"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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FIXED POINT THEOREMS FOR TWO PAIRS OF WEAKLY COMPATIBLE MAPPINGS SATISFYING A NEW TYPE OF COMMON LIMIT RANGE PROPERTY

VALERIU POPA AND ALINA-MIHAELA PATRICIU

Abstract. The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying a new type of common limit range property. In the last part of the paper, as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for ϕ - contractive mappings and ψ - weak contractive mappings generalizing Theorem 2 [27] and other known results are obtained.

1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. Jungck [11] defined S and T to be compatible if$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. This concept has been frequently used to prove the existence theorem in fixed point theory.

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Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $fx = gx = w$ and w is a point of coincidence of f and g.

By $\mathcal{C}(f, g)$ we denote the set of all coincidence points of f and g.

f and g are said to be weakly compatible if $fgx = gfx$ for all $x \in \mathcal{C}(f, g)$.

The study of noncompatible mappings in metric spaces is also interesting. The work in this regard has been initiated by Pant in [15] - [17] and in other papers.

Aamri and El - Moutawakil [1] introduced a generalization of noncompatible mappings.

Definition 1.1 ([1])**.** Let S and T be self mappings of a metric space (X, d). S and T satisfy $(E.A)$ property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Remark 1.2. *It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists a sequence $\{x_n\}$ in X such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, *for some $t \in X$ but* $\lim_{n\to\infty} d(STx_n, TSx_n)$ *is non zero or non existent.*

Therefore, two noncompatible mappings of a metric space satisfy $(E.A)$ *- property.*

Liu et al. [13] introduced the notion of common $(E.A)$ - property.

Definition 1.3 ([13])**.** Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy common $(E.A)$ - property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

In 2011, Sintunawarat and Kumam introduced the notion of common limit range property.

Definition 1.4 ([26])**.** A pair of mappings (A, S) of a metric space (X, d) is said to satisfy the common limit range property with respect to S, denoted $CLR_{(S)}$ - property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X)$.

Thus we can infer that a pair (A, S) satisfying $(E.A)$ - property along with the closedness of the subspace $S(X)$ always have $CLR_{(S)}$ - property.

Recently, Imdad et al. [8] introduced the concept of common limit range property for two pairs of mappings.

Definition 1.5 ([8]). Two pairs (A, S) and (B, T) of mappings defined on a metric space (X, d) are said to satisfy common limit range property with respect to S and T, denoted $CLR_{(S,T)}$ - property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

where $t \in S(X) \cap T(X)$.

Some results for mappings satisfying $CLR_{(S)}$ - property and $CLR_{(S,T)}$ - property are obtained in [7], [9] and in other papers.

Now, we introduce a new type of limit range property.

Definition 1.6. Let A, S and T be self mappings of a metric space (X, d). The pair (A, S) is said to satisfy common limit range property with respect to T, denoted $CLR_{(A,S),T}$ - property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X) \cap T(X)$.

Example 1.7. *Let $\mathbb{R}_+$ be the metric space with the usual metric, $Ax = \dfrac{x^2+1}{2}$, $Sx = \dfrac{x+1}{2}$ and $Tx = x + \dfrac{1}{4}$. Then $S(X) = \left[\dfrac{1}{2}, \infty\right)$, $T(X) = \left[\dfrac{1}{4}, \infty\right)$ and $S(X) \cap T(X) = \left[\dfrac{1}{2}, \infty\right)$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = 0$. Then, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \dfrac{1}{2} \in S(X) \cap T(X)$.*

Remark 1.8. *Let A, B, S and T be self mappings of a metric space (X, d). If (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, then (A, S) and T satisfy $CLR_{(A,S),T}$ - property. The converse is not true. In Example 1.7, let $Bx = x^2 + \dfrac{1}{4}$ and let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = 0$. Then $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \dfrac{1}{4} \neq \dfrac{1}{2}$.*

Hence (A, S) and (B, T) don't satisfy $CLR_{(S,T)}$ - property.

The notion of weak contractive condition in Hilbert spaces is introduced in [3] by Alber and Guerre - Delabriere.

Rhoades [22] extends this concept in metric spaces.

Other results are obtained in [5], [10] and in other papers.

