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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 27(2017), No. 1, 61-66

SOME GEOMETRICAL ASPECTS OF A HORIZONTAL DISTRIBUTION OF PFAFF SYSTEMS ON MANIFOLDS

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Abstract. We continue our investigation [7] and [8] of Pfaff systems on manifolds by studying the horizontal distributions that are supplementary to the vertical distributions on the total space of a submersion. The horizontal distribution is kernel of the Pfaff system. We point out the Berwald connection induced by a nonlinear connection and its curvature tensors.

1. Introduction

As we previously shown in [7] and [8], if X is C^∞ manifold of dimension $n+m$ and $\tau : TX \to X$ its tangent bundle, a **distribution** on X is a mapping $x \to H_x \subset T_xX$, $x \in X$ with the properties

i) H_x is a linear subspace of dimension n in T_xX and

ii) for every $x_0 \in X$, there exists an open neighborhood U and the vector fields $X_1, ..., X_n$ on U which are linearly independent on U and $H_x = \text{span}[X_1(x), ..., X_n(x)]$, $\forall x \in U$.

A distribution on X can be given by a Pfaff system called its annihilator.

Keywords and phrases: Pfaff systems, distributions, structure equations
(2010) Mathematics Subject Classification: 53C99.

A **Pfaff system** [7] on X is a mapping $E : x \to E_x \subset T_x^*X$, $x \in X$ with the properties

i) E_x is a linear subspace of dimension m in T_x^*X and
ii) for every $x_0 \in X$, there exists an open neighborhood U and the 1-forms $\omega^1, ..., \omega^m$ on U which are linearly independent on U and $E_x = \text{span}[\omega^1(x), ..., \omega^m(x)]$, $\forall x \in U$.

Given a distribution H spanned by $X_1, ..., X_n$, the set $E_x = \{\theta \mid \theta(X_i) = 0, i = 1, 2, ...n\}$ is a linear subspace in T_x^*X of dimension $d - n = m$. A basis of it is given by the 1-forms ω^a having as coefficients the fundamental solutions of the linear system $X_i^a \theta_a = 0$ in the unknown θ_a, where $\theta = \theta_a dx^a$ and $X_i = X_i^b \frac{\partial}{\partial x^b}$.

Given a Pfaff system E by the 1-forms (ω^a), $a = 1, 2, ..m$, the set $H_x = \{X \mid \omega^a(X) = 0, a = 1, 2, ...m\}$ is a linear subspace of T_xX of dimension $d-m = n$. A basis of it is made of the fundamental solutions of the linear system $\omega_a^a X^a = 0$, for $\omega^a = \omega_a^a dx^a$ and $X = X^a \frac{\partial}{\partial x^a}$. Thus the mapping $x \to H_x$ is a distribution on X. This is called the kernel of E and denoted by $\ker E$. It comes out that $\ker H^\circ = H$ and $(\ker E)^\circ = E$.

We have shown in [7] that the 1- forms (ω^a) may be replaced with the 1-forms $\delta y^a = dy^a + N_i^a(x, y)dx^i$ when the coordinates (x^{alpha}) on X are separated in (x^i, x^a) and (x^a) are denoted as (y^a).

We computed in [7] $d(\delta y^a)$ and put the result into the form $d(\delta y^a) + \omega_b^a \wedge \delta y^b = \Omega^a$ that provides a first structure equation and introduces the 2- forms of torsion Ω^a.

Exterior differentiating again we get the first Bianchi identity $d\Omega^a + \omega_b^a \wedge \Omega^b = \theta_c^a \wedge \delta y^c$, where the equality $\theta_c^a = d\omega_c^a + \omega_b^a \wedge \omega_c^b$ represents a second structure equation.

A new exterior differentiation leads to the second Bianchi identity: $d\theta_b^a + \omega_c^a \wedge \theta_b^c = \theta_c^a \wedge \omega_b^c$.

2. Main result

We consider a surjective submersion $\pi : E \to M$ with M a smooth manifold of dimension n and E a smooth manifold of dimension $n+m$. For every $x \in M$, the fiber $\pi^{-1}(x)$ is a submanifold of dimension m in E.

The mapping $V : u \to V_uE$ is a distribution on E that is called the vertical distribution. We denote by $VE = \cup u \in E$ the vertical subbundle of TE.

