"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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COMMON FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS SATISFYING A NEW TYPE OF COMMON LIMIT RANGE PROPERTY IN PARTIAL METRIC SPACES

VALERIU POPA AND ALINA-MIHAELA PATRICIU

Abstract. The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings involving altering distances and satisfying a new type of common limit range property in partial metric spaces. In the last part of the paper, as applications, some fixed point results for a sequence of mappings, for mappings satisfying contractive conditions of integral type and for ϕ - contractive mappings is partial metric spaces are obtained.

1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. In [14], Jungck defined S and T to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. The concept has been frequently used to prove the existence theorems in fixed point theory.

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Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $fx = gx$. The set of all coincidence points of f and g is denoted by $\mathcal{C}(f, g)$.

In 1994, Pant [23] introduced the notion of pairwise R - weakly commuting mappings which is equivalent to commutativity in coincidence points [25].

Jungck [15] defined f and g to be weakly compatible if $fx = gx$ implies $fgx = gfx$. Thus, f and g are weakly compatible if and only if f and g are pairwise R - weakly commuting.

The study of common fixed points for noncompatible mappings is also interesting. The work in this regard has been initiated by Pant [24], [25], [26].

Aamri and El - Moutawakil [1] introduced a generalization of noncompatible mappings.

Definition 1.1 ([1]). Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy $(E.A)$ - property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Remark 1.2. *It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or nonexistent. Therefore, noncompatible self mappings of a metric space (X, d) satisfy property $(E.A)$.

It is known [27], [28] that the notions of weakly compatible mappings and mappings satisfying property $(E.A)$ are independent.

Liu et al. [20] defined the notion of common $(E.A)$ - property.

Definition 1.3 ([20]). Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy common property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

There exists a vast literature concerning the study of fixed points for mappings satisfying $(E.A)$ - property.

In 2011, Sintunavarat and Kumam [37] introduced the notion of common limit range property.

Definition 1.4 ([37])**.** A pair (A, S) of self mappings of a metric space (X, d) is said to satisfy common limit range property with respect to S, denoted $CLR_{(S)}$ - property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X)$.

Thus we can infer that a pair (A, S) satisfying $(E.A)$ - property, along with the closedness of the subspace $S(X)$ always have $CLR_{(S)}$ - property with respect to S.

Recently, Imdad et al. [11] introduced the notion of common limit range property to two pairs of self mappings.

Definition 1.5 ([11])**.** Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy common limit range property with respect to S and T, denoted $CLR_{(S,T)}$ - property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

where $t \in S(X) \cap T(X)$.

Some results for pairs of mappings satisfying $CLR_{(S)}$ - and $CLR_{(S,T)}$ - property are obtained in [10], [12], [13], [14] and in other papers.

Now we introduce a new type of limit range property.

Definition 1.6. Let A, S, T be self mappings of a metric space (X, d). The pair (A, S) is said to satisfy common limit range property with respect to T if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X) \cap T(X)$.

Example 1.7. *Let* $\mathbf{R}_+$ *be the metric space with the usual metric,* $Ax = \frac{x^2+1}{2}$, $Sx = \frac{x+1}{2}$, $Tx = x + \frac{1}{4}$. *Then* $S(X) = \left[\frac{1}{2}, \infty\right)$, $T(X) = \left[\frac{1}{4}, \infty\right)$, $S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right)$.

Let $\{x_n\}$ *be a sequence with* $\lim_{n\to\infty} x_n = 0$. *Then,*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} \in S(X) \cap T(X).$$

Remark 1.8. *Let* A, B, S *and* T *satisfying the common limit range property with respect to* S *and* T. *Then* (A, S) *satisfy the common limit range property with respect to* T.