Definition 1.9. Let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

1) ϕ is lower semi - continuous,

2) $\phi(0) = 0$,
3) $\phi(t) > 0$, $\forall t > 0$.

In [6], Branciari established the following theorem, which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 1.10 ([6])**.** *Let (X, d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ such that for all $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt$$

whenever $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, such that $\int_0^\varepsilon h(t)dt > 0$, $\forall \varepsilon > 0$.

Then, f has a unique fixed point $z \in X$ such that for all $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [20], [21], [23] and in other papers.

The following theorem is proved in [27].

Theorem 1.11 (Theorem 2 [27])**.** *Let A, S, B and T be self mappings of a metric space (X, d) such that for all $x, y \in X$, there exists $\phi \in \Phi$ such that*

$$\int_0^{d(Ax,By)} h(t)dt \leq M(x,y) - \phi(M(x,y)),$$

where $h(t)$ is as in Theorem 1.10 and

$$M(x,y) = \int_0^{\max\left\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}} h(t)dt.$$

If the pairs (A, S) and (B, T) satisfy $CLR_{(S,T)}$ - property, then
1) $\mathcal{C}(A, S) \neq \emptyset$,
2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings (A, S) and (B, T) such that (A, S) and T satisfy $CLR_{(A,S),T}$ - property, using implicit relations. As applications, some fixed point results for mappings satisfying contractive conditions of integral type, for φ - contractive mappings and ϕ - weak contractive mappings generalizing Theorem 1.11, are obtained and also we obtain other known results.

2. Implicit relations

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function in [18], [19] and in other papers. Recently, this method is used in the study of the existence of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, convex metric spaces, ultra - metric spaces, compact metric spaces, Hilbert spaces, in two and three metric spaces, for single - valued functions, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, intuitionistic metric spaces, probabilistic metric spaces, G - metric spaces, G_p - metric spaces, partial metric spaces.

With this method the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

In 2008, Ali and Imdad [4] introduced a new class of implicit relations. Recently, Imdad and Chauhan [9] employed common limit range property to prove unified metrical common fixed point theorems in metric spaces. We introduce a new form of implicit relation.

Definition 2.1. Let $\mathcal{F}_5$ be the family of lower semi - continuous functions $F(t_1, ..., t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ such that:
$(F_1) : F(t, 0, 0, t, t) > 0$, $\forall t > 0$;
$(F_2) : F(t, 0, t, 0, t) > 0$, $\forall t > 0$;
$(F_3) : F(t, t, 0, 0, 2t) > 0$, $\forall t > 0$.

Example 2.2. $F(t_1, ..., t_5) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_5}{2}\right\}$, *where* $k \in [0, 1)$.

Example 2.3. $F(t_1, ..., t_5) = t_1 - k \max\{t_2, t_3, t_4, t_5\}$, *where* $k \in \left[0, \dfrac{1}{2}\right)$.

Example 2.4. $F(t_1, ..., t_5) = t_1 - \max\{at_2, bt_3, ct_4, dt_5\}$, *where* $a, b, c, d \geq 0$ *and* $a + b + c + 2d < 1$.

Example 2.5. $F(t_1, ..., t_5) = t_1 - at_2 - bt_3 + ct_4 - dt_5$, *where* $a,b,c,d \geq 0$ *and* $a + b + c + 2d < 1$.

Example 2.6. $F(t_1, ..., t_5) = t_1 - a\max\{t_2, t_3, t_4\} - bt_5$, *where* $a,b \geq 0$ *and* $a + 2b < 1$.

Example 2.7. $F(t_1, ..., t_5) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5^2$, *where* $a, b, c, d \geq 0$ *and* $a + b + c + 4d < 1$.

Example 2.8. $F(t_1, ..., t_5) = t_1^2 + \dfrac{t_1}{1 + t_5} - (at_2^2 + bt_3^2 + ct_4^2)$, *where* $a, b, c \geq 0$ *and* $a + b + c < 1$.

Example 2.9. $F(t_1, ..., t_5) = at_2 - bt_3 - c\max\{2t_4, t_5\}$, *where* $a, b, c \geq 0$ *and* $a + b + 2c < 1$.

