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GROWTH RATES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS IN THE DIRECTION OF THEIR RELATIVE L^*-ORDERS

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Abstract. In this paper we establish some newly developed results regarding the growth rates of composite entire and meromorphic functions on the basis of their relative L^* -order and relative L^* - lower order.

1. Introduction and Preliminaries.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function corresponding to entire f is defined as $M(r, f) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, $M_f(r)$ can not be defined as f is not analytic. In this case one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as maximum modulus function in the following manner:

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$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r)$ and $m_f(r)$ are respectively the enumerative function and the proximity function corresponding to f. For further details, one may see [17]. If f is an entire function, then the Nevanlinna's Characteristic $T_f(r)$ of f reduces to $m_f(r)$.

For a non-constant entire function f, $T_f(r)$ is strictly increasing and continuous and its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s \to \infty} T_f^{-1}(s) = \infty$.

In this connection we just recall the following definition which is relevant:

Definition 1. {[2]} *A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see* [2].

In the sequel we use the following notation : $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Taking this into account the following definition is well known:

Definition 2. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

When f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r \to \infty} \frac{\log T_f(r)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [16] defined it in the following way:

Definition 3. [16] *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(> 0)$,*

$$\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \text{ for } r \geq r(\varepsilon) \text{ and}$$

uniformly for $k\,(\geq 1)$.

Somasundaram and Thamizharasi [17] introduced the notions of L-order and L-lower order for entire functions. The more generalized concept for L-order and L-lower order for entire and meromorphic functions are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 4. [17] *The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} .$$

Bernal [2] introduced the definition of *relative order* of an entire function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .\end{aligned}$$

For $g(z) = \exp z$, the above definition coincides with the classical one {cf. [18]}.

Similarly, one can define the *relative lower order* of an entire function f with respect to another entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .$$

Along the lines of of Somasundaram and Thamizharasi { cf. [17] }, one can define the *relative L^*-order* of an entire function in the following manner:

Definition 5. {[3],[4]} *The* relative L^*-order *of an entire function f with respect to another entire function g , denoted by $\rho_g^{L^*}(f)$ is defined in the following wa*

$$\begin{aligned}\rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left\{re^{L(r)}\right\}^\mu \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} .\end{aligned}$$

Analogously, one may define the relative L^*-lower order *of an entire function f with respect to another entire function g denoted by $\lambda_g^{L^*}(f)$ as follows*

$$\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} .$$

Extending the notion of Bernal [2], Lahiri and Banerjee [14] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 6. [14] *Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} .\end{aligned}$$

The definition coincides with the classical one [14] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} .$$

Further along the lines of Somasundaram and Thamizharasi [17] and Lahiri and Banerjee [14], one may define the relative L^*-order and relative L^*-lower order of a meromorphic function f with respect to an entire function g in the following manner:

Definition 7. *The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of a meromorphic function f with respect to an entire function g are defined by*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} .$$

For entire and meromorphic functions, the notions of their growth indicators such as *order* is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the classical growth indicators. But at that time, the concepts of *relative orders* and consequently the *relative L^*-orders* of entire and meromorphic functions with respect to another entire function and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the growth of composite entire and meromorphic functions needs to be modified on the basis of their *relative order* as well as *relative* L^*-*orders* some of which has been explored in [5], [6], [7], [8], [9], [10] and [11]. In this paper we establish some newly developed results related to the growth rates of composite entire and meromorphic functions on the basis of their *relative* L^*-*orders (* respectively *relative* L^**- lower orders).* We have used the standard notations and definitions in the theory of entire and meromorphic functions which are available in [12] and [19].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] *If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) \ .$$

Lemma 2. [15] *Let f and g be any two entire functions. Then for all $r > 0$,*

$$T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1), f \right\} \ .$$

Lemma 3. [11] *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right) .$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_g^{L^*} < \rho_h^{L^*}(f) < \infty$ and $\rho_k^{L^*}(g) < \infty$. If h satisfy the Property (A), then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r) + \log M_k^{-1} M_g(r)}{T_h^{-1} T_f(r) \cdot K(r, g; L)} = 0 \ ,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \ \text{if } L(M_g(r)) = o\left\{\left[re^{L(r)}\right]^{\alpha}\right\} \ \text{as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha < \ \rho_h^{L^*}(f) \\ L(M_g(r)) \ \text{otherwise.} \end{cases}$$

