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ON A MAXIMAL OPERATOR IN REARRANGEMENT INVARIANT BANACH FUNCTION SPACES ON METRIC SPACES

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Abstract. We introduce a maximal operator for functions defined on a doubling metric measure space, belonging to a rearrangement Banach function space. We provide some estimates for the distribution function of this operator, generalizing results proved by Bastero, Milman and Ruiz (1999) in the Euclidean case and by Costea and Miranda (2012) for Newtonian Lorentz spaces on metric spaces.

1. Introduction

The maximal operators, in the first place the Hardy-Littlewood maximal operator, are an essential tool in real analysis and harmonic analysis and have applications to PDE's and geometric analysis and ergodic theory, also playing an important role in understanding differentiability properties of functions and of singular integrals. In analysis on metric measure spaces, maximal operators have been used, in the case of a doubling measure, to study pointwise behaviour of Sobolev-type functions, to prove density of Lipschitz functions in Sobolev-type spaces, etc.

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In the following, (X, d, μ) is a metric measure space, i.e. a metric space (X, d) endowed with a Borel regular measure μ, that is finite and positive on balls.

The classical Hardy-Littlewood maximal function operator is defined by $\mathcal{M}f(x) = \sup\limits_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f| \, d\mu$, where $f \in L^1_{loc}(X)$.

The corresponding non-centered maximal function operator is given by $\mathcal{M}^* f(x) = \sup\limits_{B \ni x} \frac{1}{\mu(B)} \int\limits_B |f| \, d\mu$ for $f \in L^1_{loc}(X)$, where the supremum is taken over all balls $B \subset X$ which contain x. Note that

$$\mathcal{M}^* f(x) = \sup_{B \ni x} \frac{1}{\|\chi_B\|_{L^1(X)}} \|f \chi_B\|_{L^1(X)}$$

and that $\mathcal{M}f \leq \mathcal{M}^* f$ for every $f \in L^1_{loc}(X)$. Moreover, $\mathcal{M}^* f$ and $\mathcal{M}f$ are comparable if the measure μ is doubling.

The maximal function theorem extends to doubling metric measure spaces [7, Theorem 2.2], showing that the Hardy-Littlewood maximal function operator maps $L^1(X)$ to weak-$L^1(X)$ and $L^1(X)$ and $L^p(X)$ to $L^p(X)$ for $p > 1$, i.e. the following inequalities hold:

(1.1)

$$\mu(\{x \in X : \mathcal{M}f(x) > \lambda\}) \leq \frac{C_1}{\lambda} \int\limits_X |f| \, d\mu \text{ for all } \lambda > 0, \text{ for } f \in L^1(X)$$

and

$$\int\limits_X |\mathcal{M}f|^p \, d\mu \leq C_p \int\limits_X |f|^p \, d\mu \text{ for } f \in L^p(X), \, p > 1.$$

Here the constant C_1 and C_p depend only on the doubling constant of μ and, respectively, on $p > 1$.

For the case of Lorentz spaces $L^{p,q}(X)$ with $1 \leq p, q < \infty$ Stein [13] defined the (non-centered) maximal operator $\mathcal{M}_{p,q} f(x) = \sup\limits_{B \ni x} \frac{1}{\mu(B)^{1/p}} \|f \chi_B\|_{L^{p,q}(X)}$, see also [6].

Bastero, Milman and Ruiz [2] defined for a Banach function space $\mathbf{E}$ on $\mathbb{R}^n$ the maximal operator $\mathcal{M}_{\mathbf{E}} f(x) = \sup\limits_{Q \ni x} \frac{1}{\|\chi_Q\|_{\mathbf{E}}} \|f \chi_Q\|_{\mathbf{E}}$ for $f \in \mathbf{E}$, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ which contain x and have sides parallel to the coordinate axes. Under the assumption that $\mathbf{E}$ is a rearrangement invariant space satisfying a lower $\Phi-$ estimate, where Φ is the fundamental function of $\mathbf{E}$, Bastero, Milman and Ruiz [2] generalized (1.1) proving that

$$\mu\left(\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\}\right) \leq \frac{C}{\lambda} \|f\|_{\mathbf{E}} \text{ for all } \lambda > 0.$$

In this paper we introduce a maximal operator for functions defined on a doubling metric measure space, belonging to a rearrangement Banach function space. This operator is analogue to the above operator $\mathcal{M}_{\mathbf{E}}$ introduced by Bastero, Milman and Ruiz [2]. We provide some estimates for the distribution function of this operator, generalizing the above result proved by Bastero, Milman and Ruiz [2] in the Euclidean case and a similar result proved by Costea and Miranda [6] for Newtonian Lorentz spaces on metric spaces.