3. Main results

Lemma 3.1 ([2]). *Let f and g be two weakly compatible self mappings of a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$, for some $x \in X$, then w is the unique common fixed point of f and g.*

Theorem 3.2. *Let (X, d) be a metric space and A, B, S and T be self mappings of X satisfying the inequality*

$$\begin{gathered} F(d(Ax, By), d(Sx, Ty), d(Sx, Ax), \\ d(Ty, By), d(Sx, By) + d(Ty, Ax)) \leq 0, \end{gathered} \tag{3.1}$$

for all $x, y \in X$ and some F satisfying property (F_3).

If there exist $u, v \in X$ such that $Au = Su$ and $Tv = Bv$, then there exists $t \in X$ such that t is the unique point of coincidence of A and S, as well the unique point of coincidence of B and T.

Proof. First we prove that $Su = Tv$. By (3.1) for $x = u$ and $y = v$ we get

$$\begin{gathered} F(d(Au, Bv), d(Su, Tv), d(Su, Au), \\ d(Tv, Bv), d(Su, Bv) + d(Tv, Au)) \leq 0, \\ F(d(Su, Tv), d(Su, Tv), 0, 0, 2d(Su, Tv)) \leq 0, \end{gathered}$$

a contradiction of (F_3) if $d(Su, Tv) > 0$. Hence $d(Su, Tv) = 0$, which implies $Su = Tv$. Hence, $Au = Su = Tv = Bv = t$ for some $t \in X$ and t is a point of coincidence of A and S and for B and T.

Suppose that there exists $w \neq u$ such that $Sw = Aw$. Then, by (3.1) for $x = w$ and $y = v$ we have

$$\begin{gathered} F(d(Aw, Bv), d(Sw, Tv), d(Sw, Aw), \\ d(Tv, Bv), d(Sw, Bv) + d(Tv, Aw)) \leq 0, \end{gathered}$$

$$F(d(Sw,Tv),d(Sw,Tv),0,0,2d(Sw,Tv)) \leq 0,$$

a contradiction of (F_3) if $d(Sw,Tv) > 0$. Hence, $d(Sw,Tv) = 0$, which implies $Sw = Tv = t$. Therefore t is the unique point of coincidence of A and S.

Similarly, t is the unique point of coincidence of B and T. □

Theorem 3.3. *Let (X,d) be a metric space and A, B, S and T be self mappings of X satisfying inequality (3.1) for all $x, y \in X$ and some $F \in \mathcal{F}_5$. If (A,S) and T satisfy $CLR_{(A,S),T}$ - property, then*

(i) $\mathcal{C}(A,S) \neq \emptyset$,

(ii) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Since (A,S) and T satisfy $CLR_{(A,S),T}$ - property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X) \cap T(X)$.

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$.

By (3.1) for $x = x_n$ and $y = u$ we have

$$F(d(Ax_n,Bu),d(Sx_n,Tu),d(Sx_n,Ax_n),$$
$$d(Tu,Bu),d(Sx_n,Bu)+d(Tu,Ax_n)) \leq 0.$$

Letting n tends to infinity we obtain

$$F(d(z,Bu),0,0,d(z,Bu),d(z,Bu)) \leq 0,$$

a contradiction of (F_1) if $d(z,Bu) > 0$. Hence, $d(z,Bu) = 0$, which implies $z = Bu = Tu$ and $\mathcal{C}(B,T) \neq \emptyset$.

On the other hand $z \in S(X)$, which implies $z = Sv$ for some $v \in X$. Again, by (3.1) for $x = v$ and $y = u$ we obtain

$$F(d(Av,Bu),d(Sv,Tu),d(Sv,Av),$$
$$d(Tu,Bu),d(Sv,Bu)+d(Tu,Av)) \leq 0,$$

$$F(d(Av,z),0,d(Av,z),0,d(z,Av)) \leq 0,$$

a contradiction of (F_2) if $d(Av,z) > 0$. Hence, $d(Av,z) = 0$, which implies $z = Av = Sv$ and $\mathcal{C}(A,S) \neq \emptyset$. Then $z = Av = Sv = Tu = Bu$.

By Theorem 3.2, z is the unique point of coincidence for A and S and for B and T.