Proof. Let us consider that $\beta > 2$ and $\delta > 1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, Lemma 3 and the inequality $T_g(r) \le \log M_g(r)$ {cf. [12] } for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\}T_f(M_g(r))\right] \\ i.e.,\ T_h^{-1}T_{f\circ g}(r) &\leqslant \beta\left[T_h^{-1}T_f(M_g(r))\right]^{\delta} \\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\leqslant \delta\log T_h^{-1}T_f(M_g(r)) + O(1) \end{aligned}$$

$$\log T_h^{-1}T_{f\circ g}(r) \leqslant \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\left(\log M_g(r) + L(M_g(r))\right) + O(1)\ . \tag{1}$$

Now from the definition of $\rho_g^{L^*}$, we obtain for all sufficiently large positive numbers of r that

$$\log^{[2]} M_g(r) \le \left(\rho_g^{L^*}+\varepsilon\right)\left[\log r + L(r)\right]\ . \tag{2}$$

Also from the definition of $\rho_k^{L^*}(g)$, we get for all sufficiently large positive numbers of r that

$$\log M_k^{-1}M_g(r) \le \left(\rho_k^{L^*}(g)+\varepsilon\right)\left[\log r + L(r)\right]\ . \tag{3}$$

Therefore from (1) and in view of (2), we get for all sufficiently large positive numbers of r that

$$\begin{aligned} &\log T_h^{-1}T_{f\circ g}(r) \\ \le\ &O(1) + \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\cdot\left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M_g(r))\right]\ . \end{aligned} \tag{4}$$

Now from (3) and (4), it follows for all sufficiently large positive numbers of r that

$$\begin{aligned} &\log T_h^{-1}T_{f\circ g}(r) + \log M_k^{-1}M_g(r) \\ &\le \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\cdot\left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M_g(r))\right] \\ &\quad O(1) + \left(\rho_k^{L^*}(g)+\varepsilon\right)\left[\log r + L(r)\right]\ . \end{aligned} \tag{5}$$

Also from the definition of $\rho_h^{L^*}(f)$, we obtain for a sequence of positive numbers of r tending to infinity that

$$\log T_h^{-1} T_f(r) \geq \left(\rho_h^{L^*}(f) - \varepsilon\right) \log\left[re^{L(r)}\right]$$

$$\text{(6)} \qquad i.e.,\ T_h^{-1} T_f(r) \geq \left[re^{L(r)}\right]^{\left(\rho_h^{L^*}(f)-\varepsilon\right)} .$$

Now from (5) and (6), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r) + \log M_k^{-1} M_g(r)}{T_h^{-1} T_f(r)}$$

$$\leq \frac{O(1) + \left(\rho_k^{L^*}(g) + \varepsilon\right)\left[\log r + L(r)\right]}{T_h^{-1} T_f(r)}$$

$$\text{(7)} \qquad + \frac{\delta\left(\rho_h^{L^*}(f) + \varepsilon\right) \cdot \left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)} + L\left(M_g(r)\right)\right]}{\left[re^{L(r)}\right]^{\left(\rho_h^{L^*}(f)-\varepsilon\right)}} .$$

Since $\rho_g^{L^*} < \rho_h^{L^*}(f)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\text{(8)} \qquad \rho_g^{L^*} + \varepsilon < \rho_h^{L^*}(f) - \varepsilon .$$

Case I. Let $L\left(M_g(r)\right) = o\left\{\left[re^{L(r)}\right]^{\alpha}\right\}$ as $r \to \infty$ and for some $\alpha < \rho_h^{L^*}(f)$.
As $\alpha < \rho_h^{L^*}(f)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\text{(9)} \qquad \alpha < \rho_h^{L^*}(f) - \varepsilon .$$

Since $L\left(M_g(r)\right) = o\left\{\left[re^{L(r)}\right]^{\alpha}\right\}$ as $r \to \infty$, we get on using (9) that

$$\frac{L\left(M_g(r)\right)}{\left[re^{L(r)}\right]^{\alpha}} \to 0 \text{ as } r \to \infty$$

$$\text{(10)} \qquad i.e.,\ \frac{L\left(M_g(r)\right)}{\left[re^{L(r)}\right]^{\left(\rho_h^{L^*}(f)-\varepsilon\right)}} \to 0 \text{ as } r \to \infty .$$

