"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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CONTRA m-CONTINUOUS MULTIFUNCTIONS IN FUZZY SET THEORY

ANJANA BHATTACHARYYA

Abstract. In this paper, a new type of fuzzy multifunction is introduced between a set having minimal structure [15, 16] and a fuzzy topological space (fts, for short) in the sense of Chang [6] which generalizes fuzzy contra continuous multifunction. It is also shown that the image of an m-compact space [15, 16] under this fuzzy multifunction is fuzzy s-closed [20] under certain conditions.

1. Introduction

In [15, 16], minimal structure (m-structure, for short) on a non-empty set is introduced and in [14], Papageorgiou introduced fuzzy multifunction between an ordinary topological space and an fts. In this paper, we define a fuzzy multifunction between a non-empty set having minimal structure, called an m-space and an fts.

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2. Preliminaries

A fuzzy set [21] A is a mapping from a nonempty set Y into the closed interval $I = [0, 1]$, i.e., $A \in I^Y$. The support of a fuzzy set A in Y will be denoted as $suppA$ [21] and is defined by $suppA = \{y \in Y : A(y) \neq 0\}$. A fuzzy point [18] with the singleton support $y \in Y$ and the value t $(0 < t \leq 1)$ at y will be denoted by y_t. 0_Y and 1_Y are the constant fuzzy sets taking respectively the constant values 0 and 1 on Y. The complement of a fuzzy set A in Y will be denoted by $1_Y \setminus A$ [21] and is defined by $(1_Y \setminus A)(y) = 1 - A(y)$, for all $y \in Y$. For two fuzzy sets A and B in Y, we write $A \leq B$ iff $A(y) \leq B(y)$, for each $y \in Y$, while we write AqB to mean A is quasi-coincident (q-coincident, for short) with B [18] if there exists $y \in Y$ such that $A(y) + B(y) > 1$; the negation of AqB is written as $A \not{q} B$. clA and $intA$ of a set A in X (respectively, a fuzzy set A [21] in Y) respectively stand for the closure and interior of A in X (respectively, in Y). A fuzzy set A in Y is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy β-open [2], fuzzy α-open [5], fuzzy preopen [13]) if $intclA = A$ (resp., $A \leq clintA$, $A \leq clintclA$, $A \leq intclintA$, $A \leq intclA$). The complement of a fuzzy regular open (resp., fuzzy semiopen, fuzzy β-open, fuzzy α-open, fuzzy preopen) set is called fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy β-closed [2], fuzzy α-closed [5], fuzzy preclosed [13]). The union of all fuzzy semiopen sets contained in a fuzzy set A in Y is called fuzzy semi-interior of A and is denoted by $sintA$ [1]. The intersection of all fuzzy semiclosed (resp., fuzzy β-closed, fuzzy α-closed, fuzzy preclosed) sets containing a fuzzy set A in Y is called a fuzzy semiclosure [1] (resp., fuzzy β-closure [2], fuzzy α-closure [5], fuzzy preclosure [13]) of A and is denoted by $sclA$ (resp., βclA, αclA, $pclA$). A fuzzy set A is fuzzy semiclosed (resp., fuzzy β-closed, fuzzy α-closed, fuzzy preclosed) if $A = sclA$ [1] (resp., $A = \beta clA$ [2], $A = \alpha clA$ [5], $A = pclA$ [13]).

A fuzzy set B is called a quasi-neighbourhood (q-nbd, for short) [18] of a fuzzy set A if there is a fuzzy open set U in Y such that $AqU \leq B$. If, in addition, B is fuzzy regular open, then B is called a fuzzy regular open q-nbd of A. A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A in an fts Y if every fuzzy regular open q-nbd U of x_α is q-coincident with A [7]. The union of all fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by δclA [7]. A fuzzy set A is called fuzzy δ-closed if $A = \delta clA$ [7]. The complement of a fuzzy δ-closed set is called fuzzy δ-open [7]. A fuzzy point y_t is called a fuzzy θ-cluster point of a fuzzy set A in Y if $clUqA$ for every fuzzy open set U in Y with $y_t qU$ [12]. The union of all fuzzy θ-cluster points of A is called fuzzy θ-closure of A, denoted by θclA [12]. A fuzzy set A is called fuzzy θ-closed if $A = \theta clA$ [12]. The complement of a fuzzy θ-closed set is called fuzzy θ-open [12].

