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GROWTH AND OSCILLATION OF SOLUTIONS TO HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF FINITE LOGARITHMIC ORDER

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Abstract. In this paper, we study the growth of solutions of complex higher order linear differential equations with entire or meromorphic coefficients of finite logarithmic order. We extend some precedent results due to L. Kinnunen; B. Belaïdi; J. Liu, J. Tu and L. Z. Shi; H. Hu, X. M. Zheng; T. B. Cao, K. Liu and J. Wang and others. We also consider the fixed points of solutions in this paper.

1. Introduction

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's theory (see e.g. [13, 27]). For $r \in [0, +\infty)$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. We also denote $\exp_0 r := r = \log_0 r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

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Furthermore, we define the linear measure of a set $E \subset [0,+\infty)$ by $m(E) = \int_E dt$ and the logarithmic measure of a set $F \subset [1,+\infty)$ by $m_l(F) = \int_F \frac{dt}{t}$. Now, we shall introduce the definition of meromorphic functions of $[p,q]$-order and $[p,q]$-type, where p, q are positive integers satisfying $p \geq q \geq 1$ or $2 \leq q = p+1$. In order to keep accordance with the definition of iterated order (see e.g. [21, 26]), we will give a minor modification to the original definition of $[p,q]$-order (see e.g. [19, 20]).

Definition 1.1 ([23, 24]). Let $1 \leq q \leq p$ or $2 \leq q = p+1$. The (p,q)-order of a meromorphic function f is defined by

$$\sigma_{(p.q)}(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log_q r},$$

where $T(r,f)$ is the characteristic function of Nevanlinna of the function f. If f is an entire function, then

$$\sigma_{(p.q)}(f) = \overline{\lim_{r\to+\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r},$$

where $M(r,f)$ is the maximum modulus of f in the circle $|z| = r$.

Remark 1.1 (i) By the Definition 1.1, it is clear that $\sigma_{(p,1)}(f) = \sigma_p(f)$ is the iterated p-order of f (see e.g. [21, 26]) and in particular, we have that $\sigma_{(1,1)}(f) = \sigma(f)$ and $\sigma_{(2,1)}(f) = \sigma_2(f)$ are the order and hyper-order of f, respectively (see e.g. [13, 27]).
(ii) $\sigma_{(1,2)}(f) = \sigma_{\log}(f)$ is the logarithmic order of f (see e.g. [9]).
(iii) The logarithmic order of any non-constant rational function f is one, and therefore, any transcendental meromorphic function in the plane has logarithmic order no less than one. However, a function of logarithmic order one is not necessarily a rational function. Constant functions have zero logarithmic order, while there are no meromorphic functions of logarithmic order between zero and one. In addition, any meromorphic function with finite logarithmic order in the plane is of order zero.

Definition 1.2 ([18]). Let $1 \leq q \leq p$ or $2 \leq q = p+1$. The lower (p,q)-order of a meromorphic function $f(z)$ is defined by

$$\mu_{(p.q)}(f) = \underline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log_q r}.$$

If $f(z)$ is an entire function, then

$$\mu_{(p.q)}(f) = \underline{\lim}_{r \to +\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Remark 1.2 $\mu_{(1,2)}(f) = \mu_{\log}(f)$ is the logarithmic lower order of $f(z)$.

Definition 1.3 ([21]) . The finiteness degree of growth $i(f)$ of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is rational,} \\ \min\{j \in \mathbb{N} : \sigma_{(j,1)}(f) < +\infty\}, & \text{if } f \text{ is transcendental} \\ \text{and } \sigma_{(j,1)}(f) < +\infty \text{ for some } j \in \mathbb{N}, & \\ +\infty, & \text{if } \sigma_{(j,1)}(f) = +\infty, \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.4 ([23, 24]) . Let $1 \leq q \leq p$ or $2 \leq q = p+1$. The (p,q)-type of a meromorphic function $f(z)$ with $0 < \sigma_{(p,q)}(f) < +\infty$ is defined by

$$\tau_{(p.q)}(f) = \overline{\lim_{r \to +\infty}} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

If $f(z)$ is an entire function with $0 < \sigma_{(p,q)}(f) < +\infty$, then

$$\tau_{(p.q)}(f) = \overline{\lim_{r \to +\infty}} \frac{\log_p M(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

Definition 1.5 ([18]) . Let $1 \leq q \leq p$ or $2 \leq q = p+1$. The lower (p,q)-type of a meromorphic function $f(z)$ with $0 < \mu_{(p,q)}(f) < +\infty$ is defined by

$$\underline{\tau}_{(p.q)}(f) = \underline{\lim}_{r \to +\infty} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\mu_{(p,q)}(f)}}.$$

If $f(z)$ is an entire function with $0 < \mu_{(p,q)}(f) < +\infty$, then

$$\underline{\tau}_{(p.q)}(f) = \underline{\lim}_{r \to +\infty} \frac{\log_p M(r,f)}{(\log_{q-1} r)^{\mu_{(p,q)}(f)}}.$$

Remark 1.3 $\tau_{(1,1)}(f) = \tau(f)$ is the type of $f(z)$, $\underline{\tau}_{(1,1)}(f) = \underline{\tau}(f)$ is the lower type of $f(z)$, $\tau_{(1,2)}(f) = \tau_{\log}(f)$ is the logarithmic type of $f(z)$ and $\underline{\tau}_{(1,2)}(f)$ is the logarithmic lower type of $f(z)$.

Definition 1.6 ([23, 24]) . Let $1 \leq q \leq p$ or $2 \leq q = p+1$. The (p,q)-exponent of convergence of the sequence of a-points of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}(f-a)=\overline{\lim_{r\to+\infty}}\frac{\log_p n(r,\frac{1}{f-a})}{\log_q r}$$

and the (p,q)-exponent of convergence of the sequence of distinct a-points of a meromorphic function $f(z)$ is defined by

$$\bar{\lambda}_{(p.q)}(f-a)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f-a})}{\log_q r}.$$

If $a=0$, the (p,q)-exponent of convergence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p n(r,\frac{1}{f})}{\log_q r}$$

and the (p,q)-exponent of convergence of distinct zeros of a meromorphic function $f(z)$ is defined by

$$\bar{\lambda}_{(p.q)}(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q r},$$

where $n\left(r,\frac{1}{f}\right)$ (or $\overline{n}\left(r,\frac{1}{f}\right)$) denotes the number of zeros (or distinct zeros) of f in the disc $|z|\leq r$. If $a=\infty$, the (p,q)-exponent of convergence of the sequence of poles of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}\left(\frac{1}{f}\right)=\overline{\lim_{r\to+\infty}}\frac{\log_p n(r,f)}{\log_q r}.$$

The (p,q)-exponent of convergence of the sequence of distinct fixed points of a meromorphic function $f(z)$ is defined by

$$\bar{\lambda}_{(p.q)}(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f-z})}{\log_q r}.$$

Remark 1.4 (i) $\lambda_{(1,2)}(f)=\lambda_{\log}(f)$ is the logarithmic exponent of convergence of zeros of $f(z)$, (see e.g. [9]) and $\overline{\lambda}_{(1,2)}(f)=\overline{\lambda}_{\log}(f)$ is the logarithmic exponent of convergence of distinct zeros of $f(z)$.
(ii) For the case $p\geq q\geq 1$, $\lambda_{(p.q)}(f)$ (or $\bar{\lambda}_{(p.q)}(f)$) can also be defined by using $N(r,\frac{1}{f})$ (or $\overline{N}(r,\frac{1}{f})$) instead of $n(r,\frac{1}{f})$ (or $\overline{n}(r,\frac{1}{f})$) respectively (see e.g. [24]), yet, it does not hold for the case $p<q$. For example, the logarithmic order of $N(r,\frac{1}{f})$ is equal to $\lambda_{\log}(f)+1$, (see [9], Theorem 4.1).

Definition 1.7 ([21]). The finiteness degree of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$i_{\lambda}(f)=\begin{cases}0, & \text{if } n(r,\frac{1}{f})=O(\log r),\\ \min\{j\in\mathbb{N}:\lambda_{(j,1)}(f)<+\infty\}, & \text{if } \lambda_{(j,1)}(f)<+\infty \\ & \text{for some } j\in\mathbb{N},\\ +\infty, & \text{if } \lambda_{(j,1)}(f)=+\infty, \text{ for all } j\in\mathbb{N}.\end{cases}$$

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.1}$$

where $A_0(z)\not\equiv 0$. When the coefficients $A_0(z),A_1(z),\cdots,A_{k-1}(z)$ are entire functions, it is well-known that all solutions of (1.1) are entire functions, and that if some coefficients of (1.1) are transcendental then (1.1) has at least one solution with infinite order. As much as we know, the idea of the iterated order was first introduced by Bernal [4] to express the fast growth of solutions of complex linear differential equations. From then, many authors achieved further results on the iterated order of solutions of (1.1), (see e.g. [1, 2, 3, 6, 7, 17, 21]).

Theorem A ([1]). *Let $A_j(z)$ $(j=0,1,\cdots,k-1)$ be entire functions, and let $i(A_0)=p$ $(0<p<\infty)$. Assume that either*

$$\max\{i(A_j):j=1,2,\cdots,k-1\}<p$$

or

$$\max\{\sigma_p(A_j):j=1,2,\cdots,k-1\}\leq\sigma_p(A_0):=\sigma\ (0<\sigma<\infty),$$
$$\max\{\tau_p(A_j):\sigma_p(A_j)=\sigma_p(A_0)\}<\tau_p(A_0):=\tau\ (0<\tau<\infty).$$

Then every solution $f(z)\not\equiv 0$ of (1.1) satisfies $i(f)=p+1$ and $\sigma_{p+1}(f)=\sigma_p(A_0)$.

