U) = f(intU) \leq gs^*int(f(U))$, hence $f(U)$ is fgs^*-open in Y. Let $V = f(U)$. Then $f(x_t) \in V$ and $V \leq f(U)$.

(iii) $\Rightarrow$ (i). Let U be any fuzzy open set in X and y_t be any fuzzy point in $f(U)$, i.e., $y_t \in f(U)$. Then there exists $x \in X$ such that $f(x) = y$ (as f is bijective). Then $[f(U)](y) \geq t$, hence $U(f^{-1}(y)) \geq t$, therefore $U(x) \geq t$, which implies $x_t \in U$. By (iii), there exists an fgs^*-open set V in Y such that $f(x_t) \in V$ and $V \leq f(U)$. Then $f(x_t) \in V = gs^*int(V) \leq gs^*int(f(U))$. Since x_t is taken arbitrarily and $f(U)$ is the union of all fuzzy points in $f(U)$, $f(U) \leq gs^*int(f(U))$, hence $f(U)$ is fgs^*-open in Y. It follows that the function f is fgs^*-open.

Theorem 4.16. *If $f : X \to Y$ is fgs^*-open, then the following statements are true :*
(i) For each fuzzy point x_t in X and each fuzzy open set U in X with x_tqU, there exists an fgs^*-open set V in Y with $f(x_t)qV$ such that $V \leq f(U)$,
(ii) $f^{-1}(gs^*cl(B)) \leq cl(f^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_t be any fuzzy point in X and U be any fuzzy open set in X with $x_tqU = intU$ implies $f(x_t)qf(intU) \leq gs^*int(f(U))$ (by Theorem 4.15) which shows that $f(x_t)qgs^*int(f(U))$ and hence there exists fgs^*-open set V in Y such that $f(x_t)qV$ and $V \leq f(U)$.

(ii) Let x_t be any fuzzy point in X such that $x_t \notin cl(f^{-1}(B))$ for any $B \in I^Y$. Then there exists a fuzzy open set U in X with x_tqU, $U\not{q}f^{-1}(B)$. Now

$$f(x_t)qf(V) \tag{4}$$

where $f(U)$ is fgs^*-open in Y (as f is fgs^*-open function). Now $f^{-1}(B) \leq 1_X \setminus U$ implies $B \leq f(1_X \setminus U) \leq 1_Y \setminus f(U)$ and so $B\not{q}f(U)$. Let $V = 1_Y \setminus f(U)$. Then V is fgs^*-closed in Y with $B \leq V$. We claim that $f(x_t) \notin V$. Assume that $f(x_t) \in V = 1_Y \setminus f(U)$. Then $1 - [f(U)](f(x)) \geq t$, hence $f(U)\not{q}f(x_t)$, which contradicts (4). So $f(x_t) \notin V$ implies $f(x_t) \notin gs^*cl(B)$ and therefore $x_t \notin f^{-1}(gs^*cl(B))$. Hence $f^{-1}(gs^*cl(B)) \leq cl(f^{-1}(B))$.

Theorem 4.17. *If $f : X \to Y$ is an injective, fgs^*-open function, $B \in I^Y$ and F is a fuzzy closed set in X with $f^{-1}(B) \leq F$, then there* *exists an fgs^*-closed set V in Y such that $B \leq V$ and $f^{-1}(V) \leq F$.*

Proof. Let $B \in I^Y$ and F be a fuzzy closed set in X with $f^{-1}(B) \leq F$. Then $1_X \setminus f^{-1}(B) \geq 1_X \setminus F$, where $1_X \setminus F$ is fuzzy open in X, implies $f(1_X \setminus F) \leq f(1_X \setminus f^{-1}(B)) \leq 1_Y \setminus B$ (as f is injective) where $f(1_X \setminus F)$ is an fgs^*-open in Y. Let $V = 1_Y \setminus f(1_X \setminus F)$. Then V is fgs^*-closed in Y such that $B \leq 1_Y \setminus f(1_X \setminus F) = V$. Now $f^{-1}(V) = f^{-1}(1_Y \setminus f(1_X \setminus F)) = 1_X \setminus f^{-1}(f(1_X \setminus F)) \leq F$.

5. fgs^*-Neighbourhood of a Fuzzy Point and a Fuzzy Set

In this section a new type of fuzzy neighbourhood nbd, for short) system, viz., fgs^*-nbd system of a fuzzy point is introduced which is coarser than the fuzzy nbd system.

Definition 5.1. A fuzzy set A in an fts (X, τ) is called an fgs^*-nbd of a fuzzy point x_t in X if there exists an fgs^*-open set G in X such that $x_t \in G \leq A$.

Definition 5.2. A fuzzy set A in an fts (X, τ) is called an fgs^*-nbd of a fuzzy set B in X if there exists an fgs^*-open set G in X such that $B \leq G \leq A$.