3. Some Well Known Definitions, Lemmas and Theorems

In this section, we recall some definitions, lemmas and theorems for ready references.

Definition 3.1 [15, 16]. A subfamily m_X of the power set $\mathcal{P}(X)$ of a non empty set X is called a minimal structure (m-structure, for short) on X if $\emptyset \in m_X$ and $X \in m_X$. (X, m_X) is called an m-space. The members of m_X are called m_X-open (m-open, for short) and the complement of an m_X-open set is called m_X-closed (m-closed, for short).

Definition 3.2 [10]. Let (X, m_X) be an m-space. For a subset A of X, the m_X-closure and m_X-interior of A are defined as follows :

$$mClA = \bigcap\{F : F \supset A, X \setminus F \in m_X\}$$

$$mIntA = \bigcup\{U : U \subset A, U \in m_X\}$$

.

Remark 3.3. From Definition 3.1 and Definition 3.2, it is to be noted that for a subset A of (X, m_X), $mIntA$ may not be m-open as well as $mClA$ may not be m-closed.

Lemma 3.4 [10]. Let (X, m_X) be an m-space. For two subsets A, B of X, the following properties hold :
(i) $mCl(X \setminus A) = X \setminus mIntA$, $mInt(X \setminus A) = X \setminus mClA$,
(ii) If $X \setminus A \in m_X$, then $mClA = A$ and if $A \in m_X$, then $mIntA = A$,
(iii) $mCl(\emptyset) = \emptyset$, $mInt(\emptyset) = \emptyset$, $mCl(X) = X$, $mInt(X) = X$,
(iv) If $A \subset B$, then $mCl(A) \subset mCl(B)$ and $mInt(A) \subset mInt(B)$,
(v) $A \subset mCl(A)$ and $mInt(A) \subset A$
(vi) $mCl(mClA) = mClA$ and $mInt(mIntA) = mIntA$.

Lemma 3.5 [15]. Let (X, m_X) be an m-space and A, a subset of X. Then $x \in mClA$ if and only if $U \bigcap A \neq \emptyset$ for every $U \in m_X$ containing x.

Definition 3.6 [10]. An m-structure m_X on a non empty set X is said to have property $(\mathcal{B})$ if the union of every family of subsets belonging to m_X belongs to m_X.

Lemma 3.7 [17]. Let (X, m_X) be an m-space where m_X satisfying the property $(\mathcal{B})$. For a subset A of X, the following properties hold :
(i) $A \in m_X$ iff $mInt(A) = A$
(ii) A is m-closed iff $mCl(A) = A$
(iii) $mInt(A) \in m_X$ and $mCl(A)$ is m-closed.

Definition 3.8 [15, 16]. An m-space (X, m_X) is said to be m-compact if every cover of X by m-open sets has a finite subcover.

Definition 3.9 [6]. Let A be a fuzzy set in an fts (Y, τ). A collection $\mathcal{U}$ of fuzzy sets of Y is called a fuzzy cover of A if $sup\{U(x) : U \in \mathcal{U}\} = 1$ for each $x \in suppA$. In particular, if $A = I^Y$, we get the definition of fuzzy cover of Y.

Definition 3.10 [6, 8]. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in an fts (Y, τ) is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$, i.e., $\mathcal{U}_0$ is also a fuzzy cover of A. In particular, if $A = 1_Y$, we get $\bigcup \mathcal{U}_0 = 1_Y$.

Definition 3.11 [20]. An fts (Y, τ) is said to be fuzzy s-closed if every cover of Y by fuzzy regular closed sets has a finite subcover.

Definition 3.12 [1]. An fts (Y, τ) is said to be fuzzy semi-regular if for each fuzzy open set U in Y and each fuzzy point x_α with $x_\alpha q U$, there exists $V \in \tau$ such that $x_\alpha q V \leq int(clV) \leq U$.

Theorem 3.13 [1]. An fts (Y, τ) is fuzzy semi-regular iff $\delta clA = clA$, for all $A \in I^Y$.
So in a fuzzy semi-regular space, if $A \in \tau^c$, then $A = \delta clA$. Similarly, if $A \in \tau$, then $A = \delta intA$.

Definition 3.14 [9]. An fts (Y, τ) is said to be fuzzy regular if every fuzzy open set V can be expressed as union of fuzzy open sets U_α's such that $clU_\alpha \leq V$ for each $\alpha \in \Lambda$.