For the case when the dominant coefficient in (1.1) is of finite iterated lower order, Hu and Zheng in [17], investigated the growth of solutions of (1.1) and obtained the following result.

Theorem B ([17]). *Let $A_j(z)$ $(j=0,1,\cdots,k-1)$ be entire functions of finite iterated order satisfying*

$$\max\{\sigma_p(A_j):j=1,2,\cdots,k-1\}<\mu_p(A_0)\leq\sigma_p(A_0)<\infty.$$

Then every solution $f(z)\not\equiv 0$ of (1.1) satisfies

$$\mu_p(A_0)=\mu_{p+1}(f)\leq\sigma_{p+1}(f)=\sigma_p(A_0).$$

In [6], Cao, Xu and Chen considered the growth of meromorphic solutions of equation (1.1) with meromorphic coefficients of finite iterated order, and obtained the following result when A_0 is the dominant coefficient with finite iterated order.

Theorem C ([6]). *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions in the plane, and let $i(A_0) = p$ $(0 < p < \infty)$. Assume that either $i_\lambda(\frac{1}{A_0}) < p$ or $\lambda_p(\frac{1}{A_0}) < \sigma_p(A_0)$, and that either*

$$\max\{i(A_j) : j = 1, 2, \cdots, k-1\} < p$$

or

$$\max\{\sigma_p(A_j) : j = 1, 2, \cdots, k-1\} \leq \sigma_p(A_0) := \sigma \ (0 < \sigma < \infty),$$

$$\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_0)\} < \tau_p(A_0) := \tau \ (0 < \tau < \infty).$$

Then every meromorphic solution $f(z) \not\equiv 0$, whose poles are of uniformly bounded multiplicities, of (1.1) *satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

In [24], Liu, Tu and Shi were the first to use the concepts of (p, q)-order and (p, q)-type for the case $p \geq q \geq 1$ to investigate the growth of entire solutions of (1.1), and obtained some results which improve and generalize other results.

Theorem D ([24, Theorems 2.2 and 2.3]). *Let $p \geq q \geq 1$, and let $A_0(z)$, $A_1(z), \cdots, A_{k-1}(z)$ be entire functions such that either*

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} < \sigma_{(p,q)}(A_0) < +\infty$$

or

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} \leq \sigma_{(p,q)}(A_0) < +\infty,$$

$$\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0\} < \tau_{(p,q)}(A_0).$$

Then every solution $f(z) \not\equiv 0$ of (1.1) *satisfies $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$.*

In Theorems A and D, the authors investigated the growth of the solutions of (1.1) when the coefficients are of finite iterated order or finite (p, q)-order. A natural question occurs: How about the growth of solutions of (1.1) when the coefficients are of order zero? Recently, the concept of logarithmic order has been used to investigate the growth and the oscillation of solutions of linear differential equations in the complex plane [5] and in the unit disc ([15], [16]). In [5], Cao, Liu and Wang discussed the above question and obtained the following result, when the dominant coefficient A_0 is transcendental and of order zero, by making use of the idea of the logarithmic order due to Chern [9].

Theorem E ([5]). *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions of finite logarithmic order. If $A_0(z)$ is transcendental and satisfies*

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, \cdots, k-1\} = \sigma_{(1,2)}(A_0) < \infty,$$

and

$$\max\{\tau_{(1,2)}(A_j) : \sigma_{(1,2)}(A_j) = \sigma_{(1,2)}(A_0)\} < \tau_{(1,2)}(A_0) \leq \infty,$$

then every transcendental solution $f(z)$ of equation (1.1) satisfies $\sigma_{(2,1)}(f) = 0$ and

$$1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1.$$

Furthermore, if $\sigma_{(1,2)}(A_0) > 1$, then the degree of any nonzero polynomial solution of (1.1) is not less than $\tau_{(1,2)}(A_0)$; if $\sigma_{(1,2)}(A_0) = 1$, then any nonzero solution of (1.1) can not be a polynomial.

2. Main results

Now we consider the following questions: Firstly, what do we say about the growth of solutions of (1.1) in Theorems B and C when the dominant coefficient is of finite logarithmic lower order? And secondly, can we extend the above results when the coefficients are meromorphic functions of finite logarithmic order? In this paper, we consider the above questions and obtain the following results.

Theorem 2.1 *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions satisfying*

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \cdots, k-1\} < \mu_{(1,2)}(A_0) \leq \sigma_{(1,2)}(A_0) := \sigma < \infty.$$

Then every solution $f(z) \not\equiv 0$ of (1.1) satisfies

$$1 \leq \mu_{(1,2)}(A_0) \leq \mu_{(2,2)}(f) \leq \mu_{(1,2)}(A_0) + 1,$$

$$\sigma_{(2,1)}(f) = 0 \text{ and } 1 \leq \sigma_{(1,2)}(A_0) \leq \sigma_{(2,2)}(f) = \lambda_{(2,2)}(f - \varphi) = \bar{\lambda}_{(2,2)}(f - \varphi) \leq \sigma_{(1,2)}(A_0) + 1,$$

where $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma_{(2,2)}(\varphi) < \mu_{(1,2)}(A_0)$.

Theorem 2.2 *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions of finite logarithmic order. If $A_0(z)$ is transcendental and satisfies*

$$\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \cdots, k-1\} \leq \mu_{(1,2)}(A_0) := \mu < \infty,$$

$$\overline{\lim_{r\to+\infty}}\sum_{j=1}^{k-1} m(r,A_j)/m(r,A_0)<1,$$

then every transcendental solution $f(z)$ *of* (1.1) *satisfies*

$$\mu_{(1,2)}(A_0)\le\mu_{(2,2)}(f)\le\mu_{(1,2)}(A_0)+1,$$

$$\sigma_{(2,1)}(f)=0 \text{ and } 1\le\sigma_{(1,2)}(A_0)\le\sigma_{(2,2)}(f)=\lambda_{(2,2)}(f-\varphi)$$
$$=\bar{\lambda}_{(2,2)}(f-\varphi)\le\sigma_{(1,2)}(A_0)+1,$$

where $\varphi(z)\not\equiv 0$ *is an entire function satisfying* $\sigma_{(2,2)}(\varphi)<\mu_{(1,2)}(A_0)$.

Theorem 2.3 *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$ *be entire functions of finite logarithmic order. If* $A_0(z)$ *is transcendental and satisfies*

$$\max\{\sigma_{(1,2)}(A_j): j\ne 0\}\le\mu_{(1,2)}(A_0)\le\sigma_{(1,2)}(A_0)<\infty,$$

$$\tau_1:=\max\{\tau_{(1,2)}(A_j):\sigma_{(1,2)}(A_j)=\mu_{(1,2)}(A_0), j\ne 0\}$$
$$<\underline{\tau}_{(1,2)}(A_0):=\tau<\infty,$$

then every transcendental solution $f(z)$ *of equation* (1.1)

$$\mu_{(1,2)}(A_0)\le\mu_{(2,2)}(f)\le\mu_{(1,2)}(A_0)+1,$$

$$\sigma_{(2,1)}(f)=0 \text{ and } 1\le\sigma_{(1,2)}(A_0)\le\sigma_{(2,2)}(f)=\lambda_{(2,2)}(f-\varphi)$$
$$=\bar{\lambda}_{(2,2)}(f-\varphi)\le\sigma_{(1,2)}(A_0)+1,$$

where $\varphi(z)\not\equiv 0$ *is an entire function satisfying* $\sigma_{(2,2)}(\varphi)<\mu_{(1,2)}(A_0)$.

Our next results are for the case when the coefficients $A_j(z)$ are meromorphic functions.

Theorem 2.4 *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$ *be meromorphic functions. Assume that* $\lambda_{(1,2)}\left(\frac{1}{A_0}\right)<\sigma_{(1,2)}(A_0)$, *and*

$$\max\{\sigma_{(1,2)}(A_j): j=1,2,\cdots,k-1\}<\sigma_{(1,2)}(A_0):=\sigma<\infty.$$

Then every meromorphic solution $f(z)\not\equiv 0$ *of* (1.1) *satisfies*

$$\sigma_{(2,2)}(f)\ge\sigma_{(1,2)}(A_0)\ge 1.$$

Theorem 2.5 *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$ *be meromorphic functions of finite logarithmic order. Assume that* $\lambda_{(1,2)}\left(\frac{1}{A_0}\right)<\mu_{(1,2)}(A_0)$, *and*

$$\max\{\sigma_{(1,2)}(A_j): j=1,2,\cdots,k-1\}<\mu_{(1,2)}(A_0)\le\sigma_{(1,2)}(A_0)<\infty.$$

Then every meromorphic solution $f(z) \not\equiv 0$ *of the equation* (1.1) *satisfies* $\sigma_{(2,2)}(f) \geq \mu_{(2,2)}(f) \geq \mu_{(1,2)}(A_0) \geq 1$.

Replacing the dominant coefficient A_0 by an arbitrary coefficient A_s, where $s \in \{0, 1, \cdots, k-1\}$, we obtain the following results.

Theorem 2.6 *Let* $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *be non-constant meromorphic functions. Suppose that there exists one* $A_s(z)$ $(0 \leq s \leq k-1)$ *satisfying* $\lambda_{(1,2)}\left(\frac{1}{A_s}\right) < \sigma_{(1,2)}(A_s)$, *and*

$$\max\left\{\sigma_{(1,2)}(A_j) : j \neq s\right\} < \sigma_{(1,2)}(A_s) := \sigma < \infty,$$

Then every transcendental meromorphic solution $f(z)$ *of* (1.1) *satisfies* $\sigma_{(1,2)}(f) \geq \sigma_{(1,2)}(A_s) > 1$, *and every non-transcendental meromorphic solution* $f(z) \not\equiv 0$ *of* (1.1) *is a polynomial with degree* $\deg(f) \leq s - 1$ *if* $s \geq 1$. *Furthermore, if all solutions* $f(z) \not\equiv 0$ *of* (1.1) *are meromorphic functions, then there is at least one meromorphic solution* $f_1(z)$ *which satisfies* $\sigma_{(2,2)}(f_1) \geq \sigma_{(1,2)}(A_s) > 1$.