We may take on E local coordinates $(x^1 \circ \pi, ..., x^n \circ \pi, y^1, ..., y^m)$ where $(x^1, ..., x^n)$ are local coordinates on M, $(y^1, ..., y^m)$ to be local coordinates in the fiber and the π takes the form $(x^1, ..., x^n, y^1, ..., y^n) \to (x^1, ..., x^n)$. We use the identification $x^i \equiv x^i \circ \pi$.

A section in vertical subbundle is called a vertical vector field. A vector $A = A^i \frac{\partial}{\partial x^i} + B^a \frac{\partial}{\partial y^a}$ in T_uE is vertical, that is $\pi_{*,u}(A) = 0$ if and only if $A^i = 0$. Thus the vertical vectors are tangent to fibers so they are linear combination of $\frac{\partial}{\partial y^a}$.

In other words the vertical distribution is integrable. The vertical distribution is the kernel of the Pfaff system spanned by $\{dx^i,\ i = 1, 2, ..., n\}$.

The ideal $\{dx^i\}$ is clearly a differential ideal and it results again that the vertical distribution is integrable.

We introduce a new distribution on E which is supplementary to the vertical distribution on E will be called a **horizontal distribution** on E.

We denote by H_uE the subspace in T_uE such that

$$T_uE = H_uE \oplus V_uE,\ \forall u \in E. \tag{2.1}$$

and by $HE = \cup u \in EH_uE$ the horizontal subbundle of TE.

A horizontal distribution can be given by the local vector fields

$$\delta_i = \frac{\partial}{\partial x^i} - N_i^a(x, y)\frac{\partial}{\partial y^a}. \tag{2.2}$$

We notice that $\pi_*(\delta_i) = \frac{\partial}{\partial x^i}$.

A section in the horizontal subbundle will be called a horizontal vector field on E. Any such field can be given in the form $X^i(x, y)\delta_i$.

The Pfaff system whose kernel is the horizontal distribution is spanned by the local 1-forms $\delta y^a = dy^a + N_i^a(x, y)dx^i$.

We compute: $d(\delta y^a) = dN_j^a \wedge dx^j = \frac{1}{2}(\frac{\partial N_j^a}{\partial x^i} - \frac{\partial N_i^a}{\partial x^j})dx^i \wedge dx^j + + \frac{\partial N_j^a}{\partial y^b}dy^b \wedge dx^j$. Inserting $dy^b = \delta y^b - N_i^b dx^i$ we get

$$d(\delta y^a) + \omega_b^a \wedge \delta y^b = \Omega^a, \tag{2.2}$$

where

$$\omega_b^a = \frac{\partial N_j^a}{\partial y^b}dx^j, \tag{2.2'}$$

(2.2”)

$$2\Omega^a = (\frac{\partial N_j^a}{\partial x^i} - \frac{\partial N_i^a}{\partial x^j} + \frac{\partial N_i^a}{\partial y^b}N_j^b - \frac{\partial N_j^a}{\partial y^b}N_i^b)dx^i \wedge dx^j = (\delta_i N_j^a - \delta_j N_i^a)dx^i \wedge dx^j.$$

We exterior differentiate in (2.2) and in the result we use again (2.2). We have:

$$d\Omega^a = d\omega^a_b \wedge \delta y^b - \omega^a_b \wedge d(\delta y^b) = d\omega^a_b \wedge \delta y^b - \omega^a_b \wedge \Omega^b + \omega^a_b \wedge \omega^b_c \wedge \delta y^c,$$

that is

$$d\Omega^a + \omega^a_b \wedge \Omega^b = \theta^a_c \wedge \delta y^c, \tag{2.3}$$

where

$$\theta^a_c = d\omega^a_c + \omega^a_b \wedge \omega^b_c. \tag{2.4}$$

Exterior differentiating in (2.4) one obtains

$$d\theta^a_b + \omega^a_c \wedge \theta^c_b = \theta^a_c \wedge \omega^c_b. \tag{2.5}$$

We name (2.2) and (2.4) the structure equations of the distribution H and (2.3) and (2.5) will be called the Bianchi identities.

The functions (N^a_i) have to transform under a change of coordinates $(x^i, y^i) \to (\widetilde{x}^i, \widetilde{y}^i)$ on E.