Theorem 3.15 [19]. An fts (Y, τ) is fuzzy regular iff $clA = \theta clA$, for every $A \in I^Y$.

Definition 3.16 [14]. Let (X, τ) and (Y, τ_Y) be respectively an ordinary topological space and an fts. We say that $F : X \to Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in Y.
Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

Definition 3.17 [14, 11]. For a fuzzy multifunction $F : X \to Y$, the upper inverse F^+ and lower inverse F^- are defined as follows :
For any fuzzy set A in Y, $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x)qA\}$.

There is the following relationship between the upper and the lower inverses of a fuzzy multifunction.

Theorem 3.18 [11]. For a fuzzy multifunction $F : X \to Y$, we have $F^-(1_Y \setminus A) = X \setminus F^+(A)$, for any fuzzy set A in Y.

4. Fuzzy Contra m-Continuous Multifunctions : Some Characterizations

In this section we have introduced fuzzy upper and lower contra m-continuous multifunctions between a set having minimal structure and an fts. Several characterizations are made of these fuzzy multifunctions.

Definition 4.1. A fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is called fuzzy
(i) upper contra m-continuous (f.u.c.m.c., for short) at $x \in X$ if for each fuzzy closed set A in Y with $x \in F^+(A)$, there exists $U \in m_X$ containing x such that $F(U) \subseteq A$,
(ii) lower contra m-continuous (f.l.c.m.c., for short) at $x \in X$ if for each fuzzy closed set A in Y with $x \in F^-(A)$, there exists $U \in m_X$ containing x such that $F(u)qA$, for all $u \in U$,
(iii) upper (lower) contra m-continuous on X if F has this property at each point of X.

Theorem 4.2. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$, the following statements are equivalent :
(i) F is f.u.c.m.c. on X,
(ii) $F^+(K) = mInt(F^+(K))$, for all $K \in \tau^c$,
(iii) $F^-(V) = mCl(F^-(V))$, for all $V \in \tau$.

Proof (i) $\Rightarrow$ (ii). Let $K \in \tau^c$ and $x \in F^+(K)$. Then $F(x) \leq K$. By (i), there exists $U \in m_X$ containing x such that $F(U) \leq K$ implies that $U \subseteq F^+(K)$. Since $mInt(F^+(K)) = \bigcup\{G : G \subseteq F^+(K), G \in m_X\}$ and $U \in m_X$ with $U \subseteq F^+(K)$, $x \in U \subseteq mInt(F^+(K))$ implies that $x \in mInt(F^+(K))$ and so $F^+(K) \subseteq mInt(F^+(K))$. Again by Lemma 3.4(v), $mInt(F^+(K)) \subseteq F^+(K)$. Hence $F^+(K) = mInt(F^+(K))$.

(ii) $\Rightarrow$ (iii). Let $V \in \tau$. Then $1_Y \setminus V \in \tau^c$. By (ii), $F^+(1_Y \setminus V) = X \setminus F^-(V) = mInt(F^+(1_Y \setminus V)) = X \setminus mCl(F^-(V))$ and so $F^-(V) = mCl(F^-(V))$.

(iii) $\Rightarrow$ (ii). Retracing (ii) $\Rightarrow$ (iii), we get the result.

(ii) $\Rightarrow$ (i). Let $x \in X$ and $K \in \tau^c$ with $F(x) \leq K$. Then $x \in F^+(K) = mInt(F^+(K)) = \bigcup\{U : U \subseteq F^+(K), U \in m_X\}$ implies that $x \in U$ where $U \in m_X$ and $U \subseteq F^+(K)$ and so $F(U) \leq K$.

Corollary 4.3. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$, where m_X has the property $(\mathcal{B})$, the following statements are equivalent :
(i) F is f.u.c.m.c. on X,
(ii) $F^+(K)$ is m-open for every $K \in \tau^c$,
(iii) $F^-(V)$ is m-closed for every $V \in \tau$.

Theorem 4.4. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$, the following statements are equivalent:
(i) F is f.l.c.m.c. on X,
(ii) $F^-(K) = mInt(F^-(K))$, for all $K \in \tau^c$,
(iii) $F^+(V) = mCl(F^+(V))$, for all $V \in \tau$.