The converse is not true. Let $Bx = x^2 + \frac{1}{4}$ *and* $\{x_n\}$ *a sequence such that* $\lim_{n\to\infty} x_n = 0$, *which implies*

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \frac{1}{4} \neq \frac{1}{2}.$$

Hence, (A,S) *and* (B,T) *don't satisfy* $CLR_{(S,T)}$ *- property.*

Definition 1.9 ([18])**.** An altering distance is a function $\psi : [0,\infty) \to [0,\infty)$ satisfying:
(ψ_1) : ψ is increasing and continuous,
(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving altering distances have been studied in [32], [35], [36] and in other papers.

2. Preliminaries

In 1994, Matthews [22] introduced the concept of partial metric space as a part of the study of denotional semantics of dataflow networks and proved the Banach contraction principle in such spaces. Many authors studied some contractive conditions in complete partial metric spaces. Recently, in [2] - [4], [7], [16], [17] and in other papers, some fixed point theorems under various contractive conditions are proved.

Definition 2.1 ([22])**.** Let X be a nonempty set. A function $p : X \times X \to \mathbf{R}_+$ is said to be a partial metric on X if for all $x, y, z \in X$, the following conditions hold:
(P_1) : $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,
(P_2) : $p(x,x) \leq p(x,y)$,
(P_3) : $p(x,y) = p(y,x)$,
(P_4) : $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
The pair (X,p) is called a partial metric space.

If $p(x,y) = 0$ then (P_1) and (P_2) implies $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as base the family of open p - balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ of a partial metric space (X,p) converges to a point $x \in X$ ($x_n \to x$) with respect to the topology τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the function $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X. Further, a sequence $\{x_n\}$ in (X,p) converges in (X,d_p) to a point $x \in X$ if

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

Lemma 2.2 ([2], [17]). *Let (X,p) be a partial metric space and $\{x_n\}$ a sequence in (X,p) such that $x_n \to z$ as $n \to \infty$, where $p(z,z)=0$. Then,* $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ *for every $y \in X$.*

Definition 2.3 ([22]). Let (X,p) be a partial metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence in (X,p) if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

Remark 2.4. *Let $p : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ be a partial metric defined as $p(x,y) = \max\{x,y\}$. Define $\{x_n\}$ a sequence in X as*

$$x_n = \begin{cases} 0, & n = 2k \\ 1, & n = 2k+1, k \in \mathbf{N}. \end{cases}$$

Then $\{x_n\}$ is a convergent sequence but $\lim_{n,m\to\infty} p(x_n,x_m)$ *does not exists.*

The notion of common limit range property for a pair of mappings in partial metric spaces is defined in [33].

Definition 2.5. A pair (A,S) of self mappings of a partial metric space (X,p) is said to satisfy common limit range property with respect to S, denoted $CLR_{(S)}$ - property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X)$ with $p(t,t) = 0$.

We extend this property for three mappings.

Definition 2.6. Let A, S, T be three self mappings of a partial metric space (X,p). The pair (A,S) and T satisfy the common limit range property with respect to T, denoted $CLR_{(A,S)T}$ - property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for $t \in S(X) \cap T(X)$ and $p(t,t) = 0$.

Example 2.7. *Let $X = [0,4]$ be a particular metric space with*

$$p(x,y) = \begin{cases} |x-y|, & x \in [0,2] \\ \max\{x,y\}, & x \in (2,4] \end{cases}$$

and

$$Ax = \begin{cases} 2-x, & x \in [0,1] \\ \frac{2-x}{3}, & x \in (1,4], \end{cases}$$

$$Sx = \begin{cases} \frac{3-x}{2}, & x \in [0,1] \\ \frac{x}{2}, & x \in (1,4], \end{cases}$$

$$Tx = x.$$

For an increasing sequence $\{x_n\}$ *in* X *such that* $x_n \to 1$, *then*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1.$$

Obviously, $S(X) \cap T(X) = S(X)$, $t \in S(X)$ *and* $p(t,t) = 0$.

3. Implicit relations

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function [29], [30] and other papers. Recently, the method is used in the study of fixed points in metric spaces, quasi - metric spaces, b - metric spaces, ultra metric spaces, Hilbert spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings. Recently, the method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, G - metric spaces and G_p - metric spaces.