Now in view of (7), (8) and (10), we get that

$$\text{(11)} \qquad \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r) + \log M_k^{-1} M_g(r)}{T_h^{-1} T_f(r)} = 0 .$$

Case II. If $L\left(M_g(r)\right) \neq o\left\{\left[re^{L(r)}\right]^{\alpha}\right\}$ as $r \to \infty$ and for some $\alpha < \rho_h^{L^*}(f)$ then we get from (7) for a sequence of positive numbers of r

tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)+\log M_k^{-1}M_g(r)}{T_h^{-1}T_f(r)\cdot L(M_g(r))}$$
$$\leq \frac{O(1)+\left(\rho_k^{L^*}(g)+\varepsilon\right)[\log r+L(r)]}{\left[r\exp^{[p]}L(r)\right]^{\left({}^{(m)}_{(p)}\rho_f^{L^*}-\varepsilon\right)}\cdot L(M_g(r))}$$
$$+\frac{\delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\cdot\left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)}+L(M_g(r))\right]}{\left[re^{L(r)}\right]^{\left(\rho_h^{L^*}(f)-\varepsilon\right)}\cdot L(M_g(r))}\ . \tag{12}$$

Now using (8), it follows from (12) that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)+\log M_k^{-1}M_g(r)}{T_h^{-1}T_f(r)\cdot L(M_g(r))}=0\ . \tag{13}$$

Combining (11) and (13), we obtain that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)+\log M_k^{-1}M_g(r)}{T_h^{-1}T_f(r)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(M_g(r))=o\left\{\left[re^{L(r)}\right]^{\alpha}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha<\ \rho_h^{L^*}(f) \\ L(M_g(r)) \text{ otherwise.}\end{cases}$

Thus the theorem is established. ∎

Theorem 2. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_g^{L^*}<\lambda_h^{L^*}(f)<\infty$ and $\rho_k^{L^*}(g)<\infty$. If h satisfy the Property (A), then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)+\log M_k^{-1}M_g(r)}{T_h^{-1}T_f(r)\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \textit{ if } L(M_g(r))=o\left\{\left[re^{L(r)}\right]^{\alpha}\right\} \textit{ as } r\to\infty \\ \qquad\qquad \textit{and for some } \alpha<\ \lambda_h^{L^*}(f) \\ L(M_g(r)) \textit{ otherwise.}\end{cases}$

Theorem 3. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\lambda_g^{L^*}<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$ and $\rho_k^{L^*}(g)<\infty$. If h satisfy the Property (A), then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)+\log M_k^{-1}M_g(r)}{T_h^{-1}T_f(r)\cdot K(r,g;L)}=0\ ,$$

$$\textit{where } K(r,g;L) = \begin{cases} 1 \textit{ if } L(M_g(r)) = o\left\{\left[re^{L(r)}\right]^{\alpha}\right\} \textit{ as } r \to \infty \\ \qquad\qquad \textit{and for some } \alpha < \lambda_h^{L^*}(f) \\ L(M_g(r)) \textit{ otherwise.} \end{cases}$$

Theorem 4. *Let f be a meromorphic function and g, h and k be any three entire functions such that $\rho_g^{L^*} < \lambda_h^{L^*}(f) \le \rho_h^{L^*}(f) < \infty$ and $\rho_k^{L^*}(g) < \infty$. If h satisfies the Property (A), then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r) + \log M_k^{-1}M_g(r)}{T_h^{-1}T_f(r)\cdot K(r,g;L)} = 0\ ,$$

$$\textit{where } K(r,g;L) = \begin{cases} 1 \textit{ if } L(M_g(r)) = o\left\{\left[re^{L(r)}\right]^{\alpha}\right\} \textit{ as } r \to \infty \\ \qquad\qquad \textit{and for some } \alpha < \lambda_h^{L^*}(f) \\ L(M_g(r)) \textit{ otherwise.} \end{cases}$$

The proofs of Theorem 2, Theorem 3 and Theorem 4 are omitted because those can be carried out along the lines of Theorem 1.