Proof (i) $\Rightarrow$ (ii). Let $x \in F^-(K)$ where $K \in \tau^c$. Then $F(x)qK$. By (i), there exists $U \in m_X$ containing x such that $F(u)qK$, for all $u \in U$ implies that $U \subseteq F^-(K)$ and so $x \in U = mInt(U) \subseteq mInt(F^-(K))$

hence $x \in mInt(F^-(K))$. Consequently, $F^-(K) = mInt(F^-(K))$.

(ii) $\Rightarrow$ (iii). Let $K \in \tau$. Then $1_Y \setminus K \in \tau^c$. By (ii), $F^-(1_Y \setminus K) = X \setminus F^+(K) = mInt(F^-(1_Y \setminus K)) = X \setminus mCl(F^+(K))$ implies that $F^+(K) = mCl(F^+(K))$.

(iii) $\Rightarrow$ (ii). Retracing (ii) $\Rightarrow$ (iii), we get the result.

(ii)$\Rightarrow$ (i). Let $x \in X$ and $K \in \tau^c$ with $F(x)qK$ implies $x \in F^-(K) = mInt(F^-(K)) = \bigcup\{U : U \subseteq F^-(K), U \in m_X\}$ and so $x \in U, U \in m_X$ and $U \subseteq F^-(K)$ and hence $F(u)qK$, for all $u \in U$.

Corollary 4.5. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ where m_X has the property $(\mathcal{B})$, the following statements are equivalent:
(i) F is f.l.c.m.c. on X,
(ii) $F^-(K)$ is m-open for every $K \in \tau^c$,
(iii) $F^+(V)$ is m-closed for every $V \in \tau$.

Definition 4.6. For any fuzzy set A in an fts (Y, τ), we define $ker(A) = \bigwedge\{U \in \tau : A \leq U\}$.

Lemma 4.7. For any two fuzzy sets A, B in an fts (Y, τ), the following statements are true :
(i) $x_\alpha \in ker(A)$ iff AqF for all $F \in \tau^c$ with $x_\alpha qF$,
(ii) if $A \in \tau$, then $A = ker(A)$,
(iii) if $A \leq B$, then $ker(A) \leq ker(B)$.

Proof (i). Let $x_\alpha \in ker(A)$. Let $F \in \tau^c$ with $x_\alpha qF$. Then $\alpha > 1 - F(x)$ ... (1). We have to show that AqF. If not, then $A \leq 1_Y \setminus F$ where $1_Y \setminus F \in \tau$. Then by definition, $x_\alpha \in 1_Y \setminus F$ which implies that $1 - F(x) > \alpha$, which contradicts (1).
Conversely, let AqF for all $F \in \tau^c$ with $x_\alpha qF$. Let $U \in \tau$ with $A \leq U$ ... (2). We have to show that $x_\alpha \in U$. If not, then $U(x) < \alpha$ implies that $1 - U(x) > 1 - \alpha$ and so $x_\alpha q(1_Y \setminus U)$ where $1_Y \setminus U \in \tau^c$. By hypothesis, $Aq(1_Y \setminus U)$ which implies that there exists $y \in Y$ such that $A(y) > U(y)$, which contradicts (2).

(ii) and (iii) are obvious.

Theorem 4.8. A fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is f.u.c.m.c. on X if $mCl(F^-(V)) \subseteq F^-(ker(V))$, for all $V \in I^Y$.

Proof. Let $mCl(F^-(V)) \subseteq F^-(ker(V))$, for all $V \in I^Y$. Let $V \in \tau$. By Lemma 4.7, $mCl(F^-(V)) \subseteq F^-(V)$ implies that $F^-(V) = mCl(F^-(V))$. By Theorem 4.2, F is f.u.c.m.c. on X.

Theorem 4.9. A fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is f.l.c.m.c. on X if $mCl(F^+(V)) \subseteq F^+(ker(V))$, for all $V \in I^Y$.

Proof. The proof follows from Theorem 4.4 and Lemma 4.7.

Theorem 4.10. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ where m_X has the property ($\mathcal{B}$) and (Y, τ) is fuzzy semi-regular, then the following statements are equivalent :
(i) F is f.u.c.m.c. on X,
(ii) $F^+(\delta clB)$ is m-open for all $B \in I^Y$,
(iii) $F^+(K)$ is m-open for all fuzzy δ-closed set K of Y,
(iv) $F^-(V)$ is m-closed for all fuzzy δ-open set V of Y.