With this method, the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

Some fixed point theorems for mappings satisfying implicit relations in partial metric spaces are proved in [5], [8], [9], [38].

Recently, Imdad and Chauhan [12] employed common limit range property to prove unified metrical common fixed point theorems in metric spaces using implicit relations.

Definition 3.1. Let $\mathcal{F}_p$ be the family of all lower semi - continuous functions $F : \mathbf{R}_+^6 \to \mathbf{R}$ satisfying the following conditions:

(F_1) : F is nonincreasing in variables t_3, t_4,
(F_2) : $F(t,0,0,t,t,0) > 0$, $\forall t > 0$,
(F_3) : $F(t,0,t,0,0,t) > 0$, $\forall t > 0$,
(F_4) : $F(t,t,0,t,t,t) > 0$, $\forall t > 0$,
(F_5) : $F(t,t,t,0,t,t) > 0$, $\forall t > 0$.

Example 3.2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, *where* $k \in [0,1)$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, *where* $k\in[0,1)$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$, *where* $k\in[0,1)$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + d + e < 1$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $c \in (0,1)$, $a, b \geq 0$ *and* $a + b < 1$.

Example 3.7. $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $a + b + c < 1$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - -(1-\alpha)(at_5 + bt_6)$, *where* $\alpha \in (0,1)$, $a, b \geq 0$ *and* $a + b < 1$.

Example 3.9. $F(t_1, ..., t_6) = t_1^2 - at_2t_3 - bt_3t_4 - ct_4t_5 - dt_5t_6$, *where* $a, b, c, d \geq 0$ *and* $a + b + c + d < 1$.

The purpose of this paper is to prove a general fixed point theorem involving altering distance and satisfying common limit range property of type $CLR_{(A,S)T}$. In the last part of the paper, as applications, some fixed point results for mappings satisfying contractive conditions of integral type and for φ - contractive mappings in partial metric spaces are obtained.

4. Main results

Theorem 4.1. *Let* (X, p) *be a partial metric space and* A, B, S *and* T *be self mappings of* X *such that for all* $x, y \in X$,

$$\begin{aligned} F(&\psi(p(Ax, By)), \psi(p(Sx, Ty)), \psi(p(Sx, Ax)), \\ &\psi(p(Ty, By)), \psi(p(Sx, By)), \psi(p(Ty, Ax))) \leq 0, \end{aligned} \tag{4.1}$$

where $F \in \mathcal{F}_p$ *and* ψ *is an altering distance.*

If (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property, then*

1) $\mathcal{C}(A, S) \neq \emptyset$,

2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. Since (A, S) and T satisfy $CLR_{(A,S)T}$ - property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

$z \in S(X) \cap T(X)$ and $p(z,z) = 0$.

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$. By (4.1) we obtain

$$F(\psi(p(Ax_n, Bu)), \psi(p(Sx_n, Tu)), \psi(p(Sx_n, Ax_n)), \psi(p(Tu, Bu)), \psi(p(Sx_n, Bu)), \psi(p(Tu, Ax_n))) \leq 0. \tag{4.2}$$

By (P_4), $p(Sx_n, Ax_n) \leq p(Sx_n, z) + p(z, Ax_n)$. Letting n tends to infinity, by Lemma 2.2 we obtain $\lim_{n\to\infty} p(Sx_n, Ax_n) = 0$.

Letting n tends to infinity in (4.2), by Lemma 2.2 we obtain

$$F(\psi(p(z, Bu)), 0, 0, \psi(p(z, Bu)), \psi(p(z, Bu)), 0) \leq 0,$$

a contradiction of (F_2) if $p(z, Bu) > 0$. Hence $p(z, Bu) = 0$ which implies $z = Bu = Tu$. Therefore, $\mathcal{C}(B, T) \neq \emptyset$.

Also, since $z \in S(X)$, there exists $v \in X$ such that $z = Sv$.