Theorem 2.7 *Let* $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *be non-constant meromorphic functions. Suppose that there exists one* $A_s(z)$ $(0 \leq s \leq k-1)$ *satisfying* $\lambda_{(1,2)}\left(\frac{1}{A_s}\right) < \mu_{(1,2)}(A_s)$, *and*

$$\max\left\{\sigma_{(1,2)}(A_j) : j \neq s\right\} < \mu_{(1,2)}(A_s) \leq \sigma_{(1,2)}(A_s) := \sigma < \infty.$$

Then every transcendental meromorphic solution $f(z)$ *of* (1.1) *satisfies* $\sigma_{(1,2)}(f) \geq \mu_{(1,2)}(f) \geq \mu_{(1,2)}(A_s) > 1$, *and every non-transcendental meromorphic solution* $f(z) \not\equiv 0$ *of* (1.1) *is a polynomial with degree* $\deg(f) \leq s - 1$ *if* $s \geq 1$. *Furthermore, if all solutions* $f(z) \not\equiv 0$ *of* (1.1) *are meromorphic functions, then there is at least one meromorphic solution* $f_1(z)$ *which satisfies* $\sigma_{(2,2)}(f_1) \geq \mu_{(2,2)}(f_1) \geq \mu_{(1,2)}(A_s) > 1$.

Set $g(z) = f(z) - z$. Then we have $\bar{\lambda}_{(2,2)}(f - z) = \bar{\lambda}_{(2,2)}(g)$ and $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(g)$. Many authors have investigated the problems on the fixed points and the distinct fixed points of solutions of linear differential equations and obtained some valuables results [2, 3, 6, 8, 25]. In the following results, we obtained some estimates of distinct fixed points of meromorphic solutions of (1.1).

Theorem 2.8 *Under the hypotheses of Theorem 2.4, if* $\sigma_{(1,2)}(A_0) > 1$, *then every meromorphic solution* $f(z) \not\equiv 0$ *of* (1.1) *satisfies* $\bar{\lambda}_{(2,2)}(f - z) = \sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0) > 1$.

Theorem 2.9 *Under the hypotheses of Theorem 2.6, if* $A_1(z) + zA_0(z) \not\equiv 0$ *and all solutions of* (1.1) *are meromorphic then any transcendental meromorphic solution* $f(z)$ *with* $\sigma_{(1,2)}(f) > \sigma_{(1,2)}(A_s) > 1$ *satisfies* $\bar{\lambda}_{(1,2)}(f - z) = \sigma_{(1,2)}(f)$. *Furthermore, there exists at least one solution* $f_1(z)$ *satisfying* $\bar{\lambda}_{(2,2)}(f_1 - z) = \sigma_{(2,2)}(f_1) \geq \sigma_{(1,2)}(A_s) > 1$.

3. Preliminary lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 3.1 ([14]). *Let* k *and* j *be integers such that* $k > j \geq 0$. *Let* f *be a meromorphic function in the plane* $\mathbb{C}$ *such that* $f^{(j)}$ *does not vanish identically. Then, there exists an* $r_0 > 1$ *such that*

$$m(r, \frac{f^{(k)}}{f^{(j)}}) \leq (k - j) \log^+ \frac{\rho(T(\rho, f))}{r(\rho - r)} + \log \frac{k!}{j!} + (k - j)5.3078,$$

for all $r_0 < r < \rho < +\infty$. *If* f *is of finite order* s, *then*

$$\limsup_{r\to+\infty} \frac{m(r, \frac{f^{(k)}}{f^{(j)}})}{\log r} \leq \max\{0, (k - j)(s - 1)\}.$$

Lemma 3.2 *Let* $f(z)$ *be a meromorphic function with* $\sigma_{(1,2)}(f) < \infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_1 \subset (1, +\infty)$ *having infinite logarithmic measure such that for all* $r \in E_1$, *we have*

$$\sigma_{(1,2)}(f) = \lim_{r\to+\infty, r\in E_1} \frac{\log T(r, f)}{\log\log r}.$$

Proof. By the definition of the logarithmic order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$, and

$$\lim_{r_n\to+\infty} \frac{\log T(r_n, f)}{\log\log r_n} = \sigma_{(1,2)}(f).$$

Then, for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in [r_n, (1 + \frac{1}{n})r_n]$, we have

$$\frac{\log T(r_n, f)}{\log\log(1 + \frac{1}{n})r_n} \leq \frac{\log T(r, f)}{\log\log r} \leq \frac{\log T((1 + \frac{1}{n})r_n, f)}{\log\log r_n}.$$

Set $E_1 = \bigcup_{n=n_1}^{\infty} [r_n, (1+\frac{1}{n})r_n]$. Then for any $r \in E_1$, we have

$$\lim_{r\to+\infty, r\in E_1} \frac{\log T(r,f)}{\log\log r} = \lim_{r_n\to+\infty} \frac{\log T(r_n,f)}{\log\log r_n} = \sigma_{(1,2)}(f),$$

and

$$m_l E_1 = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty.$$

Lemma 3.3 ([12]). *Let $g : [0,+\infty) \to \mathbb{R}$ and $h : [0,+\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \le h(r)$ for all $r \notin [0,1] \cup E_2$ where $E_2 \subset (1,+\infty)$ is a set of finite logarithmic measure. Then for any $\alpha > 1$, there exists an $r_0 = r_0(\alpha) > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Lemma 3.4 ([22]). *Let $f(z)$ be a transcendental entire function. Then there exists a set $E_3 \subset (1,+\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3$ and $|f(z)| = M(r,f)$, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^n (1+o(1)), \quad (n \in \mathbb{N}),$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.5 ([19]). *Let $f(z)$ be an entire function of $[p,q]$-order and lower $[p,q]$-order, and let $\nu_f(r)$ be a central index of $f(z)$. Then*

$$\overline{\lim_{r\to+\infty}} \frac{\log_p \nu_f(r)}{\log_q r} = \sigma_{(p,q)}(f), \quad \underline{\lim_{r\to+\infty}} \frac{\log_p \nu_f(r)}{\log_q r} = \mu_{(p,q)}(f).$$

Lemma 3.6 ([5]). *Let $A_j(z)$ $(j = 0,1,\cdots,k-1)$ be entire functions such that $\max\{\sigma_{(1,2)}(A_j) : j = 1,2,\cdots,k-1\} < \sigma_{(1,2)}(A_0) < +\infty$. Then any nonzero entire solution $f(z)$ of (1.1) satisfies $\sigma_{(2,2)}(f) \ge \sigma_{(1,2)}(A_0) \ge 1$.*

Lemma 3.7 ([5]). *Let $A_j(z)$ $(j = 0,1,\cdots,k-1)$ be entire functions such that $\max\{\sigma_{(1,2)}(A_j) : j = 0,1,\cdots,k-1\} \le \alpha < +\infty$. Then any solution $f(z)$ of (1.1) satisfies $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \le \alpha + 1$.*

Lemma 3.8 ([11]). *Let $f(z)$ be a transcendental meromorphic function in the plane, and let $\alpha > 1$ be a given constant. Then there exist* *a set $E_4 \subset (1,+\infty)$ that has a finite logarithmic measure, and a constant $B>0$ depending only on α and (m,n) $(m,n \in \{0,1,\cdots,k\})$ $m<n$ such that for all z with $|z|=r \notin [0,1]\cup E_4$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \le B\left(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)\log T(\alpha r,f)\right)^{n-m}.$$

Lemma 3.9 *Let $f(z)$ be an entire function with $\mu_{(1,2)}(f)<\infty$. Then for any given $\varepsilon>0$, there exists a set $E_5 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_5$, we have*

$$\mu_{(1,2)}(f)=\lim_{r\to+\infty, r\in E_5}\frac{\log T(r,f)}{\log\log r}=\lim_{r\to+\infty, r\in E_5}\frac{\log\log M(r,f)}{\log\log r}$$

and for any given $\varepsilon>0$

$$M(r,f)<\exp\{(\log r)^{\mu_{(1,2)}(f)+\varepsilon}\}.$$

Proof. By the definition of the logarithmic lower order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n<r_{n+1}$, and

$$\lim_{r_n\to+\infty}\frac{\log\log M(r_n,f)}{\log\log r_n}=\mu_{(1,2)}(f).$$

Then for any given $\varepsilon>0$, there exists an n_2 such that for $n\ge n_2$ and any $r\in\left[\frac{n}{n+1}r_n, r_n\right]$, we have

$$\frac{\log\log M(\frac{n}{n+1}r_n,f)}{\log\log r_n}\le\frac{\log\log M(r,f)}{\log\log r}\le\frac{\log\log M(r_n,f)}{\log\log\frac{n}{n+1}r_n}.$$

Set $E_5=\bigcup\limits_{n=n_2}^{\infty}\left[\frac{n}{n+1}r_n, r_n\right]$. Then for any $r\in E_5$, we have

$$\lim_{r\to+\infty, r\in E_5}\frac{\log\log M(r,f)}{\log\log r}=\lim_{r_n\to+\infty}\frac{\log\log M(r_n,f)}{\log\log r_n}=\mu_{(1,2)}(f),$$

and

$$m_l E_5=\sum_{n=n_2}^{\infty}\int_{\frac{n}{n+1}r_n}^{r_n}\frac{dt}{t}=\sum_{n=n_2}^{\infty}\log(1+\frac{1}{n})=\infty.$$

Also, we have

$$\lim_{r\to+\infty, r\in E_5}\frac{\log T(r,f)}{\log\log r}=\lim_{r\to+\infty, r\in E_5}\frac{\log\log M(r,f)}{\log\log r}.$$

Then for any given $\varepsilon>0$

$$M(r,f)<\exp\{(\log r)^{\mu_{(1,2)}(f)+\varepsilon}\}.$$

Using similar proof as in the proof of Theorem 3.4 in [5], we obtain the following for the case of logarithmic order.