Remark 5.3. The fgs^*-nbd of a fuzzy point x_t need not be an fgs^*-open set in X, as it can be seen from the following example.

Example 5.4. Consider Example 2.15 and the fuzzy point $a_{0.5}$. Here $1_X \setminus D$ is not fgs^*-open. Now $(1_X \setminus D)(a) = 0.6 > 0.5$, hence $a_{0.5} \in 1_X \setminus D$. Now $(1_X \setminus A)(a) = 0.5 \geq 0.5$ implies $a_{0.5} \in 1_X \setminus A \leq 1_X \setminus D$ where $1_X \setminus A$ is fgs^*-open in X and hence $1_X \setminus D$ is an fgs^*-nbd of $a_{0.5}$.

It is clear from definition that

Theorem 5.5. Every nbd of a fuzzy point x_t is an fgs^*-nbd of this point.

But the converse may not be true, as it can be seen from the following example.

Example 5.6. Consider Example 2.15 and a fuzzy point $a_{0.6}$. Here $1_X \setminus C$ being an fgs^*-open set with $a_{0.6} \in 1_X \setminus C$ is an fgs^*-nbd of

$a_{0.6}$. But $1_X \setminus C$ is not a fuzzy nbd of $a_{0.6}$.

Remark 5.7. An fgs^*-open set is an fgs^*-nbd of each of its points.

But the converse may not be true in general, as it can be seen from the following example.

Example 5.8. Consider Example 2.15. Here $1_X \setminus D$ is not fgs^*-open set. We claim that $1_X \setminus D$ is an fgs^*-nbd of each of its points. The points of $1_X \setminus D$ are either of the form a_t, $(0 < t \leq 0.6)$ or of the form b_{t_1}, $(0 < t_1 \leq 0.6)$. For a_t, $(0 < t \leq 0.6)$, $a_t \in 1_X \setminus C \leq 1_X \setminus D$ where $1_X \setminus C$ is an fgs^*-open set in X. For b_{t_1}, $(0 < t_1 \leq 0.6)$, $b_{t_1} \in 1_X \setminus A \leq 1_X \setminus D$ where $1_X \setminus A$ is an fgs^*-open set in X. So $1_X \setminus D$ is an fgs^*-nbd of each of its points.

Theorem 5.9. *Let $F \in I^X$ be fgs^*-closed set in X and $x_t \in 1_X \setminus F$. Then there exists an fgs^*-nbd G of x_t such that $G \not{q} F$.*

Proof. $1_X \setminus F$ is fgs^*-open in X. By Remark 5.7, there exists an fgs^*-nbd G of x_t such that $x_t \in G \leq 1_X \setminus F$ implies $G \not{q} F$.

Definition 5.10. The set of all fgs^*-nbds of a fuzzy point x_t $(0 < t \leq 1)$ in an fts (X, τ) is called the fgs^*-nbd system at x_t, to be denoted by $fgs^* - N(x_t)$.

Theorem 5.11. *For a fuzzy point x_t in an fts (X, τ), the following statements hold :*
(i) $fgs^* - N(x_t) \neq \emptyset$,
(ii) If $G \in fgs^* - N(x_t)$, then $x_t \in G$,
(iii) If $G \in fgs^* - N(x_t)$, $F \geq G$, then $F \in fgs^* - N(x_t)$,
(iv) $F, G \in fgs^* - N(x_t)$ implies $F \bigwedge G \in fgs^* - N(x_t)$,
(v) If $G \in fgs^* - N(x_t)$, then there exists $F \in fgs^* - N(x_t)$ such that $F \leq G$ and $F \in fgs^* - N(y_{t'})$ for every $y_{t'} \in F$.

Proof. (i) Since 1_X is an fgs^*-open set, it is an fgs^*-nbd of any fuzzy point x_t $(0 < t \leq 1)$ and so $fgs^* - N(x_t) \neq \emptyset$.

(ii) and (iii) follow from Definition 5.10.

(iv) Follows from Remark 2.14 (i).

(v) Follows from Definition 5.10 and Remark 5.7.

Theorem 5.12. *Let x_t be a fuzzy point in an fts (X,τ). Let $fgs^* - N(x_t)$ be a non-empty collection of fuzzy sets in X satisfying the following conditions :*
(1) $G \in fgs^* - N(x_t)$ implies $x_t \in G$,
(2) $F, G \in fgs^* - N(x_t)$ implies $F \bigwedge G \in fgs^* - N(x_t)$.
Let τ consist of 0_X and all those non zero fuzzy sets G of X having the property that for every *$x_t \in G$ there exists an $F \in fgs^* - N(x_t)$ such that $x_t \in F \leq G$. Then τ is a fuzzy topology on X.*