Theorem 5. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f) \le \rho_h^{L^*}(f) < \infty$, $0 < \lambda_g^{L^*} < \infty$. Also let h satisfy the Property (A). Then for every constant A and for any real number x,*

$$\lim_{r\to\infty} \frac{\log T_h^{-1}T_{f\circ g}(r)}{\left\{\log T_h^{-1}T_f(r^A)\right\}^{1+x}} = \infty\ .$$

Proof. If x is such that $1+x \le 0$, then the theorem is obvious. So we suppose that $1+x > 0$. Let us consider that $\beta > 2$ and $\delta > 1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2, Lemma 3 and the inequality $T_g(r) \le \log M_g(r)$ {cf. [12] } for all sufficiently large positive numbers of r that

$$T_h^{-1}T_{f\circ g}(r) \ge T_h^{-1}\left[\frac{1}{3}T_f\left\{\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right\}\right]$$

$$\textit{i.e., } T_h^{-1}T_{f\circ g}(r) \ge \left[\frac{1}{\beta}T_h^{-1}T_f\left\{\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right\}\right]^{\frac{1}{\delta}}$$

$$\textit{i.e., } \log T_h^{-1}T_{f\circ g}(r) \ge \log\left[\frac{1}{\beta}T_h^{-1}T_f\left\{\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right\}\right]^{\frac{1}{\delta}}$$

(14)

$$\textit{i.e., } \log T_h^{-1}T_{f\circ g}(r) \ge O(1) + \frac{1}{\delta}\log T_h^{-1}T_f\left\{\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right\}\ .$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}(r) \geq O(1)+\frac{1}{\delta}\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[\log\left\{\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right\}\right.$$
$$\left.+\exp L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}(r) \geq O(1)+\frac{1}{\delta}\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[\log M_g\left(\frac{r}{4}\right)+o(1)\right.$$
$$\left.+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}(r)$$
$$\geq O(1)+\frac{1}{\delta}\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[\left[\left(\frac{r}{4}\right)\exp L(r)\right]^{\lambda_g^{L^*}-\varepsilon}+o(1)\right.$$
$$\left.+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right] \tag{15}$$

where we choose $0<\varepsilon<\min\left\{\lambda_h^{L^*}(f),\lambda_g^{L^*}\right\}$.
Also for all sufficiently large positive numbers of r, we get that

$$\log T_h^{-1}T_f\left(r^A\right) \leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left[r^A\exp L\left(r^A\right)\right]$$
$$i.e.,\ \left\{\log T_h^{-1}T_f\left(r^A\right)\right\}^{1+x} \leq$$
$$\left(\rho_h^{L^*}(f)+\varepsilon\right)^{1+x}\left(\log\left[r^A\exp L\left(r^A\right)\right]\right)^{1+x}. \tag{16}$$

Therefore from (15) and (16) , it follows for all sufficiently large positive numbers of r that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\left\{\log T_h^{-1}T_f\left(r^A\right)\right\}^{1+x}}$$
$$\geq \frac{O(1)+\frac{1}{\delta}\left(\lambda_h^{L^*}(f)-\varepsilon\right)\left[\left[\left(\frac{r}{4}\right)\exp L(r)\right]^{\lambda_g^{L^*}-\varepsilon}+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)\right]}{\left(\rho_h^{L^*}(f)+\varepsilon\right)^{1+x}\left(\log\left[r^A\exp L\left(r^A\right)\right]\right)^{1+x}}.$$

Thus from above the theorem follows. ∎

Theorem 6. *Let f, g, h and k be any four entire functions such that $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$, $0<\lambda_g^{L^*}<\infty$, $0<\rho_k^{L^*}(g)<\infty$. Also let h satisfy the Property (A). Then for every constant A and for any real number x,*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\left\{\log T_k^{-1}T_g\left(r^A\right)\right\}^{1+x}}=\infty\ .$$

The proof of Theorem 6 is omitted as it can be carried out along the lines of Theorem 5.

Theorem 7. *Let f be a meromorphic function and g and h be any two entire functions satisfying the conditions that (i) $0 < \rho_h^{L^*}(f) < \infty$, (ii) $\rho_g^{L^*}$ is non zero finite and (iii) h satisfy the Property (A). Then for each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f(\exp r^A)} = 0$$

where $A > (1+\alpha)\cdot \rho_g^{L^*}$.