Proof (i)$\Rightarrow$ (ii). Let $B \in I^Y$. Then $\delta clB \in \tau^c$ (by Theorem 3.13). By Corollary 4.3, $F^+(\delta clB)$ is m-open.

(ii)$\Rightarrow$ (iii). Let K be fuzzy δ-closed in Y. Then $K = \delta clK$. By (ii), $F^+(K) = F^+(\delta clK)$ is m-open.

(iii)$\Rightarrow$ (iv). Let V be any fuzzy δ-open set in Y. Then $1_Y \setminus V$ is fuzzy δ-closed in Y. By (iii), $F^+(1_Y \setminus V) = X \setminus F^-(V)$ is m-open in X and so $F^-(V)$ is m-closed in X.

(iv)$\Rightarrow$ (i). Let $V \in \tau$. By Theorem 3.13, V is fuzzy δ-open in Y. By (iv), $F^-(V)$ is m-closed in X and hence by Corollary 4.3, F is f.u.c.m.c. on X.

Theorem 4.11. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ where m_X has the property ($\mathcal{B}$) and (Y, τ) is fuzzy semi-regular, then the following statements are equivalent :
(i) F is f.l.c.m.c. on X,
(ii) $F^-(\delta clB)$ is m-open for all $B \in I^Y$,
(iii) $F^-(K)$ is m-open for all fuzzy δ-closed set K of Y,

(iv) $F^+(V)$ is m-closed for all fuzzy δ-open set V of Y.

Proof (i)$\Rightarrow$ (ii). Let $B \in I^Y$. Then $\delta clB \in \tau^c$ (by Theorem 3.13). By Corollary 4.5, $F^-(\delta clB)$ is m-open.

(ii)$\Rightarrow$ (iii). Let K be fuzzy δ-closed in Y. Then $K = \delta clK$. By (ii), $F^-(K) = F^-(\delta clK)$ is m-open.

(iii)$\Rightarrow$ (iv). Let V be any fuzzy δ-open set in Y. Then $1_Y \setminus V$ is fuzzy δ-closed in Y. By (iii), $F^-(1_Y \setminus V) = X \setminus F^+(V)$ is m-open in X and so $F^+(V)$ is m-closed in X.

(iv)$\Rightarrow$ (i). Let $V \in \tau$. By Theorem 3.13, V is fuzzy δ-open in Y. By (iv), $F^+(V)$ is m-closed in X and hence by Corollary 4.5, F is f.l.c.m.c. on X.

Theorem 4.12. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ where m_X has the property ($\mathcal{B}$) and (Y, τ) is fuzzy regular, then the following statements are equivalent :
(i) F is f.u. (l.)c.m.c. on X,
(ii) $F^+(\theta clB)$ $(F^-(\theta clB))$ is m-open, for all $B \in I^Y$,
(iii) $F^+(K)$ $(F^-(K))$ is m-open for every fuzzy θ-closed set K of Y,
(iv) $F^-(V)$ $(F^+(V))$ is m-closed for every fuzzy θ-open set V of Y.

Proof (i)$\Rightarrow$ (ii). Let $B \in I^Y$. By Theorem 3.15, $\theta clA \in \tau^c$. By (i), $F^+(\theta clA)$ is m-open in X by Corollary 4.3.

(ii)$\Rightarrow$ (iii). Let K be fuzzy θ-closed in Y. Then $K = \theta clK$. By (ii), $F^+(K) = F^+(\theta clK)$ is m-open in X.

(iii)$\Rightarrow$ (iv). Let V be any fuzzy θ-open set in Y. Then $1_Y \setminus V$ is fuzzy θ-closed in Y. By (iii), $F^+(1_Y \setminus V) = X \setminus F^-(V)$ is m-open in X and so $F^-(V)$ is m-closed in X.

(iv)$\Rightarrow$ (iii). obvious.

(iii)$\Rightarrow$ (i). Let $K \in \tau^c$. By Theorem 3.15, $K = clK = \theta clK$. By (iii), $F^+(K)$ is m-open in X. By Corollary 4.3, F is f.u.c.m.c. on X.

Similarly we can prove the theorem for f.l.c.m.c. multifunction.

Definition 4.13. A net $\{S_n : n \in (D, \geq)\}$ in an m-space (X, m_X) with the directed set $(D, \geq)$ as the domain, is said to m-converge to a point $x \in X$ if for each $U \in m_X$ containing x, there exists $m \in D$ such that $S_n \in U$, for all $n \geq m$ $(n \in D)$.