Lemma 3.10 *Let $A_j(z)$ $(j=0,1,\cdots,k-1)$ and $F(z)\not\equiv 0$ be meromorphic functions and let $f(z)$ be a meromorphic solution of the equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F,$$

such that

$$\max\{\sigma_{(i,2)}(F),\sigma_{(i,2)}(A_j):j=0,1,\cdots,k-1\}<\sigma_{(i,2)}(f)<+\infty\ (i=1,2).$$

Then we have

$$\bar{\lambda}_{(i,2)}(f)=\lambda_{(i,2)}(f)=\sigma_{(i,2)}(f)\ \ (i=1,2).$$

Remark 3.1. The Lemma 3.10 was proved for $i=2$ in [5].

Lemma 3.11 *Let $A_j(z)$ $(j=0,1,\cdots,k-1)$ be entire functions of finite logarithmic order. If $A_0(z)$ is transcendental and satisfies*

$$\overline{\lim_{r\to+\infty}}\sum_{j=1}^{k-1}m(r,A_j)/m(r,A_0)<1,$$

then every transcendental solution $f(z)$ of (1.1) satisfies $\sigma_{(2,1)}(f)=0$ and

$$1\le\sigma_{(1,2)}(A_0)\le\sigma_{(2,2)}(f)\le\sigma_{(1,2)}(A_0)+1.$$

Proof. We divide the proof into two parts: (i) $\sigma_{(2,2)}(f)\ge\sigma_{(1,2)}(A_0)\ge 1$ and (ii) $\sigma_{(2,1)}(f)=0$, $\sigma_{(2,2)}(f)\le\sigma_{(1,2)}(A_0)+1$.
(i) Let f be a transcendental solution of (1.1). By (1.1), we have

$$-A_0(z)=\frac{f^{(k)}}{f}+A_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots+A_1(z)\frac{f'}{f}. \tag{3.1}$$

Using Lemma 3.1 and (3.1), we obtain

$$m(r,A_0)\le\sum_{j=1}^{k-1}m(r,A_j)+\sum_{j=1}^{k}m(r,\frac{f^{(j)}}{f})+O(1)$$

$$\le\sum_{j=1}^{k-1}m(r,A_j)+k^2\log^+T(2r,f)+O(1). \tag{3.2}$$

Suppose that

$$\overline{\lim_{r\to+\infty}}\sum_{j=1}^{k-1} m(r,A_j)/m(r,A_0)=\alpha<\beta<1.$$

Then, for sufficiently large r, we have

$$\sum_{j=1}^{k-1} m(r,A_j)<\beta m(r,A_0). \tag{3.3}$$

By (3.2) and (3.3), we have

$$(1-\beta)m(r,A_0)\le k^2\log^+ T(2r,f)+O(1). \tag{3.4}$$

By $\sigma_{(1,2)}(A_0):=\sigma\ge 1$ and Lemma 3.2, there exists a set $E_1\subset(1,+\infty)$ having infinite logarithmic measure such that for all z satisfying $|z|=r\in E_1$ and for any given $\varepsilon>0$, we have

$$(1-\beta)(\log r)^{\sigma-\varepsilon}\le(1-\beta)m(r,A_0)\le k^2\log^+ T(2r,f)+O(1). \tag{3.5}$$

Since $\varepsilon>0$ is arbitrary, then by (3.5), we have

$$\sigma_{(2,2)}(f)\ge\sigma=\sigma_{(1,2)}(A_0)\ge 1.$$

(ii) By (1.1), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\le|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|+|A_0(z)|. \tag{3.6}$$

By Lemma 3.4 and (3.6), there exists a set $E_3\subset(1,+\infty)$ with finite logarithmic measure such that for all z satisfying $|z|=r\notin E_3$ and $|f(z)|=M(r,f)$, we have

$$\left(\frac{\nu_f(r)}{r}\right)^k|1+o(1)|$$

$$\le(|A_{k-1}(z)|+|A_{k-2}(z)|+\cdots+|A_0(z)|)\left(\frac{\nu_f(r)}{r}\right)^{k-1}|1+o(1)|. \tag{3.7}$$

By (3.3) and $\sigma_{(1,2)}(A_0)=\sigma$, it is clear that $\sigma_{(1,2)}(A_j)\le\sigma$, $j=1,\cdots,k-1$. Then, by the definition of the logarithmic order for any given $\varepsilon>0$ and sufficiently large r, we have

$$|A_j(z)|\le M(r,A_j)\le\exp\{(\log r)^{\sigma+\varepsilon}\},\ j=1,\cdots,k-1. \tag{3.8}$$

Therefore, by (3.7) and (3.8), for all z satisfying $|z| = r \notin E_3$, and $|f(z)| = M(r, f)$, we have

$$\left(\frac{\nu_f(r)}{r}\right)^k |1 + o(1)| \le k \exp\{(\log r)^{\sigma+\varepsilon}\} \left(\frac{\nu_f(r)}{r}\right)^{k-1} |1 + o(1)|,$$

that is,

$$\nu_f(r)\,|1 + o(1)| \le kr \exp\{(\log r)^{\sigma+\varepsilon}\}\,|1 + o(1)|. \tag{3.9}$$

By Lemma 3.3, Lemma 3.5 and (3.9), we have $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. From (i) and (ii), we get $1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$.

Lemma 3.12 ([5]). *Let $\phi(r)$ be a continuous and positive increasing function, defined for r on $(0, +\infty)$ with logarithmic order $\sigma_{(1,2)}(\phi)$. Then for any subset E_6 of $[0, +\infty)$ that has finite linear measure, there exists a sequence $\{r_n\}$, $r_n \notin E_6$ such that*

$$\sigma_{(1,2)}(\phi) = \lim_{r_n \to +\infty} \frac{\log \phi(r_n)}{\log\log r_n}.$$

Lemma 3.13 ([10]). *Let $f_1, f_2, \cdots, f_k$ be linearly independent meromorphic solutions of the differential equation (1.1) with meromorphic functions $A_0, A_1, \cdots, A_{k-1}$ in the plane as the coefficients. Then*

$$m(r, A_j) = O\left\{\log\left(\max_{1\le n\le k} T(r, f_n)\right)\right\} \quad (j = 0, 1, \cdots, k-1).$$

4. Proof of Theorem 2.1

Assume that $f(z) \not\equiv 0$ is a solution of (1.1). Then, by Lemma 3.6, we have $\sigma_{(2,2)}(f) \ge \sigma_{(1,2)}(A_0) \ge 1$. On the other hand, by Lemma 3.7, we have $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. Thus, we obtain $\sigma_{(2,1)}(f) = 0$ and $1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. Now, we just need to prove (i) $1 \le \mu_{(1,2)}(A_0) \le \mu_{(2,2)}(f) \le \mu_{(1,2)}(A_0) + 1$ and (ii) $\bar{\lambda}_{(2,2)}(f - \varphi) = \lambda_{(2,2)}(f - \varphi) = \sigma_{(2,2)}(f)$.

(i) By (1.1), we have

$$|A_0(z)| \le \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{4.1}$$

Set $\alpha := \max\{\sigma_{(1,2)}(A_j) : j = 1, \cdots, k-1\} < \mu_{(1,2)}(A_0) \le \sigma_{(1,2)}(A_0) := \sigma < \infty$. Then for any given ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_0) - \alpha)$ and for sufficiently large r, we have

$$M(r, A_0) \ge \exp\{(\log r)^{\mu_{(1,2)}(A_0) - \varepsilon}\} \tag{4.2}$$

and

$$M(r, A_j) \leq \exp\{(\log r)^{\alpha+\varepsilon}\},\ j = 1, \cdots, k-1. \tag{4.3}$$

By Lemma 3.8, there exist a set $E_4 \subset (1, +\infty)$ that has a finite logarithmic measure and a constant $B > 0$ such that for all z with $|z| = r \notin [0, 1] \cup E_4$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,(T(2r, f))^{k+1},\ j = 1, \cdots, k. \tag{4.4}$$

By substituting (4.2) – (4.4) into (4.1), for the above ε, we get

$$\exp\{(\log r)^{\mu_{(1,2)}(A_0)-\varepsilon}\} \leq Bk \exp\{(\log r)^{\alpha+\varepsilon}\}\,[T(2r, f)]^{k+1}, \tag{4.5}$$

for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, $r \to \infty$ and $|A_0(z)| = M(r, A_0)$. By Lemma 3.3 and (4.5), we have $\mu_{(2,2)}(f) \geq \mu_{(1,2)}(A_0) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then

$$\mu_{(2,2)}(f) \geq \mu_{(1,2)}(A_0) \geq 1. \tag{4.6}$$

From (1.1), we get

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|. \tag{4.7}$$

By Lemma 3.4, there exists a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)),\ (j = 1, \cdots, k). \tag{4.8}$$

By Lemma 3.9, there exists a set $E_5 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$|A_0(z)| \leq M(r, A_0) \leq \exp\{(\log r)^{\mu_{(1,2)}(A_0)+\varepsilon}\}. \tag{4.9}$$