Proof. If $1+\alpha < 0$, then the theorem is trivial. So we take $1+\alpha > 0$. Now from (4), we obtain for all sufficiently large positive numbers of r that

$$\begin{aligned} & \log T_h^{-1} T_{f\circ g}(r) \\ \leq\ & O(1) + \delta\left(\rho_h^{L^*}(f) + \varepsilon\right)\cdot \left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)} + L\left(M_g(r)\right)\right] . \end{aligned}$$

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) \ \leq\ & \exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)}\cdot \delta\left(\rho_h^{L^*}(f)+\varepsilon\right) + \\ & O(1) + \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\cdot L\left(M_g(r)\right) \end{aligned}$$

$$\begin{aligned} & i.e., \ \left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha} \\ \leq\ & \left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)}\cdot \delta\left(\rho_h^{L^*}(f)+\varepsilon\right) + O(1)\right. \\ & \left. +\delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\cdot L\left(M_g(r)\right)\right]^{1+\alpha} . \qquad (17) \end{aligned}$$

Again we have for a sequence of positive numbers of r tending to infinity and for $\varepsilon(>0)$,

$$\log T_h^{-1} T_f\left(\exp r^A\right) \geq \left(\rho_h^{L^*}(f) - \varepsilon\right)\log\left[\exp\left(r^A\right)\left\{\exp L\left(\exp\left(r^A\right)\right)\right\}\right]$$

$$\begin{aligned} & i.e., \ \log T_h^{-1} T_f\left(\exp r^A\right) \\ \geq\ & \left(\rho_h^{L^*}(f) - \varepsilon\right)\left[r^A + L\left(\exp\left(r^A\right)\right)\right] . \qquad (18) \end{aligned}$$

Now let

$$\delta\left(\rho_h^{L^*}(f)+\varepsilon\right) = k_1, \delta\left(\rho_h^{L^*}(f)+\varepsilon\right)\cdot L\left(M_g(r)\right) + O(1) = k_2,$$

$$\left(\rho_h^{L^*}(f)-\varepsilon\right) = k_3 \text{ and } \left(\rho_h^{L^*}(f)-\varepsilon\right) L\left(\exp\left(r^A\right)\right) = k_4.$$

Then from (17), (18) and above, we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f\left(\exp r^A\right)} \leq \frac{\left[\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)} k_1 + k_2\right]^{1+\alpha}}{k_3 r^A + k_4}$$

$$i.e., \ \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f\left(\exp r^A\right)}$$

$$\leq \frac{\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)} \left[k_1 + \frac{k_2}{\exp\left(re^{L(r)}\right)^{\left(\rho_g^{L^*}+\varepsilon\right)}}\right]^{1+\alpha}}{k_3 r^A + k_4}$$

where k_1, k_2,k_3 and k_4 are all finite.

Since$\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha) < A$, we obtain from above that

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f\left(\exp r^A\right)} = 0$$

where we choose $\varepsilon(>0)$ in such a way that

$$0 < \varepsilon < \min\left\{\rho_h^{L^*}(f), \frac{A}{1+\alpha} - \rho_g^{L^*}\right\}.$$

This proves the theorem. ∎

Along the lines of Theorem 7, the following theorem may be proved and therefore its proof is omitted:

Theorem 8. *Let f be a meromorphic function and g and h be any two entire functions satisfying the conditions that (i) $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$, (ii) $\rho_g^{L^*}$ is non zero finite and (iii) h satisfy the Property (A). Then for each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_h^{-1} T_f\left(\exp r^A\right)} = 0$$

where $A > (1+\alpha)\cdot\rho_g^{L^*}$.

Theorem 9. *Let f be a meromorphic function and g, h and k be any three entire functions satisfying the conditions that (i) $\rho_h^{L^*}(f) < \infty$,* *(ii)* $\rho_g^{L^*}$ *is non zero finite (iii)* $\lambda_k^{L^*}(g) > 0$ *and (iv) h satisfy the Property (A). Then for each* $\alpha \in (-\infty, \infty)$,

$$\lim_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_k^{-1} T_g(\exp r^A)} = 0$$

where $A > (1+\alpha)\cdot \rho_g^{L^*}$.

Theorem 10. *Let f be a meromorphic function and g, h and k be any three entire functions satisfying the conditions that (i)* $\rho_h^{L^*}(f) < \infty$, *(ii)* $0 < \rho_g^{L^*} < \infty$, *(iii)* $\rho_k^{L^*}(g) > 0$ *and (iv) h satisfy the Property (A). Then for each* $\alpha \in (-\infty, \infty)$,

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_k^{-1} T_g(\exp r^A)} = 0 \text{ where } A > (1+\alpha)\cdot \rho_g^{L^*} .$$

The proof of Theorem 9 and Theorem 10 are omitted because those can be carried out along the lines of Theorem 8 and Theorem 7 respectively.