Theorem 4.14. A fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is f.u.c.m.c. on X iff for each point $x \in X$, if $\{x_n : n \in (D, \geq)\}$ is a net in X, m-converges to x, there exists $V \in \tau^c$ with $x \in F^+(V)$, the net is eventually in $F^+(V)$.

Proof. Let $\{x_n : n \in (D, \geq)\}$ be a net in X, m-converge to $x \in X$ and $V \in \tau^c$ be such that $x \in F^+(V)$. As F is f.u.c.m.c. at $x \in X$, there exists $U \in m_X$ containing x such that $U \subseteq F^+(V)$. Since the net m-converges to x and $U \in m_X$ containing x, there exists $m \in D$ such that $x_n \in U$, for all $n \geq m$ $(n \in D) \Rightarrow x_n \in F^+(V)$, for all $n \geq m$. Hence the net is eventually in $F^+(V)$.
Conversely, let the given condition hold, but F be not f.u.c.m.c. at some point $x \in X$. Then there exists $V \in \tau^c$ with $x \in F^+(V)$ such that for each $U \in m_X$ containing x such that $F(x_U) \not\leq V$, for some $x_U \in U$. Let $(D, \geq)$ be the directed set consisting of all pairs (x_U, U) with $(x_U, U) \geq (x_V, V)$ iff $U \subseteq V$ $(U, V \in m_X$ containing $x)$ and consider the net $S(x_U, U) = x_U$ in X. Then the net $\{S_n : n \in (D, \geq)\}$ m-converges to x, but $F(S_n) \not\leq V$, i.e., $S_n \notin F^+(V)$, for each $n \in D$, contradicting our hypothesis.

Theorem 4.15. A fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is f.l.c.m.c. on X iff for each point $x \in X$, if $\{S_n : n \in (D, \geq)\}$ is a net in an m-space (X, m_X) m-converges to x, then for each $V \in \tau^c$ with $F(x)qV$, there exists $m \in D$ such that $F(S_n)qV$, for all $n \geq m$ $(n \in D)$.

Proof. Let $\{S_n : n \in (D, \geq)\}$ be a net in $(X.m_X)$ m-converge to $x \in X$ and $V \in \tau^c$ with $F(x)qV$. Then as F is f.l.c.m.c. at x, there exists $U \in m_X$ containing x such that $U \subseteq F^-(V)$. Since the net m-converges to x and $U \in m_X$ containing x, by Definition 4.13, there exists $m \in D$ such that $S_n \in U$, for all $n \geq m$ $(n \in D)$ and so $S_n \subseteq F^-(V)$, for all $n \geq m$ and hence $F(S_n)qV$, for all $n \geq m$.
Conversely assume that the given condition holds, but F is not

f.l.c.m.c. at some point $x \in X$. Then there exists $V \in \tau^c$ with $F(x)qV$ such that for each $U \in m_X$ containing x we have $F(x_U) \not{q}V$, for some $x_U \in U$. Let $(D, \geq)$ be the directed set consisting of all pairs (x_U, U) with $(x_U, U) \geq (x_W, W)$ iff $U \subseteq W$ $(U, W \in m_X$ containing x and $F(x_U) \not{q}U$, $F(x_W) \not{q}W)$ and consider the net $S(x_U, U) = x_U$ in X. Then evidently the net $\{S_n : n \in (D, \geq)\}$ m-converges to x but $F(S_n) \not{q}V$, for each $n \in D$ contradicting our hypothesis.

5. Applications

Definition 5.1. For a fuzzy multifunction $F : X \to Y$, $clF : X \to Y$ [3] (resp., $sclF : X \to Y$ [3], $pclF : X \to Y$ [4], $\alpha clF : X \to Y$ [4], $\beta clF : X \to Y$) is defined by $(clF)(x) = clF(x)$ (resp., $(sclF)(x) = sclF(x)$, $(pclF)(x) = pclF(x)$, $(\alpha clF)(x) = \alpha clF(x)$, $(\beta clF)(x) = \beta clF(x)$) for all $x \in X$.

Lemma 5.2. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$, the following statements are true :
(i) $(clF)^-(U) = F^-(U)$, for all $U \in \tau$,
(ii) $(clF)^+(U) = F^+(U)$, for all $U \in \tau^c$.