Hence, by (4.3), (4.7) – (4.9), for all $|z| = r \in E_5 \backslash E_3$, we get

$$\left(\frac{\nu_f(r)}{r}\right)^k |1 + o(1)| \leq k \exp\{(\log r)^{\mu_{(1,2)}(A_0)+\varepsilon}\} \left(\frac{\nu_f(r)}{r}\right)^{k-1} |1 + o(1)|,$$

that is

$$\nu_f(r)\,|1 + o(1)| \leq kr\,|1 + o(1)| \exp\{(\log r)^{\mu_{(1,2)}(A_0)+\varepsilon}\}. \tag{4.10}$$

Since $\varepsilon > 0$ is arbitrary, then by Lemma 3.3, Lemma 3.5 and (4.10), we have

$$\mu_{(2,2)}(f) \leq \mu_{(1,2)}(A_0) + 1. \tag{4.11}$$

By (4.6) and (4.11), we obtain

$$1 \le \mu_{(1,2)}(A_0) \le \mu_{(2,2)}(f) \le \mu_{(1,2)}(A_0) + 1.$$

(ii) Now, we prove that $\bar{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f)$. Set $g = f - \varphi$. Since $\sigma_{(2,2)}(\varphi) < \mu_{(1,2)}(A_0) \le \sigma_{(1,2)}(A_0)$, then we have $\sigma_{(2,2)}(g) = \sigma_{(2,2)}(f)$, $\lambda_{(2,2)}(g) = \lambda_{(2,2)}(f-\varphi)$ and $\bar{\lambda}_{(2,2)}(g) = \bar{\lambda}_{(2,2)}(f-\varphi)$. Substituting $f = g + \varphi$ into (1.1), we obtain

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g$$

$$= -[\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi]. \tag{4.12}$$

If $G(z) = \varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi \equiv 0$, then by part (i), we have $\sigma_{(2,2)}(\varphi) \ge \mu_{(2,2)}(\varphi) \ge \mu_{(1,2)}(A_0)$, which is a contradiction. Hence $G(z) \not\equiv 0$. Since $G(z) \not\equiv 0$ and $\sigma_{(2,2)}(G) \le \sigma_{(2,2)}(\varphi) < \mu_{(1,2)}(A_0) \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(g)$, then by Lemma 3.10 and (4.12), we have

$$\begin{aligned}1 \le \sigma_{(1,2)}(A_0) \le \bar{\lambda}_{(2,2)}(g) &= \lambda_{(2,2)}(g) = \sigma_{(2,2)}(g)\\ &= \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1.\end{aligned}$$

That is,

$$1 \le \sigma_{(1,2)}(A_0) \le \bar{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0)+1.$$

5. Proof of Theorem 2.2

Assume that $f(z)$ is a transcendental solution of (1.1). Then, by Lemma 3.11, we have $\sigma_{(2,1)}(f) = 0$ and $1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. Now it only remains to prove (i) $\mu_{(1,2)}(A_0) \le \mu_{(2,2)}(f) \le \mu_{(1,2)}(A_0) + 1$ and (ii) $\bar{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f)$.

(i) By $\mu_{(1,2)}(A_0) := \mu$, for any given $\varepsilon > 0$ and sufficiently large r, we get

$$m(r, A_0) \ge (\log r)^{\mu-\varepsilon}. \tag{5.1}$$

By (3.4) and (5.1), for the above ε and sufficiently large r, we have

$$(1-\beta)(\log r)^{\mu-\varepsilon} \le k^2 \log^+ T(2r, f) + O(1). \tag{5.2}$$

Since $\varepsilon > 0$ is arbitrary, then by (5.2), we obtain

$$\mu_{(2,2)}(f) \ge \mu = \mu_{(1,2)}(A_0).$$

On the other hand, since $\max\{\sigma_{(1,2)}(A_j) : j = 1, 2, \cdots, k-1\} \le \mu_{(1,2)}(A_0) := \mu$, then for any given $\varepsilon > 0$ and sufficiently large r, we have

$$|A_j(z)| \le \exp\{(\log r)^{\mu+\varepsilon}\},\ j = 1, 2, \cdots, k-1. \tag{5.3}$$

By Lemma 3.9, there exists a set $E_5 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$|A_0(z)| \leq \exp\{(\log r)^{\mu+\varepsilon}\}. \tag{5.4}$$

Hence, by (4.7), (4.8), (5.3) and (5.4), for sufficiently large $|z| = r \in E_5 \backslash E_3$, we have

$$\left(\frac{\nu_f(r)}{r}\right)^k |1 + o(1)| \leq k \exp\{(\log r)^{\mu+\varepsilon}\} \left(\frac{\nu_f(r)}{r}\right)^{k-1} |1 + o(1)|,$$

that is

$$\nu_f(r)\,|1 + o(1)| \leq kr\,|1 + o(1)| \exp\{(\log r)^{\mu+\varepsilon}. \tag{5.5}$$

Since $\varepsilon > 0$ is arbitrary, then by Lemma 3.3, Lemma 3.5 and (5.5), we have

$$\mu_{(2,2)}(f) \leq \mu + 1 = \mu_{(1,2)}(A_0) + 1.$$

Thus, we have

$$\mu_{(1,2)}(A_0) \leq \mu_{(2,2)}(f) \leq \mu_{(1,2)}(A_0) + 1.$$

(ii) We prove that $\bar{\lambda}_{(2,2)}(f - \varphi) = \lambda_{(2,2)}(f - \varphi) = \sigma_{(2,2)}(f)$. Set $g = f - \varphi$. Since $\sigma_{(2,2)}(\varphi) < \mu_{(1,2)}(A_0) \leq \sigma_{(1,2)}(A_0)$, then we have $\sigma_{(2,2)}(g) = \sigma_{(2,2)}(f)$, $\lambda_{(2,2)}(g) = \lambda_{(2,2)}(f-\varphi)$ and $\bar{\lambda}_{(2,2)}(g) = \bar{\lambda}_{(2,2)}(f-\varphi)$. Substituting $f = g + \varphi$ into (1.1), we obtain

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g$$

$$= -[\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi]. \tag{5.6}$$

If $G(z) = \varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi \equiv 0$, then by part (i), we have $\sigma_{(2,2)}(\varphi) \geq \mu_{(2,2)}(\varphi) \geq \mu_{(1,2)}(A_0)$, which is a contradiction. Hence $G(z) \not\equiv 0$. Since $G(z) \not\equiv 0$ and $\sigma_{(2,2)}(G) < \sigma_{(2,2)}(g)$, then by Lemma 3.10 and (5.6), we have

$$1 \leq \sigma_{(1,2)}(A_0) \leq \bar{\lambda}_{(2,2)}(g) = \lambda_{(2,2)}(g) = \sigma_{(2,2)}(g)$$

$$= \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1.$$

That is,

$$1 \leq \sigma_{(1,2)}(A_0) \leq \bar{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0)+1.$$

6. Proof of Theorem 2.3

Assume that $f(z)$ is a transcendental solution of (1.1). First, we prove that $\sigma_{(2,1)}(f) = 0$ and $1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. By Lemma 3.7, we have $\sigma_{(2,1)}(f) = 0$ and $\sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1$. Now it only remains to prove (i) $\sigma_{(2,2)}(f) \ge \sigma_{(1,2)}(A_0) \ge 1$, $\mu_{(1,2)}(A_0) \le \mu_{(2,2)}(f) \le \mu_{(1,2)}(A_0)+1$ and (ii) $\overline{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f)$.
(i) We set $d := \max\{\sigma_{(1,2)}(A_j) : \sigma_{(1,2)}(A_j) < \sigma_{(1,2)}(A_0)\}$. If $\sigma_{(1,2)}(A_j) < \mu_{(1,2)}(A_0) \le \sigma_{(1,2)}(A_0)$ or $\sigma_{(1,2)}(A_j) \le \mu_{(1,2)}(A_0) < \sigma_{(1,2)}(A_0)$ then for any given ε $(0 < 2\varepsilon < \sigma_{(1,2)}(A_0) - d)$ and for sufficiently large r, we have

$$M(r, A_j) \le \exp\{(\log r)^{d+\varepsilon}\} \le \exp\{(\log r)^{\sigma_{(1,2)}(A_0)-\varepsilon}\}. \tag{6.1}$$

If $\sigma_{(1,2)}(A_j) = \mu_{(1,2)}(A_0) = \sigma_{(1,2)}(A_0)$, then we have $\tau_{(1,2)}(A_j) \le \tau_1 < \tau_2 := \tau_{(1,2)}(A_0)$, so for sufficiently large r and for any given ε $(0 < 2\varepsilon < \tau_2 - \tau_1)$ we have

$$M(r, A_j) \le \exp\{(\tau_1 + \varepsilon)(\log r)^{\sigma_{(1,2)}(A_0)}\}, \tag{6.2}$$

$$M(r, A_0) \ge \exp\{(\tau_2 - \varepsilon)(\log r)^{\sigma_{(1,2)}(A_0)}\}. \tag{6.3}$$

By (6.1)$-$(6.3), (4.1) and (4.4), for sufficiently large $|z| = r \notin [0,1] \cup E_4$ and for the above ε, we obtain

$$\exp\{(\tau_2-\varepsilon)(\log r)^{\sigma_{(1,2)}(A_0)}\} \le Bk\,[T(2r, f)]^{k+1} \exp\{(\tau_1+\varepsilon)(\log r)^{\sigma_{(1,2)}(A_0)}\}. \tag{6.4}$$