2. Preliminaries

Denote the open balls, respectively the closed balls in the metric space (X, d) by $B(x, r) = \{y \in X : d(y, x) < r\}$ and $\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$. We use the abbreviation $\lambda B(x, r) = B(x, \lambda r)$, whenever $x \in X$ and $r, \lambda > 0$.

The measure μ on the metric space (X, d) is said to be doubling if there exists a constant $C_\mu \geq 1$ such that

$$\mu(2B) \leq C_\mu \mu(B)$$

for every ball $B \subset X$. A doubling measure can have as atoms only singletons consisting of isolated points [9]. Under the assumption that the measure μ is doubling, the space (X, d, μ) is *proper* (i.e. every closed ball of the space is compact) if and only if it is complete.

We use the definition given by Bennet and Sharpley [3] for Banach function spaces over the $\sigma-$ finite measure space (X, μ). Let (X, μ) be a $\sigma-$finite measure space and $\boldsymbol{M}^+(X)$ be the set of $\mu-$measurable non-negative functions on X.

Definition 1. [3] *A function $\rho : \boldsymbol{M}^+(X) \to [0, \infty]$ is called a Banach function norm if, for all functions f, g, f_n $(n \geq 1)$ in $\boldsymbol{M}^+(X)$, for all constants $a \geq 0$ and for all measurable sets $A \subset X$, the following properties hold: (P1)$\rho(f) = 0$ if and only if $f = 0$ $\mu-$a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$.*

(P2) If $0 \leq g \leq f$ $\mu-$a.e., then $\rho(g) \leq \rho(f)$.

(P3) If $0 \leq f_n \uparrow f$ $\mu-$a.e., then $\rho(f_n) \uparrow \rho(f)$.

(P4) If $\mu(A) < \infty$, then $\rho(\chi_A) < \infty$.

(P5) If $\mu(A) < \infty$, then $\int\limits_E f\, d\mu \leq C_A \rho(f)$, for some constant $C_A \in (0, +\infty)$ depending only on A and ρ.

The collection $\mathbf{E}$ of all $\mu-$measurable functions $f : X \to [-\infty, +\infty]$ for which $\rho(|f|) < \infty$ is called a *Banach function space* on X. For $f \in \mathbf{E}$ define

$$\|f\|_{\mathbf{B}} = \rho(|f|).$$

We identify two functions that coincide $\mu-$a.e. If $f, g : X \to \overline{\mathbb{R}}$ such that $\rho(|f|) < \infty$ and $f = g$ $\mu-$a.e, then g is $\mu-$measurable and $\rho(|g|) = \rho(|f|) < \infty$. Moreover, by Definition 1 (P5) and the $\sigma-$finiteness of μ, it follows that f and g are finite $\mu-$a.e., hence $f - g = 0$ $\mu-$a.e. and therefore $\|f - g\|_{\mathbf{E}} = 0$.

Orlicz spaces and Lorentz spaces are important examples of Banach function spaces, both generalizing Lebesgue spaces.

Definition 2. [3, Definition I.3.1] *A function $f \in \mathbf{E}$ is said to have absolutely continuous norm in $\mathbf{E}$ if and only if $\|f\chi_{E_k}\|_{\mathbf{E}} \to 0$ for every sequence $(E_k)_{k\geq 1}$ of measurable sets satisfying*

$$\mu\left(\limsup_{k\to\infty} E_k\right) = 0.$$

The space $\mathbf{E}$ is said to have absolutely continuous norm if every $f \in \mathbf{E}$ has absolutely continuous norm.