Proof (i). Let $U \in \tau$ and $x \in (clF)^-(U)$. Then $(clF)(x)qU$ ... (1). We claim that $F(x)qU$. If not, then $F(x) \ /qU$ where $U \in \tau$ implies that $(clF)(x) \ /qU$, which contradicts (1). Hence $F(x)qU$ and so $x \in F^-(U)$ and then $(clF)^-(U) \subseteq F^-(U)$. Obviously, $F^-(U) \subseteq (clF)^-(U)$. So $F^-(U) = (clF)^-(U)$.

(ii). Let $x \in (clF)^+(U)$ for $U \in \tau^c$. Then $(clF)(x) \leq U$ and so $F(x) \leq U$ implies that $x \in F^+(U)$ which implies that $(clF)^+(U) \subseteq F^+(U)$.
Conversely, let $y \in F^+(U)$. Then $F(y) \leq U$ and then $clF(y) \leq clU = U$ which implies that $y \in (clF)^+(U)$ and hence $F^+(U) \subseteq (clF)^+(U)$. Consequently,$(clF)^+(U) = F^+(U)$, for $U \in \tau^c$.

In a similar manner we can easily state the following lemma.

Lemma 5.3. For a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$, the following statements are true :
(i) $(G)^-(U) = F^-(U)$, for each $U \in \tau$,
(ii) $(G)^+(U) = F^+(U)$, for each $U \in \tau^c$,
where G stands for $sclF$, $pclF$, αclF, βclF.

Theorem 5.4. Let $F : (X, m_X) \to (Y, \tau)$ be a fuzzy multifunction where m_X has the property $(\mathcal{B})$. Then the following statements are equivalent :
(i) F is f.u.c.m.c. on X,
(ii) G is f.u.c.m.c. on X where G stands for clF, scF, $pclF$, αclF, βclF.

Proof (i)$\Rightarrow$ (ii). Let $K \in \tau^c$. By Lemma 5.2 and Lemma 5.3, $G^+(K) = F^+(K)$. By Corollary 4.3, $F^+(K)$ is m-open and so $G^+(K)$ is m-open. Hence G is f.u.c.m.c. on X.

(ii)$\Rightarrow$ (i). Let $K \in \tau^c$. By Lemma 5.2 and Lemma 5.3, $F^+(K) = G^+(K)$. As G is f.u.c.m.c. on X, by Corollary 4.3, $G^+(K)$ is m-open in X and so $F^+(K)$ is m-open in X. Hence F is f.u.c.m.c. on X.

Theorem 5.5. Let $F : (X, m_X) \to (Y, \tau)$ be f.u.c.m.c. surjective multifunction and $F(x)$ is fuzzy s-closed for each $x \in X$. If (X, m_X) is m-compact, then (Y, τ) is fuzzy s-closed.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of Y by fuzzy regular closed sets of Y. Now for each $x \in X$, $F(x)$ is fuzzy s-closed in Y and so there exists a finite subset Λ_x of Λ such that $F(x) \leq \bigcup\{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \bigcup\{A_\alpha : \alpha \in \Lambda_x\}$. Then A_x being fuzzy regular closed in Y is fuzzy closed in Y with $F(x) \leq A_x$. As F is f.u.c.m.c. surjective multifunction, there exists $U \in m_X$ containing x such that $F(U_x) \leq A_x$. Then $\{U_x : x \in X\}$ is an m-open cover of X, As X is m-compact, there exist finitely many members $x_1, x_2, ..., x_n$ of X such that $X = \bigcup\{U_{x_i} : i = 1, 2, ..., n\}$.

As F is surjective, $1_Y = F(X) = F(\bigcup\{U_{x_i} : i = 1, 2, ..., n\}) = \bigcup_{i=1}^{n} F(U_{x_i}) \leq \bigcup_{i=1}^{n} A_{x_i} \leq \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{x_i}} A_\alpha$. Hence Y is fuzzy s-closed.

Definition 5.6 [17]. Let (X, m_X) be an m-space and A, a subset of X. Then m_X-frontier of A, denoted by $mFr(A)$, is defined by $mFr(A) = mClA \setminus mIntA$.