Theorem 11. *Let f be a meromorphic function and g, h and k be any three entire functions satisfying the conditions that (i)* $\rho_h^{L^*}(f) < \infty$, *(ii)* $0 < \rho_g^{L^*} < \infty$, *(iii)* $0 < \lambda_k^{L^*}(g) < \infty$ *and (iv) h satisfy the Property (A). Then*
(a) if $L(M_g(r)) = o\left\{\log T_k^{-1} T_g(r)\right\}$ *then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} \le \frac{\rho_g^{L^*}}{\lambda_k^{L^*}(g)}$$

and (b) if $\log T_k^{-1} T_g(r) = o\{L(M_g(r))\}$ *then*

$$\lim_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} = 0 .$$

Proof. Using $\log\left[1 + \frac{L(M_g(r))+O(1)}{\log M_g(r)}\right] \sim \frac{L(M_g(r))+O(1)}{\log M_g(r)}$, we obtain from (1) for all sufficiently large positive numbers of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta\left(\rho_h^{L^*}(f) + \varepsilon\right) \log M_g(r) \left[1 + \frac{L(M_g(r)) + O(1)}{\log M_g(r)}\right]$$

$$\begin{aligned} i.e., \ \log^{[2]} T_h^{-1} T_{f\circ g}(r) \le & \log \delta\left(\rho_h^{L^*}(f) + \varepsilon\right) + \log^{[2]} M_g(r) \\ & + \log\left[1 + \frac{L(M_g(r)) + O(1)}{\log M_g(r)}\right] \end{aligned}$$

$$i.e., \ \log^{[2]} T_h^{-1} T_{f \circ g}(r) \leq \log \delta \left(\rho_h^{L^*}(f) + \varepsilon \right) + \left(\rho_g^{L^*} + \varepsilon \right) \log \left[r \exp L(r) \right] + \log \left[1 + \frac{L(M_g(r)) + O(1)}{\log M_g(r)} \right]$$

$i.e.,$ $\log^{[2]} T_h^{-1} T_{f \circ g}(r) \leq$

$$O(1) + \left(\rho_g^{L^*} + \varepsilon \right) \left[\log r + L(r) \right] + \frac{L(M_g(r)) + O(1)}{\log M_g(r)} \ . \tag{19}$$

Again from the definition of relative L^*-lower order, we get for all sufficiently large positive numbers of r that

$$\log T_k^{-1} T_g(r) \geq \left(\lambda_k^{L^*}(g) - \varepsilon \right) \log \left[r \exp L(r) \right]$$

$$i.e., \ \left[\log r + L(r) \right] \leq \frac{\log T_k^{-1} T_g(r)}{(\lambda_k^{L^*}(g) - \varepsilon)} \ . \tag{20}$$

Hence from (19) and (20), it follows for all sufficiently large positive numbers of r that

$$\log^{[2]} T_h^{-1} T_{f \circ g}(r) \leq O(1) + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_k^{L^*}(g) - \varepsilon} \right) \cdot \log T_k^{-1} T_g(r) + \frac{L(M_g(r)) + O(1)}{\log M_g(r)}$$

$$i.e, \ \frac{\log^{[2]} T_h^{-1} T_{f \circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))}$$
$$\leq \frac{O(1)}{\log T_k^{-1} T_g(r) + L(M_g(r))} + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_k^{L^*}(g) - \varepsilon} \right) \cdot \frac{\log T_k^{-1} T_g(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} + \frac{L(M_g(r)) + O(1)}{\left[\log T_k^{-1} T_g(r) + L(M_g(r)) \right] \log M_g(r)}$$

$$i.e, \ \frac{\log^{[2]} T_h^{-1} T_{f \circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))}$$
$$\leq \frac{\frac{O(1)}{L(M_g(r))}}{\frac{\log T_k^{-1} T_g(r)}{L(M_g(r))} + 1} + \frac{\left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_k^{L^*}(g) - \varepsilon} \right)}{1 + \frac{L(M_g(r))}{\log T_k^{-1} T_g(r)}} + \frac{1}{\left[1 + \frac{\log T_k^{-1} T_g(r)}{L(M_g(r))} \right] \log M_g(r)} \ . \tag{21}$$