We recall that an Orlicz space $L^{\Psi}(X)$ has absolutely continuous norm if the Young function Ψ is doubling and that the $(p,q)-$norm of a Lorentz space $L^{p,q}(X)$ with $1 < p < \infty$ and $1 \leq q < \infty$ is absolutely continuous [6], while in $L^{\infty}(X)$ the only function having an absolutely continuous norm is the null function.

3. Rearrangement invariant Banach function spaces and lower $\Phi-$estimates

The notion of rearrangement invariant Banach function space, set up by Hardy and Littlewood, has many applications in real and harmonic analysis, playing a central role in interpolation theory and being involved in the study of singular integrals, isoperimetric inequalities, in the theory of elliptic second order PDE's.

Let $f : X \to \overline{\mathbb{R}}$ be a $\mu-$measurable function. *The distribution function* of f is defined by $d_f(t) = \mu(\{x \in X : |f(x)| > t\})$, where $t \geq 0$. The *nonincreasing rearrangement* of f is $f^*(t) = \inf\{s \geq 0 : d_f(s) \leq t\}$, $t \geq 0$.

A Banach function space $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is said to be *rearrangement invariant* if $f^* = g^*$ implies $\|f\|_{\mathbf{E}} = \|g\|_{\mathbf{E}}$ whenever $f, g \in \mathbf{E}$.

Let $\mathbf{E}$ be a rearrangement invariant space over (X,μ). *The fundamental function* of $\mathbf{E}$ is defined by $\Phi_{\mathbf{E}}(t) = \|\chi_A\|_{\mathbf{E}}$, $t \geq 0$, where $A \subset X$ is any $\mu-$measurable set with $\mu(A) = t$. The definition is unambiguous, since the characteristic functions of two sets with equal measures have the same distribution function.

Lemma 1. [3, Corollary II. 5.3]*Let* $\mathbf{E}$ *be a rearrangement invariant space over a resonant measure space* (X,μ). *Then the fundamental function* $\Phi_{\mathbf{E}}$ *satisfies:* $\Phi_{\mathbf{E}}$ *is increasing, vanishes only at the origin, is continuous (except perhaps at the origin) and* $t \mapsto \frac{\Phi_{\mathbf{E}}(t)}{t}$ *is decreasing.*

Recall that (X,μ) is resonant if μ is $\sigma-$finite and nonatomic.

The most well-known examples of rearrangement invariant Banach function spaces are the Orlicz spaces $L^{\Psi}(X)$ and the Lorentz spaces $L^{p,q}(X)$ with $1 \leq q \leq p < \infty$ [3, Theorem IV.4.3]. Let $A \subset X$ be a measurable set of finite positive measure. If $\mathbf{E} = L^{\Psi}(X)$ is an Orlicz space with the Luxemburg norm, then $\|\chi_A\|_{\mathbf{E}} = 1/\Psi^{-1}(1/\mu(A))$, see [11, page 17, Example 9]. If $\mathbf{E} = L^{p,q}(X)$ is a Lorentz space with the $p,q-$ norm, where $1 \leq q \leq p < \infty$, then $\|\chi_A\|_{\mathbf{E}} = c(p,q)\,\mu(A)^{1/p}$, where $c(p,q) = (p/q)^{1/q}$. If $\mathbf{E} = L^{\infty}(X)$, then $\|\chi_A\|_{\mathbf{E}} = sgn\ \mu(A)$, this time even for a null set.

Definition 3. [2] *Let* $\Phi : [0,\infty) \to [0,\infty)$ *be an increasing bijection. The rearrangement invariant Banach function space* $\mathbf{E}$ *is said to satisfy a lower* $\Phi-$*estimate if there exists a positive constant* $M < \infty$ *such that for every finite family* $\{f_i : i = 1,...,n\} \subset \mathbf{E}$ *of functions with disjoint supports,*

$$\left\| \sum_{i=1}^{n} f_i \right\|_{\mathbf{E}} \geq M\Phi\left(\sum_{i=1}^{n} \Phi^{-1}\left(\|f_i\|_{\mathbf{E}}\right) \right). \tag{3.1}$$

It is easy to see that $\mathbf{E}$ satisfies a lower $\Phi-$estimate if and only if there exists a positive constant $M < \infty$ such that for every pairwise disjoint finite family $\{A_i : i = 1,...,n\}$ of measurable sets and every function $f \in \mathbf{E}$ we have

$$\|f\chi_A\|_{\mathbf{E}} \geq M\Phi\left(\sum_{i=1}^{n} \Phi^{-1}\left(\|f\chi_{A_i}\|_{\mathbf{E}}\right) \right),$$

where $A = \bigcup_{i=1}^{n} A_i$.