By Lemma 3.3 and (6.4), we have

$$1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f).$$

Therefore, $\sigma_{(2,1)}(f) = 0$ and

$$1 \le \sigma_{(1,2)}(A_0) \le \sigma_{(2,2)}(f) \le \sigma_{(1,2)}(A_0) + 1.$$

Now, we return to prove that $\mu_{(1,2)}(A_0) \le \mu_{(2,2)}(f) \le \mu_{(1,2)}(A_0) + 1$. On one hand, we set $b := \max\{\sigma_{(1,2)}(A_j) : \sigma_{(1,2)}(A_j) < \mu_{(1,2)}(A_0)\}$. If $\sigma_{(1,2)}(A_j) < \mu_{(1,2)}(A_0)$, then for any given ε $(0 < 2\varepsilon < \min\{\mu_{(1,2)}(A_0) - b, \tau - \tau_1\})$ and for sufficiently large r, we have

$$M(r, A_j) \le \exp\{(\log r)^{b+\varepsilon}\} \le \exp\{(\log r)^{\mu_{(1,2)}(A_0)-\varepsilon}\}. \tag{6.5}$$

If $\sigma_{(1,2)}(A_j) = \mu_{(1,2)}(A_0)$, $\tau_{(1,2)}(A_j) \le \tau_1 < \tau := \underline{\tau}_{(1,2)}(A_0)$, then for sufficiently large r, we have

$$M(r, A_j) \le \exp\{(\tau_1 + \varepsilon)(\log r)^{\mu_{(1,2)}(A_0)}\}, \tag{6.6}$$

$$M(r, A_0) \ge \exp\{(\tau - \varepsilon)(\log r)^{\mu_{(1,2)}(A_0)}\}. \tag{6.7}$$

By (6.5)−(6.7), (4.1) and (4.4), for sufficiently large $|z| = r \notin [0,1] \cup E_4$ and for the above ε, we obtain

(6.8)
$$\exp\{(\tau-\varepsilon)(\log r)^{\mu_{(1,2)}(A_0)}\} \leq Bk\,[T(2r,f)]^{k+1}\exp\{(\tau_1+\varepsilon)(\log r)^{\mu_{(1,2)}(A_0)}\}.$$

By Lemma 3.3 and (6.8), we have

$$\mu_{(1,2)}(A_0) \leq \mu_{(2,2)}(f).$$

On the other hand, by Lemma 3.9, there exists a set $E_5 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$|A_0(z)| \leq M(r,A_0) \leq \exp\{(\log r)^{\mu_{(1,2)}(A_0)+\varepsilon}\}. \tag{6.9}$$

By (4.7), (4.8), (6.5), (6.6) and (6.9), for sufficiently large $|z| = r \in E_5 \backslash E_3$, we have

$$\nu_f(r)\,|1+o(1)| \leq kr\,|1+o(1)|\exp\{(\log r)^{\mu_{(1,2)}(A_0)+\varepsilon}\}. \tag{6.10}$$

Since ε $(0 < 2\varepsilon < \min\{\mu_{(1,2)}(A_0) - b, \tau - \tau_1\})$ is arbitrary, then by Lemma 3.3, Lemma 3.5 and (6.10), we have

$$\mu_{(2,2)}(f) \leq \mu_{(1,2)}(A_0) + 1.$$

Therefore,

$$\mu_{(1,2)}(A_0) \leq \mu_{(2,2)}(f) \leq \mu_{(1,2)}(A_0) + 1.$$

(ii) We prove that $\bar{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f)$. Set $g = f - \varphi$. Since $\sigma_{(2,2)}(\varphi) < \mu_{(1,2)}(A_0) \leq \sigma_{(1,2)}(A_0)$, then we have $\sigma_{(2,2)}(g) = \sigma_{(2,2)}(f)$, $\lambda_{(2,2)}(g) = \lambda_{(2,2)}(f-\varphi)$ and $\bar{\lambda}_{(2,2)}(g) = \bar{\lambda}_{(2,2)}(f-\varphi)$. Substituting $f = g + \varphi$ into (1.1), we obtain

$$\begin{aligned} &g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g \\ &= -[\varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi]. \end{aligned} \tag{6.11}$$

If $G(z) = \varphi^{(k)} + A_{k-1}(z)\varphi^{(k-1)} + \cdots + A_1(z)\varphi' + A_0(z)\varphi \equiv 0$, then by part (i), we have $\sigma_{(2,2)}(\varphi) \geq \mu_{(2,2)}(\varphi) \geq \mu_{(1,2)}(A_0)$, which is a contradiction. Hence $G(z) \not\equiv 0$. Since $G(z) \not\equiv 0$ and $\sigma_{(2,2)}(G) < \sigma_{(2,2)}(g)$, then by Lemma 3.10 and (6.11), we have

$$\begin{aligned} 1 \leq \sigma_{(1,2)}(A_0) \leq \bar{\lambda}_{(2,2)}(g) = \lambda_{(2,2)}(g) = \sigma_{(2,2)}(g) \\ = \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0) + 1. \end{aligned}$$

That is,

$$1 \leq \sigma_{(1,2)}(A_0) \leq \bar{\lambda}_{(2,2)}(f-\varphi) = \lambda_{(2,2)}(f-\varphi) = \sigma_{(2,2)}(f) \leq \sigma_{(1,2)}(A_0)+1.$$

7. Proof of Theorem 2.4

Assume that $f(z) \not\equiv 0$ is a meromorphic solution of (1.1). By (1.1), we have

$$-A_0(z) = \frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f}. \tag{7.1}$$

By Lemma 3.1 and (7.1), we have

$$m(r, A_0) \le \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) + O(1)$$

$$\le \sum_{j=1}^{k-1} m(r, A_j) + k^2 \log^+ T(2r, f) + O(1). \tag{7.2}$$

Hence, we obtain

$$T(r, A_0) = N(r, A_0) + m(r, A_0)$$

$$\le N(r, A_0) + \sum_{j=1}^{k-1} m(r, A_j) + k^2 \log^+ T(2r, f) + O(1). \tag{7.3}$$

By $\lambda_{(1,2)}(\frac{1}{A_0}) < \sigma_{(1,2)}(A_0) := \sigma$, for any given ε $(0 < 2\varepsilon < \sigma - \lambda_{(1,2)}(\frac{1}{A_0}))$, we have

$$N(r, A_0) \le (\log r)^{\lambda_{(1,2)}(\frac{1}{A_0})+\varepsilon}. \tag{7.4}$$

Set $b := \max\{\sigma_{(1,2)}(A_j) : j = 1, \cdots, k-1\} < \sigma_{(1,2)}(A_0) := \sigma$. Then for any given ε $(0 < 2\varepsilon < \sigma - b)$, we have

$$m(r, A_j) \le T(r, A_j) \le (\log r)^{b+\varepsilon}, \ (j = 1, \cdots, k-1). \tag{7.5}$$

Since $\sigma_{(1,2)}(A_0) := \sigma \ge 1$, then by Lemma 3.12, for any subset E_6 of $[0, +\infty)$ that has finite linear measure, there exists a sequence $\{r_n\}$, $r_n \to \infty$, such that for all $r_n \notin E_6$ and for any given ε $(0 < 2\varepsilon < \sigma - b)$ we have

$$T(r_n, A_0) \ge (\log r_n)^{\sigma-\varepsilon}. \tag{7.6}$$

Hence, by substituting (7.4) – (7.6) into (7.3), for all $|z| = r_n \notin E_6$, we obtain

$$(\log r_n)^{\sigma-\varepsilon} \le (\log r_n)^{\lambda_{(1,2)}(\frac{1}{A_0})+\varepsilon} + (k-1)(\log r_n)^{b+\varepsilon}$$
$$+ k^2 \log^+ T(2r_n, f) + O(1),$$

that is

$$(1 - o(1)) (\log r_n)^{\sigma-\varepsilon} \leq k^2 \log^+ T(2r_n, f). \tag{7.7}$$

Since ε $\left(0 < 2\varepsilon < \min\left\{\sigma - b, \sigma - \lambda_{(1,2)}(\frac{1}{A_0})\right\}\right)$ is arbitrary, then by (7.7), we have

$$\sigma_{(2,2)}(f) \geq \sigma_{(1,2)}(A_0) \geq 1.$$

8. Proof of Theorem 2.5

Assume that $f(z) \not\equiv 0$ is a meromorphic solution of (1.1). By Lemma 3.1 and (7.1), we have

$$m(r, A_0) \leq \sum_{j=1}^{k-1} m(r, A_j) + k^2 \log^+ T(2r, f) + O(1). \tag{8.1}$$

By $\lambda_{(1,2)}(\frac{1}{A_0}) < \mu_{(1,2)}(A_0)$, we have

$$N(r, A_0) = o(T(r, A_0)), \quad r \to +\infty. \tag{8.2}$$

Therefore, by (8.2) we have

$$\mu_{(1,2)}(A_0) = \varliminf_{r\to+\infty} \frac{\log m(r, A_0)}{\log\log r}. \tag{8.3}$$

By (8.3), for sufficiently large r, we have

$$m(r, A_0) \geq (\log r)^{\mu_{(1,2)}(A_0)-\varepsilon}. \tag{8.4}$$

Set $c := \max\{\sigma_{(1,2)}(A_j) : j \neq 0\} < \mu_{(1,2)}(A_0)$. Then for sufficiently large r and any given ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_0) - c)$

$$m(r, A_j) \leq (\log r)^{c+\varepsilon}. \tag{8.5}$$

From (8.1), (8.4) and (8.5), it follows that

$$(\log r)^{\mu_{(1,2)}(A_0)-\varepsilon} \leq (k - 1)(\log r)^{c+\varepsilon} + k^2 \log^+ T(2r, f) + O(1). \tag{8.6}$$

By (8.6) and ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_0)-c)$, we have $\sigma_{(2,2)}(f) \geq \mu_{(2,2)}(f) \geq \mu_{(1,2)}(A_0) \geq 1$.