Since $L(M_g(r)) = o\left\{ \log T_k^{-1} T_g(r) \right\}$ as $r \to \infty$ and $\varepsilon\,(> 0)$, is arbitrary we obtain from (21) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T_h^{-1} T_{f \circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} \leq \frac{\rho_g^{L^*}}{\lambda_k^{L^*}(g)} \ . \tag{22}$$

Again if $\log T_k^{-1} T_g(r) = o\{L(M_g(r))\}$ then from (21), we get that

$$\lim_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} = 0 \ . \tag{23}$$

Thus from (22) and (23) the theorem is established. ∎

Along the lines of Theorem 11, the following theorem may be proved and therefore its proof is omitted:

Theorem 12. *Let f be a meromorphic function and g, h and k be any three entire functions satisfying the conditions (i) $\rho_h^{L^*}(f) < \infty$, (ii) $\rho_g^{L^*} < \infty$ and (iii) $\rho_k^{L^*}(g) > 0$. Also h satisfy the Property (A). Then*
(a) if $L(M_g(r)) = o\left\{\log T_k^{-1} T_g(r)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} \leq \frac{\rho_g^{L^*}}{\rho_k^{L^*}(g)}$$

and (b) if $\log T_k^{-1} T_g(r) = o\{L(M_g(r))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L(M_g(r))} = 0 \ .$$

Now we state the following three theorems without their proofs as those can be carried out along the lines of Theorem 11 and Theorem 12:

Theorem 13. *Let f be a meromorphic function and g and h be any two entire functions satisfying the conditions (i) $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and (ii) $\rho_g^{L^*} < \infty$. Also h satisfy the Property (A). Then*
(a) if $L(M_g(r)) = o\left\{\log T_h^{-1} T_f(r)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L(M_g(r))} \leq \frac{\rho_g^{L^*}}{\lambda_h^{L^*}(f)}$$

and (b) if $\log T_h^{-1} T_f(r) = o\{L(M_g(r))\}$ then

$$\lim_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L(M_g(r))} = 0 \ .$$

Theorem 14. *Let f be a meromorphic function and g and h be any two entire functions satisfying the conditions (i) $0 < \rho_h^{L^*}(f) < \infty$ and (ii) $\rho_g^{L^*} < \infty$. Also h satisfy the Property (A). Then* *(a) if* $L(M_g(r)) = o\left\{\log T_h^{-1} T_f(r)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L(M_g(r))} \leq \frac{\rho_g^{L^*}}{\rho_h^{L^*}(f)}$$

and (b) if $\log T_h^{-1} T_f(r) = o\{L(M_g(r))\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L(M_g(r))} = 0 \ .$$

Theorem 15. *Let* f *be a meromorphic function and* g *and* h *be any two entire functions satisfying the conditions (i)* $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ *and (ii)* $\lambda_g^{L^*} < \infty$. *Also* h *satisfy the Property (A). Then*
(a) if $L(M_g(r)) = o\left\{\log T_h^{-1} T_f(r)\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L(M_g(r))} \leq \frac{\lambda_g^{L^*}}{\lambda_h^{L^*}(f)}$$

and (b) if $\log T_h^{-1} T_f(r) = o\{L(M_g(r))\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L(M_g(r))} = 0 \ .$$

Theorem 16. *Let* f, g *and* h *be any three entire functions such that* $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ *and* $\rho_g^{L^*} > 0$. *Also let* h *satisfy the Property (A). Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\rho_g^{L^*}}{\rho_h^{L^*}(f)} \ .$$

Proof. Now from (14), we have for all sufficiently large positive numbers of r that

$$\log T_h^{-1} T_{f\circ g}(r) \geq O(1) + \frac{1}{\delta}\left(\lambda_h^{L^*}(f) - \varepsilon\right)\left[\log\left\{\frac{1}{8} M_g\left(\frac{r}{4}\right) + o(1)\right\} + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e.,\ \log T_h^{-1} T_{f\circ g}(r) \geq O(1) + \frac{1}{\delta}\left(\lambda_h^{L^*}(f) - \varepsilon\right) \left[\log\left\{\frac{1}{8} M_g\left(\frac{r}{4}\right)\left(1 + \frac{o(1)}{\frac{1}{8} M_g\left(\frac{r}{4}\right)}\right)\right\} + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}(r) \geq \frac{1}{\delta}\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log M_g\left(\frac{r}{4}\right)\cdot \left\{\frac{\log M_g\left(\frac{r}{4}\right)+\log\left(1+\frac{o(1)}{\frac{1}{8}M_g\left(\frac{r}{4}\right)}\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)}{\log M_g\left(\frac{r}{4}\right)}\right\}$$