Moreover, if (A,S) and (B,T) are weakly compatible, by Lemma 3.1, z is the unique common fixed point for A, B, S and T. □

By Theorem 3.3 and Example 2.2 we obtain

Theorem 3.4. *Let (X,d) be a metric space and A,B,S and T be self mappings of X such that for all $x,y\in X$*

$$d(Ax,By)\leq k\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)+d(Ty,Ax)}{2}\}\leq 0.$$

If (A,S) and T satisfy $CLR_{(A,S),T}$ - property, then
(i) $\mathcal{C}(A,S)\neq\emptyset$,
(ii) $\mathcal{C}(B,T)\neq\emptyset$.
Moreover, if (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Example 3.5. *Let $X=[0,1]$ be and d be the usual metric on X. Consider the following mappings:*

$$Ax=0,Sx=\frac{x}{x+2},Bx=\frac{x}{3},Tx=x.$$

Then

$$S(X)=\left[0,\frac{1}{3}\right],T(X)=[0,1],S(X)\cap T(X)=\left[0,\frac{1}{3}\right].$$

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}x_n=0$. Then, $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=0\in S(X)\cap T(X)$. Hence (A,S) and T satisfy $CLR_{(A,S),T}$ - property.

On the other hand $Sx=Ax$ implies $\mathcal{C}(A,S)=\{0\}$. Hence $AS0=SA0=0$ and hence A and S are weakly compatible.

$Bx=Tx$ implies $\mathcal{C}(B,T)=\{0\}$ and $BT0=TB0=0$, hence B and T are weakly compatible. As

$$d(Ax,By)=\frac{y}{3}\text{ and }d(Ty,By)=\frac{2y}{3},$$

then

$$d(Ax,By)\leq kd(Ty,By),$$

where $k\in\left[\frac{1}{2},1\right)$, which implies

$$d(Ax,By)\leq k\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)+d(Ax,Ty)}{2}\}.$$

By Theorem 3.4, A,B,S and T have a unique common fixed point $x=0$.

4. Applications

4.1. **Fixed points for mappings satisfying a contractive conditions of integral type.**

Definition 4.1. An altering distance is a function $\theta : [0, \infty) \to [0, \infty)$ satisfying

(θ_1) : θ is increasing and continuous,

(θ_2) : $\theta(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving altering distances are proved in [21], [24], [25] and in other papers.

Lemma 4.2 ([21])**.** *Let $h : [0, \infty) \to [0, \infty)$ as in Theorem 1.10. Then $\theta(t) = \int_0^t h(x)dx$ is an altering distance.*

The following functions using altering distance satisfy properties $(F_1) - (F_3)$.

Example 4.3.

$$F(t_1, ..., t_5) = \theta(t_1) - k \max\left\{\theta(t_2), \theta(t_3), \theta(t_4), \theta\left(\frac{t_5}{2}\right)\right\},$$

where $k \in [0, 1)$.

Example 4.4.

$$F(t_1, ..., t_5) = \theta(t_1) - a\theta(t_2) - b\theta(t_3) - c\theta(t_4) - d\theta\left(\frac{t_5}{2}\right),$$

where $a, b, c, d \geq 0$ and $a + b + c + d < 1$.

Example 4.5.

$$F(t_1, ..., t_5) = [\theta(t_1)]^2 + \frac{\theta(t_1)}{1 + \theta(t_5)} - \theta(t_1)[a\theta(t_2) + b\theta(t_3) + c\theta(t_4)],$$

where $a, b, c \geq 0$ and $a + b + c < 1$.

Example 4.6.

$$F(t_1, ..., t_5) = \theta(t_1) - k \max\left\{\theta(t_2), \sqrt{\theta(t_3)\theta(t_4)}, \sqrt{\theta(t_4)\theta(t_5)}\right\},$$

where $k \in \left[0, \frac{1}{2}\right)$.

Example 4.7.

$$F(t_1, ..., t_5) = \theta(t_1) - k\left(\frac{\sqrt{\theta(t_3)\theta(t_4)} + \sqrt{\theta(t_2)\theta(t_4)} + \sqrt{\theta(t_4)\theta(t_5)}}{1 + \sqrt{\theta(t_3)\theta(t_4)}}\right),$$

where $k \in [0,1)$.