9. Proof of Theorem 2.6

Suppose that $f(z) \not\equiv 0$ is a rational solution of (1.1). If either $f(z)$ is a rational function which has a pole at $z_0 \in \mathbb{C}$ of order $\alpha(\geq 1)$, or $f(z)$ is a polynomial with degree $\deg(f) \geq s$, then $f^{(s)}(z) \not\equiv 0$. Set $b := \max\{\sigma_{(1,2)}(A_j) : j \neq s \text{ and } j = 0, 1, \cdots, k-1\} < \sigma_{(1,2)}(A_s) := \sigma$. Then, for any given ε $(0 < 2\varepsilon < \sigma - b)$, we have

$$m(r, A_j) \leq T(r, A_j) \leq (\log r)^{b+\varepsilon}, \ (j \in \{0, 1, \cdots, k-1\}\backslash\{s\}). \tag{9.1}$$

Since $\sigma_{(1,2)}(A_s) := \sigma > 1$, by Lemma 3.12, for any subset E_6 of $[0, +\infty)$ that has finite linear measure, there exists a sequence $\{r_n\}$, $r_n \to \infty$, such that for any given ε $(0 < 2\varepsilon < \sigma - b)$ and for all $r_n \notin E_6$, we have

$$T(r_n, A_s) \geq (\log r_n)^{\sigma-\varepsilon}. \tag{9.2}$$

By $\lambda_{(1,2)}(\frac{1}{A_s}) < \sigma_{(1,2)}(A_s) := \sigma$, for any given ε $(0 < 2\varepsilon < \sigma - \lambda_{(1,2)}(\frac{1}{A_s}))$, we have

$$N(r, A_s) \leq (\log r)^{\lambda_{(1,2)}(\frac{1}{A_s})+\varepsilon}. \tag{9.3}$$

By (1.1), we can write

$$-A_s(z) = \frac{f}{f^{(s)}}\left[\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f}\right.$$

$$\left. + A_{s-1}(z)\frac{f^{(s-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right]. \tag{9.4}$$

Noting that

$$m(r, \frac{f}{f^{(s)}}) \leq m(r, f) + m(r, \frac{1}{f^{(s)}}) \leq T(r, f) + T(r, \frac{1}{f^{(s)}})$$

$$= T(r, f) + T(r, f^{(s)}) + O(1) = O(\log r) + O(1). \tag{9.5}$$

By (9.4) and (9.5) , we obtain

$$m(r, A_s) \leq m(r, \frac{f}{f^{(s)}}) + \sum_{j\neq s} m(r, \frac{f^{(j)}}{f}) + \sum_{j\neq s} m(r, A_j) + O(1)$$

$$\leq \sum_{j\neq s} m(r, A_j) + O(\log r) + O(1). \tag{9.6}$$

By (9.6), we have

$$T(r, A_s) = N(r, A_s) + m(r, A_s)$$

$$\leq N(r, A_s) + \sum_{j\neq s} m(r, A_j) + O(\log r) + O(1). \tag{9.7}$$

Therefore, by substituting (9.1) – (9.3) into (9.7), for all $r_n \notin E_6$, we get

$$(\log r_n)^{\sigma-\varepsilon} \leq (\log r_n)^{\lambda_{(1,2)}(\frac{1}{A_s})+\varepsilon} + k\,(\log r_n)^{b+\varepsilon} + O\,(\log r_n) + O(1).$$

Since $\sigma > 1$ and ε $(0 < 2\varepsilon < \min\left\{\sigma - b, \sigma - \lambda_{(1,2)}(\frac{1}{A_s})\right\})$, then we obtain a contradiction. Therefore, if $f(z) \not\equiv 0$ is a non-transcendental meromorphic solution of (1.1), then it must be a polynomial with degree $\deg(f) \leq s-1$ if $s \geq 1$.

Now we assume that $f(z)$ is a transcendental meromorphic solution of (1.1). Note that

$$m(r, \frac{f}{f^{(s)}}) \leq m(r, f) + m(r, \frac{1}{f^{(s)}}) \leq T(r, f) + T(r, \frac{1}{f^{(s)}})$$

$$= T(r, f) + T(r, f^{(s)}) + O(1) \leq T(r, f) + (s+1)\,T(r, f)$$

$$+o\left(T\left(r, f\right)\right) + O(1) = (s+2)T(r, f) + o\left(T\left(r, f\right)\right) + O(1). \tag{9.8}$$

By Lemma 3.1, (9.4) and (9.8), we get

$$m(r, A_s) \leq m(r, \frac{f}{f^{(s)}}) + \sum_{j\neq s} m(r, \frac{f^{(j)}}{f}) + \sum_{j\neq s} m(r, A_j) + O(1)$$

$$\leq \sum_{j\neq s} m(r, A_j) + k^2 \log^+ T(2r, f) + (s+2)T(r, f) + o\left(T\left(r, f\right)\right) + O(1).$$

Hence,

$$T(r, A_s) = N(r, A_s) + m(r, A_s)$$

$$\leq N(r, A_s) + \sum_{j\neq s} m(r, A_j) + (s+2)\left(1 + o(1)\right) T(r, f) + O(1). \tag{9.9}$$

Then, by substituting (9.1), (9.2) and (9.3) into (9.9), for all $r_n \notin E_6$, we have

$$\left(1 - o(1)\right)\left(\log r_n\right)^{\sigma-\varepsilon} \leq (s+2)\left(1 + o(1)\right) T(r_n, f). \tag{9.10}$$

Since ε $\left(0 < 2\varepsilon < \min\left\{\sigma - b, \sigma - \lambda_{(1,2)}(\frac{1}{A_s})\right\}\right)$ is arbitrary, then by (9.10), we get

$$\sigma_{(1,2)}(f) \geq \sigma_{(1,2)}(A_s) > 1.$$

Suppose that all solutions of (1.1) are meromorphic and that (1.1) has a meromorphic solution base $\{f_1, f_2, \cdots, f_k\}$. By Lemma 3.13, we get

$$m(r, A_s) = O\left\{\log\left(\max_{1\leq n\leq k} T(r, f_n)\right)\right\}.$$

If $N(r, A_s) > m(r, A_s)$ holds for all sufficiently large r, then

$$T(r, A_s) = N(r, A_s) + m(r, A_s) < 2N(r, A_s),$$

that is $\sigma_{(1,2)}(A_s) \leq \lambda_{(1,2)}(\frac{1}{A_s})$, which contradicts the assumption $\lambda_{(1,2)}(\frac{1}{A_s}) < \sigma_{(1,2)}(A_s)$. Then, for all sufficiently large r, there holds

$$N(r, A_s) \leq m(r, A_s).$$

Hence, we obtain

$$T(r, A_s) = O\left\{\log\left(\max_{1\le n\le k} T(r, f_n)\right)\right\}.$$

This means that there exists one of $\{f_1, f_2, \cdots, f_k\}$, say f_1, satisfying $T(r, A_s) = O\{\log T(r, f_1)\}$. Hence, we have

$$\sigma_{(2,2)}(f_1) \ge \sigma_{(1,2)}(A_s) > 1.$$

10. Proof of Theorem 2.7

Suppose that $f(z) \not\equiv 0$ is a rational solution of (1.1). If either $f(z)$ is a rational function which has a pole at $z_0 \in \mathbb{C}$ of order $\alpha(\ge 1)$, or $f(z)$ is a polynomial with degree $\deg(f) \ge s$ then $f^{(s)}(z) \not\equiv 0$. Set $b := \max\{\sigma_{(1,2)}(A_j) : j \ne s \text{ and } j = 0, 1, \cdots, k-1\} < \mu_{(1,2)}(A_s) \le \sigma_{(1,2)}(A_s) := \sigma < \infty$. Then, for any given ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_s) - b)$ and for all sufficiently large r, we have

$$m(r, A_j) \le T(r, A_j) \le (\log r)^{b+\varepsilon}, \ (j \in \{0, 1, \cdots, k-1\}\backslash\{s\}), \tag{10.1}$$

$$T(r, A_s) \ge (\log r)^{\mu_{(1,2)}(A_s)-\varepsilon}. \tag{10.2}$$

By $\lambda_{(1,2)}(\frac{1}{A_s}) < \mu_{(1,2)}(A_s)$, we have

$$N(r, A_s) = o\left(T\left(r, A_s\right)\right), \ r \to +\infty. \tag{10.3}$$

By (9.4) and (9.5), we obtain

$$m(r, A_s) \le m(r, \frac{f}{f^{(s)}}) + \sum_{j\ne s} m(r, \frac{f^{(j)}}{f}) + \sum_{j\ne s} m(r, A_j) + O(1)$$

$$\le \sum_{j\ne s} m(r, A_j) + O(\log r) + O(1). \tag{10.4}$$

By (10.4), we get

$$T(r, A_s) = N(r, A_s) + m(r, A_s)$$

$$\le N(r, A_s) + \sum_{j\ne s} m(r, A_j) + O(\log r) + O(1). \tag{10.5}$$

By (10.3), it follows from (10.5) that

$$\left(1 - o\left(1\right)\right) T(r, A_s) \le \sum_{j\ne s} m(r, A_j) + O(\log r) + O(1). \tag{10.6}$$

Therefore, by substituting (10.1) and (10.2) into (10.6), for any given ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_s) - b)$ and for all sufficiently large r, we have

$$(\log r)^{\mu_{(1,2)}(A_s)-\varepsilon} \leq k\,(\log r)^{b+\varepsilon} + O\,(\log r) + O(1).$$

By $\mu_{(1,2)}(A_s) > 1$ and ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_s) - b)$, we obtain a contradiction. Therefore, if $f(z) \not\equiv 0$ is a non-transcendental meromorphic of (1.1), then it must be a polynomial with degree $\deg(f) \leq s - 1$ if $s \geq 1$.