By (4.1) we obtain

$$F(\psi(p(Av, Bu)), \psi(p(Sv, Tu)), \psi(p(Sv, Av)), \psi(p(Tu, Bu)), \psi(p(Sv, Bu)), \psi(p(Tu, Av))) \leq 0,$$

$$F(\psi(p(Av, z)), 0, \psi(p(z, Av)), 0, 0, \psi(p(z, Av))) \leq 0,$$

a contradiction of (F_3) if $p(z, Av) > 0$. Hence $p(z, Av) = 0$ which implies $z = Av = Sv$. Hence, $\mathcal{C}(A, S) \neq \emptyset$.

Therefore,

$$z = Av = Sv = Bu = Tu. \tag{4.3}$$

Moreover, if (A, S) and (B, T) are weakly compatible, then $Sz = SAv = ASv = Az$ and $Tz = TBu = BTu = Bz$. By (4.1) we obtain

$$F(\psi(p(Av, Bz)), \psi(p(Sv, Tz)), \psi(p(Sv, Av)), \psi(p(Tz, Bz)), \psi(p(Sv, Bz)), \psi(p(Tz, Av))) \leq 0,$$

$$F(\psi(p(z, Bz)), \psi(p(z, Bz)), 0, \psi(p(Bz, Bz)), \psi(p(z, Bz)), \psi(p(z, Bz))) \leq 0. \tag{4.4}$$

By (P_2), $p(Bz, Bz) \leq p(z, Bz)$. By (F_1) and (4.4) we obtain

$$F(\psi(p(z, Bz)), \psi(p(z, Bz)), 0, \psi(p(z, Bz)), \psi(p(z, Bz)), \psi(p(z, Bz))) \leq 0.$$

If $p(z, Bz) > 0$, we obtain a contradiction of (F_4). Hence $p(z, Bz) = 0$ which implies $z = Bz = Tz$ and z is a common fixed point of B and T.

Similarly, by (4.1) we obtain

$$F(\psi\left(p(Az,Bu)\right),\psi\left(p(Sz,Tu)\right),\psi\left(p(Sz,Az)\right),\\ \psi\left(p(Tu,Bu)\right),\psi\left(p(Sz,Bu)\right),\psi\left(p(Tu,Az)\right))\leq 0,$$

$$F(\psi\left(p(Az,z)\right),\psi\left(p(Az,z)\right),\psi\left(p(Az,Az)\right),\\ 0,\psi\left(p(Az,z)\right),\psi\left(p(z,Az)\right))\leq 0. \tag{4.5}$$

By (P_2), $p\left(Az,Az\right)\leq p\left(Az,z\right)$. Then by (4.5) and (F_1) we obtain

$$F(\psi\left(p(Az,z)\right),\psi\left(p(Az,z)\right),\psi\left(p(Az,z)\right),\\ 0,\psi\left(p(Az,z)\right),\psi\left(p(Az,z)\right))\leq 0,$$

a contradiction of (F_5) if $p\left(Az,z\right)>0$. Hence $p\left(Az,z\right)=0$ which implies $z=Az=Sz$. Hence z is a common fixed point of A,B,S and T.

Suppose that w is another fixed point of A,B,S and T. Then by (4.1) we obtain

$$F(\psi\left(p(Az,Bw)\right),\psi\left(p(Sz,Tw)\right),\psi\left(p(Sz,Az)\right),\\ \psi\left(p(Tw,Bw)\right),\psi\left(p(Sz,Bw)\right),\psi\left(p(Tw,Az)\right))\leq 0,$$

$$F(\psi\left(p(z,w)\right),\psi\left(p(z,w)\right),0,\\ \psi\left(p(w,w)\right),\psi\left(p(z,w)\right),\psi\left(p(z,w)\right))\leq 0. \tag{4.6}$$