$$i.e.,\ \log^{[2]} T_h^{-1}T_{f\circ g}(r) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+ \log\left\{\frac{\log M_g\left(\frac{r}{4}\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)}{\log M_g\left(\frac{r}{4}\right)}\right\}$$

$$i.e.,\ \log^{[2]} T_h^{-1}T_{f\circ g}(r) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+ \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right) - \log\left[\exp\left\{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)\cdot L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right\}\right] + \log\left\{\frac{\log M_g\left(\frac{r}{4}\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)}{\log M_g\left(\frac{r}{4}\right)}\right\}$$

$$i.e.,\ \log^{[2]} T_h^{-1}T_{f\circ g}(r) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+ \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right) + \log\left\{\frac{\log M_g\left(\frac{r}{4}\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)}{\exp\left\{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)\cdot L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right\}\cdot\log M_g\left(\frac{r}{4}\right)}\right\}$$

$$i.e.,\ \log^{[2]} T_h^{-1}T_{f\circ g}(r) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+ \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right) .$$

Now from above, it follows for a sequence of positive numbers of r tending to infinity that

$$\log^{[2]} T_h^{-1} T_{f\circ g}(r) \geq \left(\rho_g^{L^*} - \varepsilon\right) \log\left[\frac{r}{4}\exp L\left(\frac{r}{4}\right)\right] + \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right) L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right). \tag{24}$$

Further, we get for all sufficiently large positive numbers of r that

$$\log T_h^{-1} T_f(r) \leq \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left[r \exp L(r)\right]$$

$$i.e., \ \log T_h^{-1} T_f(r) \leq \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left[\frac{r}{4}\exp L\left(\frac{r}{4}\right)\right] + \log 4. \tag{25}$$

Hence from (24) and (25) , it follows for a sequence of positive numbers of r tending to infinity that

$$i.e., \ \log^{[2]} T_h^{-1} T_{f\circ g}(r) \geq \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right)\left(\log T_h^{-1} T_f(r) - \log 4\right) + \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right) L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)$$

$$i.e., \ \log^{[2]} T_h^{-1} T_{f\circ g}(r) \geq \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right)\left[\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)\right] - \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right) \log 4$$

$$i.e., \ \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right) - \frac{\left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_h^{L^*}(f) + \varepsilon}\right) \log 4}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)}\ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\rho_g^{L^*}}{\rho_h^{L^*}(f)}\ .$$

This proves the theorem. ∎

Along the lines of Theorem 16, the following two theorems may be proved and therefore their proofs are omitted:

Theorem 17. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f) < \infty$ and $\lambda_g^{L^*} > 0$. Also let h satisfy the Property (A). Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_g^{L^*}}{\lambda_h^{L^*}(f)} .$$

Theorem 18. *Let f, g, h and k be any four entire functions such that $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ and $\lambda_g^{L^*} > 0$. Also let h satisfy the Property (A). Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_g^{L^*}}{\rho_h^{L^*}(f)} .$$

Now we state the following three theorems without their proofs as those can be carried out along the lines of Theorem 16 and Theorem 18:

Theorem 19. *Let f, g, h and k be any four entire functions such that $\lambda_h^{L^*}(f) > 0$, $0 < \rho_g^{L^*} < \infty$ and $0 < \rho_k^{L^*}(g) < \infty$. Also let h satisfy the Property (A). Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\rho_g^{L^*}}{\rho_k^{L^*}(g)} .$$

Theorem 20. *Let f, g, h and k be any four entire functions such that $\lambda_h^{L^*}(f) > 0$, $0 < \lambda_g^{L^*} < \infty$ and $0 < \lambda_k^{L^*}(g) < \infty$. Also let h satisfy the Property (A). Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_g^{L^*}}{\lambda_k^{L^*}(g)} .$$

Theorem 21. *Let f, g, h and k be any four entire functions such that $\lambda_h^{L^*}(f) > 0$, $0 < \lambda_g^{L^*} < \infty$ and $0 < \rho_k^{L^*}(g) < \infty$. Also let h satisfy the Property (A). Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_k^{-1} T_g(r) + L\left(\frac{1}{8} M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_g^{L^*}}{\rho_k^{L^*}(g)} .$$

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