Example 4.8.

$$F(t_1, ..., t_5) = \theta(t_1) - k \max\left\{\begin{array}{c} \sqrt{\theta(t_2)\theta(t_4)}, \sqrt{\theta(t_4)\theta(t_5)}, \\ \sqrt{\theta(t_3)\theta(t_5)}, \sqrt{\theta(t_2)\theta(t_3)} \end{array}\right\},$$

where $k \in [0,1)$.

By Theorem 3.2 and Example 4.3 we obtain

Theorem 4.9. *Let* A, B, S *and* T *be self mappings of a metric space* (X,d) *such that for all* $x, y \in X$

$$\theta(d(Ax,By)) \leq k \max\{\theta(d(Sx,Ty)), \theta(d(Sx,Ax)), \theta(d(Ty,By)), \theta\left(\frac{d(Sx,By)+d(Ty,Ax)}{2}\right)\} \leq 0,$$

where $k \in [0,1)$ *and* θ *is an altering distance.*

If (A,S) *and* T *satisfy* $CLR_{(A,S),T}$ *- property, then*

(i) $\mathcal{C}(A,S) \neq \emptyset$,

(ii) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

By Theorem 4.9 and Lemma 4.2 we obtain

Theorem 4.10. *Let* A, B, S *and* T *be self mappings of a metric space* (X,d) *such that for all* $x, y \in X$

$$\int_0^{d(Ax,By)} h(t)\,dt \leq k \max\{\int_0^{d(Sx,Ty)} h(t)\,dt, \int_0^{d(Sx,Ax)} h(t)\,dt, \int_0^{d(Ty,By)} h(t)\,dt, \int_0^{\frac{d(Sx,By)+d(Ty,Ax)}{2}} h(t)\,dt\} \leq 0,$$

where $k \in [0,1)$ *and* $h(t)$ *as in Theorem 1.10.*

If (A,S) *and* T *satisfy* $CLR_{(A,S),T}$ *- property, then*

(i) $\mathcal{C}(A,S) \neq \emptyset$,

(ii) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Remark 4.11. *This theorem is a generalization of Theorem 1.11.*

4.2. **Fixed points for mappings satisfying** φ **- contractive conditions.** As in [14], let Φ be the set of all real nondecreasing continuous functions $\varphi : [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \varphi^n(t) = 0$, for all $t \in [0,\infty)$.

If $\varphi \in \Phi$, then

1) $\varphi(t) < t$ for all $t > 0$,
2) $\varphi(0) = 0$.

The following functions $F(t_1, ..., t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ satisfy conditions $(F_1) - (F_3)$.

Example 4.12. $F(t_1, ..., t_5) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \dfrac{t_5}{2}\right\}\right)$.

Example 4.13.

$$F(t_1, ..., t_5) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5),$$

where $a, b, c, d \geq 0$ *and* $a + b + c + 2d < 1$.

Example 4.14. $F(t_1, ..., t_5) = t_1 - \varphi\left(\max\left\{t_2, \sqrt{t_3 t_5}, \sqrt{t_4 t_5}\right\}\right)$.

Example 4.15.

$$F(t_1, ..., t_5) = t_1 - \varphi(at_2 + b\max\{t_3 + t_5, t_4 - t_5\}),$$

where $a, b \geq 0$ *and* $a + 2b \leq 1$.

Example 4.16.

$$F(t_1, ..., t_5) = t_1 - \varphi(aL(x, y) + (1 - a) M(x, y)),$$

where $a \in (0, 1)$,

$$L(x, y) = \max\{t_2, t_3, t_4\}, \ M(x, y) = [\max\{t_2^2, t_2 t_3, t_3 t_4, t_4 t_5\}]^{1/2}.$$

Example 4.17.

$$F(t_1, ..., t_5) = (t_1 + at_2) t_1 - \varphi\left(\max\{t_2^2, t_2 t_3, t_3 t_4, t_4 t_5\}\right).$$

By Theorem 3.3 and Example 4.12 we obtain

Theorem 4.18. *Let* A, B, S *and* T *be self mappings of a metric space* (X, d) *such that for all* $x, y \in X$

$$d(Ax, By) \leq \varphi(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{d(Sx, By) + d(Ty, Ax)}{2}\}).$$

If A, S *and* T *satisfy* $CLR_{(A,S),T}$ *- property, then*
(i) $\mathcal{C}(A, S) \neq \emptyset$,
(ii) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Remark 4.19. *By Theorem 3.3 and Examples 4.13 - 4.17 we obtain new particular results.*

4.3. **Fixed point theorems for mappings satisfying ϕ - weak contractive conditions.** The following functions $F(t_1,...,t_5):\mathbb{R}_+^5\to\mathbb{R}$ satisfy conditions $(F_1)-(F_3)$.