By (P_2), $p\left(w,w\right)\leq p\left(z,w\right)$. By (F_1) and (4.6) we obtain

$$F(\psi\left(p(z,w)\right),\psi\left(p(z,w)\right),0,\\ \psi\left(p(z,w)\right),\psi\left(p(z,w)\right),\psi\left(p(z,w)\right))\leq 0,$$

a contradiction of (F_4) if $p\left(z,w\right)\neq 0$. Hence $p\left(z,w\right)=0$ which implies $z=w$ and z is the unique common fixed point of A,B,S and T. □

If $\psi\left(t\right)=t$ we obtain

Theorem 4.2. *Let (X,p) be a partial metric space and A,B,S and T be self mappings of X such that for all $x,y\in X$*

$$F(p(Ax,By),p(Sx,Ty),p(Sx,Ax),\\ p(Ty,By),p(Sx,By),p(Ty,Ax))\leq 0, \tag{4.7}$$

where $F\in\mathcal{F}_p$.

If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}\left(A,S\right)\neq\emptyset$,

2) $\mathcal{C}\left(B,T\right)\neq\emptyset$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Example 4.3. *Let* $X = [0, 1]$ *be a partial metric space with* $p(x, y) = \max\{x, y\}$. *We define the following mappings*

$Ax = 0$, $Sx = \frac{x}{x+1}$, $Bx = \frac{x}{3}$, $Tx = x$. *Then* $S(X) = \left[0, \frac{1}{2}\right]$, $T(X) = [0, 1]$, $T(X) \cap S(X) = \left[0, \frac{1}{2}\right]$. *Let* $\{x_n\}$ *be a sequence such that* $x_n \to 0$. *Then* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0 \in S(X) \cap T(X) \to 2$. *Since* $Ax = Sx$ *implies* $x = 0$, *then* $AS0 = SA0 = 0$ *and* (A, S) *is weakly compatible. Similarly,* (B, T) *is weakly compatible. On the other hand,*

$$p(Ax, By) = p\left(0, \frac{y}{3}\right) = \frac{y}{3},$$
$$p(Ty, By) = y.$$

Hence

$$p(Ax, By) \leq kp(Ty, By)$$

if $k \in \left[\frac{1}{3}, \frac{1}{2}\right)$, *which implies*

$$p(Ax, By) \leq k \max\{p(Sx, Ty), p(Sx, Ax), p(Ty, By), p(Sx, By), p(Ax, Ty)\},$$

where $k \in \left[\frac{7}{3}, 1\right)$.

By Theorem 4.2, A, B, S and T have a unique common fixed point $x = 0$.

For a function $f : (X, p) \to (X, p)$ we denote

$$pFix(f) = \{x \in X : x = fx,\ p(x, x) = 0\}.$$

Theorem 4.4. *Let* A, B, S *and* T *be self mappings of a partial metric space* (X, p). *If the inequality (4.1) holds for all* $x, y \in X$ *and* $F \in \mathcal{F}_p$, *then*

$$[pFix(S) \cap pFix(T)] \cap pFix(A) = [pFix(S) \cap pFix(T)] \cap pFix(B).$$

Proof. Let $x \in [pFix(S) \cap pFix(T)] \cap pFix(A)$. Then $x = Sx = Tx = Ax$ and $p(x, x) = 0$. By (4.1) we have

$$F(\psi(p(Ax, Bx)), \psi(p(Sx, Tx)), \psi(p(Sx, Ax)), \psi(p(Tx, Bx)), \psi(p(Sx, Bx)), \psi(p(Tx, Ax))) \leq 0,$$

$$F(\psi(p(x, Bx)), 0, 0, \psi(p(x, Bx)), \psi(p(x, Bx)), 0) \leq 0,$$

a contradiction of (F_2) if $p(x, Bx) > 0$. Hence, $x = Bx$ and $p(x, x) = 0$. Hence,

$$[pFix(S) \cap pFix(T)] \cap pFix(A) \subset [pFix(S) \cap pFix(T)] \cap pFix(B).$$

Similarly, if $x \in [pFix\,(S) \cap pFix\,(T)] \cap pFix\,(B)$, then by (4.1) and (F_3) we obtain $x = Ax$ and $p\,(x,x) = 0$. Hence,

$$[pFix\,(S) \cap pFix\,(T)] \cap pFix\,(B) \subset [pFix\,(S) \cap pFix\,(T)] \cap pFix\,(A)\,.$$

□

Theorems 4.1 and 4.4 imply the following one.