For $\Phi(t) = t^{1/p}$ a lower $\Phi-$estimate is known as a lower $p-$estimate [8]. In the case where $\mathbf{E} = L^{p,q}(X)$ is a Lorentz space, then $\mathbf{E}$ satisfies a lower $\Phi-$estimate with $M=1$, for $\Phi(t)=t^{1/p}$ if $1\leq q\leq p$ as proved by Chung, Hunt and Kurtz [4], respectively for $\Phi(t)=t^{1/q}$ if $1<p<q<\infty$ as proved by Costea and Maz'ya [5].

Now we consider the case of an Orlicz space $\mathbf{E}=L^{\Psi}(X)$ with the Luxemburg norm $\|\cdot\|_{L^{\Psi}(X)}$, where the Young function $\Psi:[0,\infty)\to[0,\infty)$ is bijective. The fundamental function of the given Orlicz space is $\Phi_{\mathbf{E}}(t)=\frac{1}{\Psi^{-1}\left(\frac{1}{t}\right)}$ and has the inverse $\Phi_{\mathbf{E}}^{-1}(s)=\frac{1}{\Psi\left(\frac{1}{s}\right)}$. Denote $F(t)=\frac{1}{\Psi\left(\frac{1}{t}\right)}$ for $t\in(0,\infty)$. Then the lower $\Phi-$estimate (3.1) for $\Phi=\Phi_{\mathbf{E}}$ has the form

$$F\left(\frac{1}{M}\left\|\sum_{i=1}^{n}f_i\right\|_{L^{\Psi}(X)}\right)\geq\sum_{i=1}^{n}F\left(\|f_i\|_{L^{\Psi}(X)}\right). \tag{3.2}$$

Recall that an $N-$function is a continuous Young function $\Psi:[0,\infty)\to[0,\infty)$ satisfying $\Psi(t)=0$ only if $t=0$,

$$\lim_{t\to\infty}\frac{\Psi(t)}{t}=\infty \text{ and } \lim_{t\to0}\frac{\Psi(t)}{t}=0.$$

Lemma 2. *If $\Psi:[0,\infty)\to[0,\infty)$ is an N-function and $F(t)=\frac{1}{\Psi\left(\frac{1}{t}\right)}$ for $t\in(0,\infty)$, then inequality (3.2) holds with $M=\frac{1}{4}$ for every finite family $\{f_i:i=1,...,n\}\subset L^{\Psi}(X)$ of functions with disjoint supports.*

Proof. Since $\Psi:[0,\infty)\to[0,\infty)$ is convex, the function $t\mapsto\frac{\Psi(t)}{t}$ is increasing (not necessarily strictly increasing, in other terms is nondecreasing) on $[0,\infty)$. Then the function $t\mapsto\frac{F(t)}{t}$ is increasing, therefore F is superadditive. It follows that

$$\sum_{i=1}^{n}F\left(\|f_i\|_{L^{\Psi}(X)}\right)\leq F\left(\sum_{i=1}^{n}\|f_i\|_{L^{\Psi}(X)}\right) \tag{3.3}$$

Let $\{f_i:i=1,...,n\}\subset L^{\Psi}(X)$ be a family of functions with disjoint supports. By Lemma 3.2 in [1],

$$4\left\|\sum_{i=1}^{n}f_i\right\|_{L^{\Psi}(X)}\geq\sum_{i=1}^{n}\|f_i\|_{L^{\Psi}(X)}.$$

Since F is increasing, by the above inequality and by (3.3) we get

$$\sum_{i=1}^{n}F\left(\|f_i\|_{L^{\Psi}(X)}\right)\leq F\left(4\left\|\sum_{i=1}^{n}f_i\right\|_{L^{\Psi}(X)}\right),$$

q.e.d. ∎

Corollary 1. *If $\Psi : [0,\infty) \to [0,\infty)$ is an N-function, then the Orlicz space $\mathbf{E} = L^{\Psi}(X)$ satisfies a lower $\Phi_{\mathbf{E}}-$estimate.*