Example 4.20.

$$F(t_1,...,t_5)=t_1-\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}+\phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}\right).$$

Example 4.21.

$$F(t_1,...,t_5)=t_1-\max\left\{t_2,\sqrt{t_3t_4},\sqrt{t_3t_5}\right\}+\phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}\right).$$

Example 4.22.

$$F(t_1,...,t_5)=t_1-\left(at_2+bt_3+ct_4-d\frac{t_5}{2}\right)+\phi\left(\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}\right),$$

where $a,b,c,d\geq 0$ and $a+b+c+2d<1$.

Example 4.23.

$$F(t_1,...,t_5)=t_1^2-t_1(at_2+bt_3+ct_4)-\phi(\max\{t_2,t_3,t_4\}).$$

Example 4.24.

$$F(t_1,...,t_5)=t_1-k\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}+\phi(\max\{t_2,t_3,t_4\}),$$

where $k\in[0,1)$.

Example 4.25.

$$F(t_1,...,t_5)=t_1-\frac{\sqrt{t_3t_5}+\sqrt{t_2t_5}+\sqrt{t_4t_5}}{1+\sqrt{t_3t_4}}+\phi\left(\max\left\{t_2,\sqrt{t_3t_4},\sqrt{t_4t_5}\right\}\right).$$

By Theorem 3.3 and Example 4.20 we obtain

Theorem 4.26. *Let A,B,S and T be self mappings of a metric space (X,d) such that for all $x,y\in X$*

$$\begin{aligned}d(Ax,By)\leq{}&\max\{d(Sx,Ty),d(Sx,Ax),\\&d(Ty,By),\frac{d(Sx,By)+d(Ty,Ax)}{2}\}-\\&-\phi\begin{pmatrix}d(Sx,Ty),d(Sx,Ax),\\d(Ty,By),\frac{d(Sx,By)+d(Ty,Ax)}{2}\end{pmatrix}.\end{aligned}$$

If A,S and T satisfy $CLR_{(A,S),T}$ - property, then
(i) $\mathcal{C}(A,S)\neq\emptyset$,
(ii) $\mathcal{C}(B,T)\neq\emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Remark 4.27. *1) If A, B, S and T satisfy $CLR_{(A,S),T}$ - property in symmetric spaces, a similar result with Theorem 2* [10] *is obtained.*

2) By Theorem 3.3 and Examples 4.21 - 4.25 new particular results are obtained.

4.4. **Fixed point theorems for mappings satisfying** (θ, ϕ) **- weak contractive conditions.** The following functions $F(t_1, ..., t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ satisfy conditions $(F_1) - (F_3)$.

Example 4.28.

$$F(t_1, ..., t_5) = \theta(t_1) - \theta\left(\max\left\{t_2, t_3, t_4, \tfrac{t_5}{2}\right\}\right) + \phi\left(\theta\left(\max\left\{t_2, t_3, t_4, \tfrac{t_5}{2}\right\}\right)\right).$$

Example 4.29.

$$\begin{aligned} F(t_1, ..., t_5) &= \theta(t_1) - \theta\left(\max\left\{t_1, t_2, t_3, t_4, \tfrac{t_5}{2}\right\}\right) + \\ &+ \phi\left(\max\left\{\theta(t_1), \theta(t_2), \theta(t_3), \theta(t_4), \theta\left(\tfrac{t_5}{2}\right)\right\}\right). \end{aligned}$$

Example 4.30.

$$F(t_1, ..., t_5) = \theta(t_1) - \theta\left(\max\left\{t_2, \sqrt{t_3 t_4}, \sqrt{t_4 t_5}\right\}\right) + \phi(\max\{t_2, t_3, t_4\}).$$