Now we assume that $f(z)$ is a transcendental meromorphic solution of (1.1). By Lemma 3.1, (9.4) and (9.8), we get

$$m(r, A_s) \leq m(r, \frac{f}{f^{(s)}}) + \sum_{j\neq s} m(r, \frac{f^{(j)}}{f}) + \sum_{j\neq s} m(r, A_j) + O(1)$$

$$\leq \sum_{j\neq s} m(r, A_j) + k^2 \log^+ T(2r, f) + (s+2)T(r, f) + o\,(T\,(r, f)) + O(1).$$

Hence,

$$\begin{aligned} T(r, A_s) &= N(r, A_s) + m(r, A_s) \\ &\leq N(r, A_s) + \sum_{j\neq s} m(r, A_j) + (s+2)\,(1 + o(1))\,T(r, f) + O(1). \end{aligned}$$

By (10.3) , it follows that

$$(1 - o\,(1))\,T(r, A_s) \leq \sum_{j\neq s} m(r, A_j) + (s+2)\,(1 + o(1))\,T(r, f) + O(1). \tag{10.7}$$

Then, by substituting (10.1) and (10.2) into (10.7), for any given ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_s) - b)$ and for all sufficiently large r, we have

$$(1 - o(1))\,(\log r)^{\mu_{(1,2)}(A_s)-\varepsilon} \leq (s+2)\,(1 + o(1))\,T(r, f). \tag{10.8}$$

Since ε $(0 < 2\varepsilon < \mu_{(1,2)}(A_s) - b)$ is arbitrary and (10.8), we obtain

$$\sigma_{(1,2)}(f) \geq \mu_{(1,2)}(f) \geq \mu_{(1,2)}(A_s) > 1.$$

Suppose that all solutions of (1.1) are meromorphic and that (1.1) has a meromorphic solution base $\{f_1, f_2, \cdots, f_k\}$. By Lemma 3.13, we get

$$m(r, A_s) = O\left\{\log\left(\max_{1\leq n\leq k} T(r, f_n)\right)\right\}.$$

By (10.3) for all sufficiently large r, we have

$$T(r, A_s) = N(r, A_s) + m(r, A_s) = o\,(T\,(r, A_s)) + m(r, A_s).$$

It follows that for all sufficiently large r

$$(1 - o\,(1))\,T(r, A_s) = m(r, A_s).$$

Hence, we obtain

$$T(r, A_s) = O\left\{\log\left(\max_{1\le n\le k} T(r, f_n)\right)\right\}.$$

This means that there exists one of $\{f_1, f_2, \cdots, f_k\}$, say f_1, satisfying $T(r, A_s) = O\{\log T(r, f_1)\}$. Hence, we have

$$\sigma_{(2,2)}(f_1) \ge \mu_{(2,2)}(f_1) \ge \mu_{(1,2)}(A_s) > 1.$$

11. Proof of Theorem 2.8

By Theorem 2.4, we have $\sigma_{(2,2)}(f) \ge \sigma_{(1,2)}(A_0) \ge 1$. Set $g(z) = f(z) - z$. Clearly, we have $\bar{\lambda}_{(2,2)}(f - z) = \bar{\lambda}_{(2,2)}(g)$ and $\sigma_{(2,2)}(f) = \sigma_{(2,2)}(g)$. Equation (1.1) gets

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g = -(A_1(z) + zA_0(z)).$$

If $\sigma_{(1,2)}(A_0) > 1$, then $\sigma_{(1,2)}(A_1(z) + zA_0(z)) = \sigma_{(1,2)}(A_0) > 1$ and thus $A_1(z) + zA_0(z) \not\equiv 0$. Then, by Lemma 3.10, we have $\bar{\lambda}_{(2,2)}(g) = \sigma_{(2,2)}(g)$. Therefore, we have $\bar{\lambda}_{(2,2)}(f - z) = \sigma_{(2,2)}(f) \ge \sigma_{(1,2)}(A_0) > 1$.

12. Proof of Theorem 2.9

By Theorem 2.6, we have $\sigma_{(1,2)}(f) \ge \sigma_{(1,2)}(A_s) > 1$. Suppose that f is a transcendental meromorphic solution of (1.1) with $\sigma_{(1,2)}(f) > \sigma_{(1,2)}(A_s) > 1$. Set $g(z) = f(z) - z$. Then, we have $\sigma_{(1,2)}(f) = \sigma_{(1,2)}(g)$. Equation (1.1) gets

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g = -(A_1(z) + zA_0(z)).$$

Since $\sigma_{(1,2)}(f) > \sigma_{(1,2)}(A_s) > 1$, then

$$\max\{\sigma_{(1,2)}(A_j)\ (j = 0, 1, \cdots, k-1), \sigma_{(1,2)}(-A_1 - zA_0)\} \le \sigma_{(1,2)}(A_s)$$

$$< \sigma_{(1,2)}(f).$$

Suppose that $A_1(z) + zA_0(z) \not\equiv 0$. Then, by Lemma 3.10 we have that $\bar{\lambda}_{(1,2)}(g) = \sigma_{(1,2)}(g)$ and thus $\bar{\lambda}_{(1,2)}(f - z) = \sigma_{(1,2)}(f)$. Again by Theorem 2.6, there exists at least one solution f_1 satisfying $\sigma_{(2,2)}(f_1) \ge \sigma_{(1,2)}(A_s) > 1$ and thus again by applying Lemma 3.10, we obtain $\bar{\lambda}_{(2,2)}(f_1 - z) = \sigma_{(2,2)}(f_1) \ge \sigma_{(1,2)}(A_s) > 1$.

13. Conclusion

Throughout this article, by using the concept of logarithmic order due to Chern [9] , we study the growth and oscillation of solutions of complex higher order linear differential equations with entire or meromorphic coefficients of finite logarithmic order. We obtain some results which improve and extend results due to Kinnunen; Belaïdi; Liu, Tu and Shi; Hu, Zheng; Cao, Liu and Wang and others.

References

[1] B. Belaïdi, *Growth and oscillation of solutions to linear differential equations with entire coefficients having the same order*, Electron. J. Differential Equations 2009, No. 70, 1–10.
[2] B. Belaïdi, *On the iterated order and the fixed points of entire solutions of some complex linear differential equations*, Electron. J. Qual. Theory Differ. Equ. 2006, No. 9, 1–11.
[3] B. Belaïdi, *Oscillation of fixed points of solutions of some linear differential equations*, Acta Math. Univ. Comenian. (N.S.) 77 (2008), no. 2, 263–269.
[4] L.G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equation*, Proc. Amer. Math. Soc. 101 (1987), no. 2, 317–322.
[5] T. B. Cao, K. Liu and J. Wang, *On the growth of solutions of complex differential equations with entire coefficients of finite logarithmic order*, Math. Rep. (Bucur.) 15(65) (2013), no. 3, 249–269.
[6] T. B. Cao, J. F. Xu and Z. X. Chen, *On the meromorphic solutions of linear differential equations on the complex plane*, J. Math. Anal. Appl. 364 (2010), no. 1, 130–142.
[7] T. B. Cao, H. X. Yi, *On the complex oscillation of higher order linear differential equations with meromorphic coefficients*, J. Syst. Sci. Complex. 20 (2007), no. 1, 135–148.
[8] Z. X. Chen, C. C. Yang, *Quantitative estimations on the zeros and growths of entire solutions of linear differential equations*, Complex Variables Theory Appl. 42 (2000), no. 2, 119–133.
[9] Peter T.Y. Chern, *On meromorphic functions with finite logarithmic order*, Trans. Amer. Math. Soc. 358 (2006), no. 2, 473–489.
[10] G. Frank, S. Hellerstein, *On the meromorphic solutions of nonhomogeneous linear differential equations with polynomial coefficients*, Proc. London Math. Soc. (3) 53 (1986), no. 3, 407–428.

[11] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) 37(1988), no. 1, 88–104.
[12] G. G. Gundersen, *Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. 305 (1988), no. 1, 415–429.
[13] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.
[14] J. Heittokangas, R. Korhonen and J. Rättyä, *Generalized logarithmic derivative estimates of Gol'dberg-Grinshtein type*, Bull. London Math. Soc. 36 (2004), no. 1, 105–114.
[15] J. Heittokangas, Z. T. Wen, *Functions of finite logarithmic order in the unit disc*, Part I. J. Math. Anal. Appl. 415 (2014), no. 1, 435–461.
[16] J. Heittokangas, Z. T. Wen, *Functions of finite logarithmic order in the unit disc*, Part II. Comput. Methods Funct. Theory 15 (2015), no. 1, 37–58.
[17] H. Hu, X. M. Zheng, *Growth of solutions to linear differential equations with entire coefficients*, Electron. J. Differential Equations 2012, No. 226, 1–15.
[18] H. Hu, X. M. Zheng, *Growth of solutions of linear differential equations with meromorphic coefficients of $[p,q]-order$*, Math. Commun. 19(2014), 29–42.
[19] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the [p, q]-order and lower [p, q]-order of an entire function*, J. Reine Angew. Math. 282 (1976), 53–67.
[20] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the [p, q]-type and lower [p, q]-type of an entire function*, J. Reine Angew. Math. 290 (1977), 180–190.
[21] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. 22 (1998), no. 4, 385–405.
[22] I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.
[23] L. M. Li, T. B. Cao, *Solutions for linear differential equations with meromorphic coefficients of [p, q]-order in the plane*, Electron. J. Diff. Equ. 2012 No. 195 (2012), 1–15.
[24] J. Liu, J. Tu and L. Z. Shi, *Linear differential equations with entire coefficients of [p, q]-order in the complex plane*, J. Math. Anal. Appl. 372 (2010), 55–67.

[25] M. S. Liu, X. M. Zhang, *Fixed points of meromorphic solutions of higher order linear differential equations*, Ann. Acad. Sci. Fenn. Math. 31 (2006), no. 1, 191–211.
[26] D. Sato, *On the rate of growth of entire functions of fast growth,* Bull. Amer. Math. Soc. 69 (1963), 411–414.
[27] C. C. Yang, H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.

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