4. Estimates for the distribution function of the maximal operator

Recall that Bastero, Milman and Ruiz [2] defined for a Banach function space $\mathbf{E}$ on $\mathbb{R}^n$ the maximal operator $\mathcal{M}_{\mathbf{E}} f(x) = \sup\limits_{Q \ni x} \frac{1}{\|\chi_Q\|_{\mathbf{E}}} \|f\chi_Q\|_{\mathbf{E}}$ for $f \in \mathbf{E}$, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ which contain x and have sides parallel to the coordinate axes.

They have shown that $\mathcal{M}_{\mathbf{E}} f \geq \mathcal{M} f$ for every $f \in \mathbf{E}$, since the conditional expectation operator is a norm one projection in any rearrangement invariant space.

We will introduce an analogue of the maximal operator defined by Bastero, Milman and Ruiz [2] , corresponding to a Banach function space.

Definition 4. *The (non-centered) maximal operator associated to a Banach function space $\mathbf{E}$ on a metric measure space (X, d, μ) is given by*

$$\mathcal{M}_{\mathbf{E}} f(x) = \sup_{B \ni x} \frac{1}{\|\chi_B\|_{\mathbf{E}}} \|f\chi_B\|_{\mathbf{E}}$$

for $f \in L^1_{loc}(X)$, where the supremum is taken over all balls $B \subset X$ which contain x.

In order to compare the above maximal operator and the non-centered Hardy-Littlewood operator, we need the following well-known result, which is Theorem II.4.8 from [3].

Theorem 1. [3] *Let (X, μ) be a resonant measure space and let $(A_i)_{i \geq 1}$ be a countable (an at most countable) collection of pairwise disjoint subsets of X, each with finite positive measure. Let $A = X \setminus \bigcup\limits_{i=1}^{\infty} A_i$. For each extended-real-valued function g which is measurable and finite $\mu-$a.e., let*

$$\mathcal{T} g = g\chi_A + \sum_{i=1}^{\infty} \left(\frac{1}{\mu(A_i)} \int_{A_i} g \, d\mu \right) \chi_{A_i}.$$

Let $\mathbf{E}$ be a rearrangement invariant Banach function space over (X, μ). Then

$$\|\mathcal{T} g\|_{\mathbf{E}} \leq \|g\|_{\mathbf{E}}$$

for every $g \in \mathbf{E}$.

Proposition 1. *If* (X, d, μ) *is a metric measure space such that* (X, μ) *is resonant and if* $\mathbf{E}$ *is a rearrangement invariant Banach function space on* X, *then* $\mathcal{M}_{\mathbf{E}} f \geq \mathcal{M}^* f$ *for every* $f \in \mathbf{E}$.

Proof. Fix $f \in \mathbf{E}$. Let $B \subset X$ be a ball. Take $A_1 = B$ and let $A = X \setminus B$. Let $g = |f| \chi_B$. Then $g \in \mathbf{E}$ and $\mathcal{T} g = \left(\frac{1}{\mu(B)} \int\limits_B |f| \, d\mu \right) \chi_B$. According to Theorem 1,

$$\frac{1}{\mu(B)} \int\limits_B |f| \, d\mu \leq \frac{1}{\|\chi_B\|_{\mathbf{E}}} \|f \chi_B\|_{\mathbf{E}} .$$

Let $x \in X$. Taking in the above inequality the supremum over all the balls $B \ni x$, we get $\mathcal{M}^* f(x) \leq \mathcal{M}_{\mathbf{E}} f(x)$, as desired. ∎

The following theorem extends Theorem 1 from [2] to the setting of metric measure spaces.