Example 4.31.

$$F(t_1, ..., t_5) = \theta(t_1) - \theta(\max\{t_2, t_3, t_4\}) + \phi\left(\max\left\{t_2, t_3, t_4, \tfrac{t_5}{2}\right\}\right).$$

Example 4.32.

$$F(t_1, ..., t_5) = \theta(t_1) - \theta\left(\tfrac{\sqrt{t_3 t_5} + \sqrt{t_2 t_4} + \sqrt{t_4 t_5}}{1 + \sqrt{t_3 t_4}}\right) + \phi(\max\{t_2, t_3, t_4\}).$$

Example 4.33.

$$F(t_1, ..., t_5) = \theta(t_1) - \theta\left(\max\left\{t_2 + t_3, t_3 + t_4, \tfrac{t_5}{2}\right\}\right) + \phi(\max\{t_2, t_3, t_4\}).$$

By Theorem 3.3 and Example 4.28 we obtain

Theorem 4.34. *Let A, B, S and T be self mappings of a metric space (X, d) such that for all $x, y \in X$*

$$\begin{aligned} \theta(d(Ax, By)) \leq \theta(\max\{&d(Sx, Ty), d(Sx, Ax), \\ &d(Ty, By), \frac{d(Sx, By) + d(Ty, Ax)}{2}\}) - \\ -\phi\left(\theta\left(\begin{array}{c} \max\{d(Sx, Ty), d(Sx, Ax), \\ d(Ty, By), \dfrac{d(Sx, By) + d(Ty, Ax)}{2}\} \end{array}\right)\right)&, \end{aligned}$$

where θ is an altering distance and $\phi \in \Phi$.

If (A,S) *and* T *satisfy* $CLR_{(A,S),T}$ *- property, then*

(i) $\mathcal{C}(A,S) \neq \emptyset$,

(ii) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

By Theorem 4.34 and Lemma 4.2 we obtain

Theorem 4.35. *Let* A,B,S *and* T *be self mappings of a metric space* (X,d) *such that for all* $x,y \in X$

$$\int_0^{d(Ax,By)} h(t)\,dt \leq \int_0^{\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)+d(Ty,Ax)}{2}\}} h(t)\,dt - \\ -\phi\left(\int_0^{\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)+d(Ty,Ax)}{2}\}} h(t)\,dt\right),$$

where $h(t)$ *is as in Theorem 1.10 and* $\phi \in \Phi$.

If (A,S) *and* T *satisfy* $CLR_{(A,S),T}$ *- property, then*

(i) $\mathcal{C}(A,S) \neq \emptyset$,

(ii) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover, if (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Remark 4.36. *1) This theorem is a generalization of Theorem 1.11.*

2) By Examples 4.29 - 4.33 we obtained new particular results.

References

[1] A. Aamri and D. El - Moutawakil, **Some new common fixed point theorems under strict contractive conditions**, Math. Anal. Appl., 270 (2002), 181 - 188.

[2] M. Abbas and B. E. Rhoades, **Common fixed points results for noncommuting mappings without continuity in generalized metric spaces**, Appl. Math. Comput., 215 (2009), 262 - 269.

[3] Ya. I. Alber and S. Guerre - Delabriere, **Principle of weakly contractive maps in Hilbert spaces, New results in operator theory and its applications**, Adv. Appl. Math. (Ed. by Y. Gahbery and Yu. Lybrch), Birkhauser Verlag Basel, 98 (1997), 7 - 22.

[4] J. Ali and M. Imdad, **An implicit function implies several contractive conditions**, Sarajevo J. Math., 17, 4 (2008), 269 - 285.

[5] V. Berinde, **Approximating fixed points of weak φ-contractions using the Picard iteration**, Fixed Point Theory, 4 (2003), 131 - 142.

[6] A. Branciari, **A fixed point theorem for mappings satisfying a general contractive condition of integral type**, Intern. J. Math. Math. Sci., 29 (9) (2002) 531 - 536.

[7] M. Imdad, M. Pant and S. Chauhan, **Fixed point theorems in Manger spaces using $CLR_{(S,T)}$ - property and applications**, J. Nonlinear Anal. Optim. 312, 2 (2012), 225 - 237.