Theorem 5.7. Let (X, m_X) be an m-space and (Y, τ) be an fts and $F : (X, m_X) \to (Y, \tau)$ be a fuzzy multifunction. Let $A = \{x \in X : F$ is not f.u.(l.)c.m.c. at $x\}$, $B = \bigcup\{mFr(F^+(K)) : F(x) \leq K, K \in \tau^c\}$ (resp. $B = \bigcup\{mFr(F^-(K)) : F(x)qK, K \in \tau^c\}$). Then $A = B$.

Proof. Let $x \in A$. Then F is not f.u.c.m.c. at x. Then there exists $K \in \tau^c$ with $F(x) \leq K$, but for all $U \in m_X$ containing x, $U \not\subseteq F^+(K)$. Then $X \setminus U \not\supseteq X \setminus F^+(K)$ implies that $U \bigcap (X \setminus F^+(K)) \neq \emptyset$ and so $x \in mCl(X \setminus F^+(K)) = X \setminus mInt(F^+(K))$ and hence $x \notin mInt(F^+(K))$. Now as $x \in F^+(K)$, $x \in mCl(F^+(K))$ and so $x \in mFr(F^+(K))$, where $F(x) \leq K, K \in \tau^c$ and so $x \in B$ and consequently $A \subseteq B$.
Conversely, let $x \notin A$. Then F is f.u.c.m.c. at x. Then for any $K \in \tau^c$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \leq K$ implies $U \subseteq F^+(K)$. Since $U \in m_X, x \in U = mIntU \subseteq mInt(F^+(K))$ implies that $x \notin mFr(F^+(K))$ and hence $x \notin B$. Contrapositively, $x \in B$ implies that $x \in A$ and so $B \subseteq A$. Combing these two, we get $A = B$.
Similarly we can prove the theorem for f.l.c.m.c. multifunction.

Definition 5.8 [3]. For a fuzzy multifunction $F : X \to Y$, the fuzzy graph multifunction $G_F : X \to X \times Y$ of F is defined as $G_F(x) =$ the fuzzy set $x_1 \times F(x)$ of $X \times Y$ where x_1 is the fuzzy set in X whose value is 1 at $x \in X$ and 0 at other points of X. We shall write $\{x\} \times F(x)$ for $x_1 \times F(x)$.

Lemma 5.9 [3]. The following hold for a fuzzy multifunction $F : X \to Y$:
(a) $(G_F)^+(A \times B) = A \bigcap F^+(B)$ and
(b) $(G_F)^-(A \times B) = A \bigcap F^-(B)$ for every subset A of X and every $B \in I^Y$.

Theorem 5.10. Let X be product related to Y. Then a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is f.l.c.m.c. on X if $G_F : X \to X \times Y$ is so.

Proof. Let G_F be f.l.c.m.c. on X. Let $x \in X$ and $V \in \tau^c$ with $F(x)qV$. Then there exists $y \in Y$ such that $[F(x)](y) + V(y) > 1$. Now $1_X \times V$ is closed in $X \times Y$ such that $[\{x\} \times F(x)](x, y) + (1_X \times V)(x, y) > 1$ and so $G_F(x)q(1_X \times V)$. By hypothesis there exists $U \in m_X$ containing x such that $U \subseteq (G_F)^-(1_X \times V) = 1_X \bigcap F^-(V) = F^-(V)$ which shows that F is f.l.c.m.c. on X.

Theorem 5.11. Let X be product related to Y. Then a fuzzy multifunction $F : (X, m_X) \to (Y, \tau)$ is f.u.c.m.c. on X if $G_F : X \to X \times Y$ is so.

Proof. Let G_F be f.u.c.m.c. on X. Let $x \in X$ and $V \in \tau^c$ with $F(x) \le V$. Then $G_F(x) \le 1_X \times V$ and $1_X \times V$ is closed in $X \times Y$. By hypothesis, there exists $U \in m_X$ containing x such that $G_F(U) \le 1_X \times V$. Now for any $z \in U$ and for any $y \in Y$, $[F(z)](y) = [G_F(z)](z, y) \le (1_X \times V)(z, y) = V(y)$ implies that $[F(z)](y) \le V(y)$, for all $y \in Y$ and so $F(Z) \le V$, for all $z \in U$. Hence $F(U) \le V$ implies that F is f.u.c.m.c. on X.

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Victoria Institution (College),
Department of Mathematics,
78B, A.P.C. Road,
Kolkata-700009, India
e-mail: anjanabhattacharyya@hotmail.com