Such a change of coordinates is given by

$$\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n),\ \widetilde{y}^a = \widetilde{y}^a(x, y),\ \frac{\partial(\widetilde{x}^i, \widetilde{y}^a)}{\partial(x^i, y^a)} \neq 0. \tag{2.6}$$

It follows that $A^i_j = \frac{\partial \widetilde{x}^i}{\partial x^j}$, $A^i_b = 0$ and the law of transformation of (N^a_i) is

$$\widetilde{N}^a_j \frac{\partial \widetilde{x}^j}{\partial x^i} = \frac{\partial \widetilde{y}^a}{\partial y^b} N^b_i - \frac{\partial \widetilde{y}^a}{\partial x^i}. \tag{2.7}$$

The matrix (B^a_b) reduces to $(\frac{\partial \widetilde{y}^a}{\partial y^b})$ and we obtain

$$\widetilde{\Omega}^a_{kh} = \frac{\partial \widetilde{y}^a}{\partial y^b} \Omega^b_{ij} \frac{\partial x^i}{\partial \widetilde{x}^k} \frac{\partial x^j}{\partial \widetilde{x}^h}, \tag{2.8$_1$}$$

$$\widetilde{\omega}^a_{ci} \frac{\partial \widetilde{x}^i}{\partial x^k} \frac{\partial \widetilde{y}^c}{\partial y^b} = \frac{\partial \widetilde{y}^a}{\partial y^c} \omega^c_{bk} - \frac{\delta}{\delta x^k} (\frac{\partial \widetilde{y}^a}{\partial y^b}). \tag{2.8$_2$}$$

So, the horizontal distribution is integrable if and only if $\Omega^a_{ij}(x, y) = 0$.

Now the fibers are linear spaces and the changes of coordinates on E have the form

$$\begin{gathered} \widetilde{x}^i = \widetilde{x}^i(x^1, ...x^n), \widetilde{y}^a = M^a_b(x) y^b, \\ \operatorname{rank}(\tfrac{\partial \widetilde{x}^i}{\partial x^j}) = n, \operatorname{rank}(M^a_b(x)) = m. \end{gathered} \tag{2.9}$$

Then, the horizontal distribution on E is named a nonlinear connection and it is an essential tool in the geometry of the total space E, [4].

The functions $(N_i^a(x,y))$ are called the local coefficients of the given nonlinear connection and they transform under a change of coordinates (2.9) by the law

$$\widetilde{N}_j^a \frac{\partial \widetilde{x}^j}{\partial x^i} = M_b^a N_i^b - \frac{\partial M_b^a}{\partial x^i} y^b. \tag{2.10}$$

The formulae (2.8_1) and (2.8_2) reduce respectively to

$$\widetilde{\Omega}_{kh}^a = M_b^a \Omega_{ij}^b \frac{\partial x^i}{\partial \widetilde{x}^k} \frac{\partial x^j}{\partial \widetilde{x}^h}, \tag{2.11_1}$$

$$\widetilde{\omega}_{ci}^a \frac{\partial \widetilde{x}^i}{\partial x^k} M_b^c = M_c^a \omega_{bk}^c - \frac{\partial M_b^a(x)}{\partial x^k}. \tag{2.11_2}$$

By (2.11_2), the set of functions $\omega_{bk}^a = \frac{\partial N_k^a(x,y)}{\partial y^b}$ behaves like the coefficients of a linear connection in the vertical subbundle on E.

So, we can define a linear connection D in the said subbundle by

$$D_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial y^b} = \frac{\partial N_k^a}{\partial y^b} \frac{\partial}{\partial y^a}, \; D_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial y^b} = 0. \tag{2.12}$$

called the Berwald connection induced by a nonlinear connection.

Now, inserting $\omega_b^a = \omega_{bi}^a dx^i$ in the 2-form of curvature θ_b^a, one gets

$$\theta_b^a = \frac{1}{2} \theta_{bij}^a dx^i \wedge dx^j + \psi_{bcj}^a \delta y^c \wedge dx^j \tag{2.13}$$

where

$$\begin{aligned} \theta_{b\ ij}^a &= \delta j \omega_b^a - \delta_i \omega_b^a + \omega_{ci}^a \omega_{bj}^c - \omega_{cj}^a \omega_{bj}^c, \\ \psi_{bcj}^a &= \tfrac{\partial \omega_{bj}^a}{\partial y^c}. \end{aligned} \tag{2.13'}$$

For a vector bundle one we have the local coefficients θ_{bij}^a and ψ_{bcj}^a given by (2.13') have mixed tensorial character, that is they change as tensors with the matrix $(\frac{\partial \widetilde{x}^i}{\partial x^j})$.

These functions are curvature tensors for the Berwald connection [4]. We also observe that (θ_b^a) deserve the name of curvature 2-forms and that it is more natural to call (Ω^a) the torsion 2-forms of a distribution.

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