Theorem 2. *Let* $\mathbf{E}$ *be a rearrangement invariant Banach function space on a doubling metric measure space* (X, d, μ) *without* isolated points. *Let* $\Phi_{\mathbf{E}}$ *and* $\mathcal{M}_{\mathbf{E}}$ *the corresponding fundamental function and maximal operator, respectively. If* $\mathbf{E}$ *satisfies a lower* $\Phi_{\mathbf{E}}-$*estimate, then there exists a positive constant* $C < \infty$ *such that for every* $f \in \mathbf{E}$ *we have*

$$\Phi_{\mathbf{E}} \left(\mu \left(\{ x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda \} \right) \right) \leq \frac{C}{\lambda} \|f\|_{\mathbf{E}} . \tag{4.1}$$

Moreover, if $\mathbf{E}$ *has absolutely continuous norm, then*

$$\lim_{\lambda \to \infty} \lambda \Phi_{\mathbf{E}} \left(\mu \left(\{ x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda \} \right) \right) = 0. \tag{4.2}$$

Proof. Since μ is doubling and (X, d) has no isolated point, the measure μ is nonatomic [9], but μ is also $\sigma-$finite, therefore (X, μ) is resonant.

For each $R > 0$ consider the restricted maximal function operator defined by

$$\mathcal{M}_{\mathbf{E}}^R f(x) = \sup \left\{ \frac{1}{\|\chi_B\|_{\mathbf{E}}} \|f \chi_B\|_{\mathbf{E}} : x \in B, \, B \subset X \text{ is a ball, } diam B \leq R \right\}.$$

Fix $\lambda > 0$. Denote $G_\lambda = \{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\}$ and, given $R > 0$, $G_\lambda^R = \left\{ x \in X : \mathcal{M}_{\mathbf{E}}^R f(x) > \lambda \right\}$. Note that $0 < R_1 \leq$

R_2 implies $G_\lambda^{R_1} \subset G_\lambda^{R_2}$ and that $G_\lambda = \bigcup\limits_{R>0} G_\lambda^R$, hence, by the continuity from below of the measure μ, it follows that

$$\chi_{G_\lambda} = \lim_{R\to\infty} \chi_{G_\lambda^R} \text{ and } \mu(G_\lambda) = \lim_{R\to\infty} \mu\left(G_\lambda^R\right). \tag{4.3}$$

Let $R > 0$.

For each $x \in G_\lambda^R$, by the definitions of G_λ^R and $\mathcal{M}_{\mathbf{E}}^R f$, we can find a ball $B_x = B(y_x, r_x)$, containing x and with diameter at most R, such that $\|f\chi_{B_x}\|_{\mathbf{E}} > \lambda \|\chi_{B_x}\|_{\mathbf{E}}$. Note that $B_x \subset G_\lambda^R$.

By the definition of the fundamental function of $\mathbf{E}$, the above inequality writes as $\|f\chi_{B_x}\|_{\mathbf{E}} > \lambda \Phi_{\mathbf{E}}(\mu(B_x))$, hence

$$\mu(B_x) < \Phi_{\mathbf{E}}^{-1}\left(\frac{1}{\lambda}\|f\chi_{B_x}\|_{\mathbf{E}}\right). \tag{4.4}$$

The family $\left\{B_x : x \in G_\lambda^R\right\}$ is a cover of G_λ^R. According to the basic $(5r-)$ covering theorem [7, Theorem 1.2], this family has a countable subfamily $\{B_{x_i} : i \geq 1\}$ of pairwise disjoint balls such that $\left\{5B_{x_i} : x \in G_\lambda^R\right\}$ is a cover of G_λ^R.

Then

$$\begin{aligned}\mu\left(G_\lambda^R\right) &\leq \sum_{i=1}^{\infty} \mu(5B_{x_i}) \leq (C_\mu)^3 \sum_{i=1}^{\infty} \mu(B_{x_i}) \leq \\ &\quad (C_\mu)^3 \sum_{i=1}^{\infty} \Phi_{\mathbf{E}}^{-1}\left(\frac{1}{\lambda}\left\|f\chi_{B_{x_i}}\right\|_{\mathbf{E}}\right).\end{aligned}$$