[8] M. Imdad, S. Chauhan and Z. Kadelburg, **Fixed point theorems for mappings with common limit range property satisfying generalized (ψ, φ) - weak contractive conditions**, Math. Sci., 7 (16) (2013) doi:10.1186/2251-7456-7-16.

[9] M. Imdad and S. Chauhan, **Employing common limit range property to prove unified metrical common fixed point theorems**, Intern. J. Anal., vol. 2013, Article ID 763261.

[10] M. Imdad, S. Chauhan, Z. Kadelburg, **Fixed point theorems for nonself mappings in symmetric spaces under φ - weak contractive conditions and an application to functional equation in dynamic programming**, Appl. Math. Comput., 227 (2014), 469 - 479.

[11] G. Jungck, **Compatible mappings and common fixed points**, Intern. J. Math. Math. Sci., 9 (4) (1986), 771 - 779.

[12] M. S. Khan, M. Swaleh and S. Sessa, **Fixed point theorems by altering distance between two points**, Bull. Austral. Math. Sci., 30 (1984), 1 - 9.

[13] Y. Liu, J. Wu, and Z. Li, **Common fixed points of single - valued and multivalued maps**, Intern. J. Math. Math. Sci., 19 (2005), 3045 - 3055.

[14] J. Matkowski, **Fixed point theorems for mappings with a contractive iterate at a point**, Proc. Amer. Math. Soc., 62, 2 (1997), 344 - 348.

[15] R. P. Pant, $R-$ **weak commutativity and common fixed points of noncompatible maps**, Ganita, 99 (1998), 19 - 26.

[16] R. P. Pant, **Common fixed point theorems for four mappings**, Bull. Calcutta Math. Soc., 9 (1998), 281 - 287.

[17] R. P. Pant, $R-$ **weak commutativity and common fixed points**, Soochow J. Math., 25 (1999), 37 - 42.

[18] V. Popa, **Fixed point theorems for implicit contractive mappings**, Stud. Cerc. Ştiinţ., Ser. Mat., Univ. Bacău, 7 (1997), 127 - 134.

[19] V. Popa, **Some fixed point theorems for compatible mappings satisfying an implicit relation**, Demonstr. Math., 32, 1 (1999) 157-163.

[20] V. Popa and M. Mocanu, **A new viewpoint in the study of fixed points for mappings satisfying a contractive condition of integral type**, Bul. Inst. Politeh. Iaşi, Sec. Mat. Mec. Teor. Fiz., 53 (57) (2007), 269 - 286.

[21] V. Popa and M. Mocanu, **Altering distances and common fixed points under implicit relations**, Hacet. J. Math. Stat., 38 (3) (2009), 329 - 337.

[22] B. E. Rhoades, **Some theorems of weakly contractive maps**, Nonlinear Anal., 47 (2001), 2683 - 2693.

[23] B. E. Rhoades, **Two fixed point theorems for mappings satisfying a general contractive condition of integral type**, Intern. J. Math. Math. Sci., 63 (2003), 4007 - 4013.

[24] K. P. Sastri and G. V. R. Babu, **Fixed point theorems in metric spaces by altering distances**, Bull. Calcutta Math. Soc., 90 (1998), 175 - 182.

[25] K. P. Sastri and G. V. R. Babu, **Some fixed point theorems by altering distances between two points**, Indian J. Pure Appl. Math., 30 (1999), 641 - 647.

[26] W. Sintunavarat and P. Kumam, **Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces**, J. Appl. Math. (2011), Article ID 637958, 14 pages

[27] C. Vetro, S. Chauhan, E. Karapinar and W. Shatanawi, **Fixed points of weakly compatible mappings satisfying generalized φ - weak contractions**, Bull. Malays. Math. Sci. Soc., DOI 101.1007/s40840-014-0074-0.

"Vasile Alecsandri" University of Bacău,
157 Calea Mărăşeşti, Bacău, 600115, ROMANIA
E-mail address: vpopa@ub.ro

"Dunărea de Jos" University of Galaţi,
Faculty of Sciences and Environment,
Department of Mathematics and Computer Sciences,
111 Domnească Street, Galaţi, 800201, ROMANIA
E-mail address: Alina.Patriciu@ugal.ro