Proof. (i) By hypothesis, $0_X \in \tau$.
(ii) It is clear from the given property of τ that $1_X \in \tau$ as 1_X is an $fgs^* - N(x_t)$ for every fuzzy point $x_t (0 < t \leq 1)$ in an fts X by (1).
(iii) Let $G_1, G_2 \in \tau$. If $G_1 \bigwedge G_2 = 0_X$, then $G_1 \bigwedge G_2 \in \tau$. If $G_1 \bigwedge G_2 \neq 0_X$, let $x_t \in G_1 \bigwedge G_2$ (where $0 < t \leq 1$). Then $G_1(x) \geq t, G_2(x) \geq t$. Since $G_1, G_2 \in \tau$, there exist $F_1, F_2 \in fgs^* - N(x_t)$ such that $x_t \in F_1 \leq G_1$, $x_t \in F_2 \leq G_2$. Then $x_t \in F_1 \bigwedge F_2 \leq G_1 \bigwedge G_2$ and by (2), $F_1 \bigwedge F_2 \in fgs^* - N(x_t)$ implies $G_1 \bigwedge G_2 \in \tau$ by construction of τ.
(iv) Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ where $G_\alpha \in \tau$, for all $\alpha \in \Lambda$. Let $x_t \in \bigvee_{\alpha \in \Lambda} G_\alpha$. Then $x_t \in G_\beta$, for some $\beta \in \Lambda$. By construction of τ, there exists $F_\beta \in fgs^* - N(x_t)$ such that $x_t \in F_\beta \leq G_\beta \leq \bigvee_{\alpha \in \Lambda} G_\alpha$ implies $\bigvee_{\alpha \in \Lambda} G_\alpha \in \tau$.
It follows that τ is a fuzzy topology on X.

6. Applications

In this section it is shown that fuzzy normal and fuzzy compact spaces remain invariant under fgs^*-continuous functions.

Definition 6.1 [11]. An fts (X,τ) is called fuzzy normal if for any two fuzzy closed sets A, B in X with $A \not{q} B$, there exist $U, V \in \tau$ such that $A \leq U, B \leq V$ and $U \not{q} V$.

Theorem 6.2. Let $f : (X,\tau) \to (Y,\tau_1)$ be a bijective, fgs^*-continuous, open function. If X is fuzzy normal and fT_{gs^*}-space, then Y is also fuzzy normal.

Proof. Let $A, B \in \tau_1^c$ be such that $A \not{q} B$. Then $f^{-1}(A) \not{q} f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are fgs^*-closed in X. As X is fT_{gs^*}-space, $f^{-1}(A), f^{-1}(B) \in \tau^c$. As X is fuzzy normal, there exist $U, V \in \tau$ such that $f^{-1}(A) \leq U, f^{-1}(B) \leq V$ and $U \not{q} V$. As f is bijective, $A \leq f(U), B \leq f(V)$ and $f(U) \not{q} f(V)$ where $f(U), f(V) \in \tau_1$ and hence Y is fuzzy normal space.

We now recall the following definition from [3] for easy reference

Definition 6.3. An fts (X, τ) is called fT_b-space if every fgs-closed set is fuzzy closed.

Theorem 6.4. *Let $f : (X, \tau) \to (Y, \tau_1)$ be a bijective, fgs^*-continuous, open function. If X is fuzzy normal and fT_b-space, then Y is fuzzy normal.*

Proof. The proof is similar to that of Theorem 6.2.

Let us now recall the following two definitions from [10] for ready reference.

Definition 6.5. Let (X, τ) be an fts. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, all the members of $\mathcal{U}$ are fuzzy open in X, $\mathcal{U}$ is called a fuzzy open cover of X.

Definition 6.6. An fts (X, τ) is said to be fuzzy compact if every fuzzy open cover $\mathcal{U}$ has a finite subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.

Theorem 6.7. *Let $f : (X, \tau) \to (Y, \tau_1)$ be an fgs^*-continuous function from a fuzzy compact, fT_{gs^*}-space X onto an fts Y. Then Y is fuzzy compact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of Y. Then $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ being an fgs^*-open cover is a fuzzy open cover of X (as f is fgs^*-continuous function and also X is an fT_{gs^*}-space). Since X is fuzzy compact, there exists a finite subset Λ_0 of Λ

such that $\bigcup_{\alpha\in\Lambda_0} f^{-1}(U_\alpha) = 1_X$. Hence $1_Y = f(1_X) = f(\bigcup_{\alpha\in\Lambda_0} f^{-1}(U_\alpha)) = \bigcup_{\alpha\in\Lambda_0} ff^{-1}(U_\alpha) \leq \bigcup_{\alpha\in\Lambda_0} U_\alpha \Rightarrow Y$ is fuzzy compact space.

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[1]

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Department of Mathematics, Victoria Institution (College)
78 B, A.P.C. Road , Kolkata - 700009, INDIA
e-mail: anjanabhattacharyya@hotmail.com