We used the doubling property of the measure μ and (4.4). Since $\Phi_{\mathbf{E}}$ is increasing and $t \mapsto \frac{\Phi_{\mathbf{E}}(t)}{t}$ is decreasing, the above inequalities imply

$$\begin{aligned}\Phi_{\mathbf{E}}\left(\mu\left(G_\lambda^R\right)\right) &\leq \Phi_{\mathbf{E}}\left((C_\mu)^3 \sum_{i=1}^{\infty} \Phi_{\mathbf{E}}^{-1}\left(\frac{1}{\lambda}\left\|f\chi_{B_{x_i}}\right\|_{\mathbf{E}}\right)\right) \\ &\leq (C_\mu)^3 \Phi_{\mathbf{E}}\left(\sum_{i=1}^{\infty} \Phi_{\mathbf{E}}^{-1}\left(\frac{1}{\lambda}\left\|f\chi_{B_{x_i}}\right\|_{\mathbf{E}}\right)\right).\end{aligned}$$

Since $\mathbf{E}$ satisfies a lower $\Phi_{\mathbf{E}}-$estimate (with a constant M) and the balls $B_{x_i} : i \geq 1$ are pairwise disjoint, we have

$$\Phi_{\mathbf{E}}\left(\sum_{i=1}^{\infty} \Phi_{\mathbf{E}}^{-1}\left(\frac{1}{\lambda}\left\|f\chi_{B_{x_i}}\right\|_{\mathbf{E}}\right)\right) \leq \frac{1}{M}\left\|\sum_{i=1}^{\infty} \frac{1}{\lambda} f\chi_{B_{x_i}}\right\|_{\mathbf{E}} \leq \frac{1}{\lambda M}\left\|f\chi_{G_\lambda^R}\right\|_{\mathbf{E}}.$$

It follows that

$$\Phi_{\mathbf{E}}\left(\mu\left(G_\lambda^R\right)\right) \leq \frac{C}{\lambda}\left\|f\chi_{G_\lambda^R}\right\|_{\mathbf{E}}, \tag{4.5}$$

where we denoted $C = (C_\mu)^3 / M$.

Now we let $R \to \infty$ in (4.5). Using the second part of (4.3) and the continuity of $\Phi_{\mathbf{E}}$ (Lemma 1) in the left-hand side, respectively the first part of (4.3) and the property (P3) of a Banach function norm in the right-hand side, we obtain

$$\Phi_{\mathbf{E}}(\mu(G_\lambda)) \leq \frac{C}{\lambda} \|f\chi_{G_\lambda}\|_{\mathbf{E}} . \tag{4.6}$$

Since $\|f\chi_{G_\lambda}\|_{\mathbf{E}} \leq \|f\chi_{G_\lambda}\|_{\mathbf{E}}$, by property (P2) of a Banach function norm, (4.1) follows from (4.6).

The estimate (4.1) implies $\lim\limits_{\lambda\to\infty} \Phi_{\mathbf{E}}(\mu(G_\lambda)) = 0$, hence $\lim\limits_{\lambda\to\infty} \mu(G_\lambda) = 0$, by Lemma 1.

Assuming that $\mathbf{E}$ has absolutely continuous norm, it follows that $\lim\limits_{\lambda\to\infty} \|f\chi_{G_\lambda}\|_{\mathbf{E}} = 0$. Now, using (4.6), we get (4.2). ∎

Remark 1. *For* $\mathbf{E} = L^{p,q}(X)$ *with* $1 \leq q \leq p$ *the above theorem becomes Lemma 6.3 from* [6], *which for* $p = q = 1$ *yields Lemma 4.2 from* [12], *in view of Proposition 1. Also, Lemma 4.2 from* [12] *yields* $\lim\limits_{\lambda\to\infty} \lambda^p \mu(\{x \in X : \mathcal{M}^* f(x) > \lambda\}) = 0$ *for every* $f \in L^p(X)$, *a fact that was generalized in Lemma 6.18 from* [14], *which implies* $\lim\limits_{\lambda\to\infty} \Psi(\lambda)\mu(\{x \in X : \mathcal{M}^* f(x) > \lambda\}) = 0$ *for every* $f \in L^\Psi(X)$, *where* Ψ *is a doubling Young function and* μ *is a doubling measure.*

Remark 2. *By Corollary 1, every Orlicz space* $\mathbf{E} = L^\Psi(X)$ *corresponding to an* $N-$*function* Ψ *satisfies a lower* $\Phi_{\mathbf{E}}-$*estimate.*

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