Theorem 4.5. *Let S, T and $\{A_i\}_{i\in\mathbf{N}^*}$ be self mappings of a partial metric space (X,p) satisfying the inequality*

$$F(\psi\,(p(A_ix, A_{i+1}y))\,, \psi\,(p(Sx,Ty))\,, \psi\,(p(Sx,A_ix))\,,\\ \psi\,(p(Ty,A_{i+1}y))\,, \psi\,(p(Sx,A_{i+1}y))\,, \psi\,(p(Ty,A_ix))) \leq 0,$$

for all $x, y \in X$, $F \in \mathcal{F}_p$ and ψ is an altering distance, $i \in \mathbf{N}^*$. If (A_1,S) and T satisfy $CLR_{(A,S)T}$ - property and $(A_1,S), (A_2,T)$ are weakly compatible, then S, T and $\{A_i\}_{i\in\mathbf{N}^*}$ have a unique fixed point z with $p\,(z,z) = 0$.

If $\psi\,(t) = t$, by Theorem 4.5 we obtain

Theorem 4.6. *Let S, T and $\{A_i\}_{i\in\mathbf{N}^*}$ be self mappings of a partial metric space (X,p) satisfying the inequality*

$$F(p(A_ix, A_{i+1}y), p(Sx,Ty), p(Sx,A_ix),\\ p(Ty,A_{i+1}y), p(Sx,A_{i+1}y), p(Ty,A_ix)) \leq 0,$$

for all $x, y \in X$ and $F \in \mathcal{F}_p$.

If (A_1,S) and T satisfy $CLR_{(A,S)T}$ - property and (A_1,S) and (A_2,T) are weakly compatible, then S, T and $\{A_i\}_{i\in\mathbf{N}^*}$ have a unique fixed point z with $p\,(z,z) = 0$.

5. Applications

5.1. **Fixed points for mappings satisfying contractive conditions of integral type.** In [6], Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 5.1 ([6]). *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ such that for all $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c\int_0^{d(x,y)} h(t)dt$$

where $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,\infty)$, such that $\int_0^{\varepsilon} h(t)dt > 0$, for all $\varepsilon > 0$.

Then, f has a unique fixed point z such that for all $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Theorem 5.1 has been extended to a pair of compatible mappings in [19]. Some fixed point results for mappings satisfying contractive condition of integral type are obtained in [31], [32], [34] and in other papers.

Lemma 5.2 ([32])**.** *Let $h : [0,\infty) \to [0,\infty)$ as in Theorem 5.1. Then $\psi(t) = \int_0^t h(x)dx$ is an altering distance.*

Theorem 5.3. *Let (X, p) be a partial metric space and A, B, S and T be self mappings of X such that*

$$(5.1)\qquad \begin{array}{c} F(\int_0^{p(Ax,By)} h(t)dt, \int_0^{p(Sx,Ty)} h(t)dt, \int_0^{p(Sx,Ax)} h(t)dt, \\ \int_0^{p(Ty,By)} h(t)dt, \int_0^{p(Sx,By)} h(t)dt, \int_0^{p(Ty,Ax)} h(t)dt) \leq 0, \end{array}$$

for all $x, y \in X$, where $F \in \mathcal{F}_p$ and $h(t)$ is as in Theorem 5.1.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A, S) \neq \emptyset$,

2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. As in Lemma 5.2,

$$\begin{aligned} \int_0^{p(Ax,By)} h(t)dt &= \psi(p(Ax, By)), \int_0^{p(Sx,Ty)} h(t)dt = \psi(p(Sx, Ty)), \\ \int_0^{p(Sx,Ax)} h(t)dt &= \psi(p(Sx, Ax)), \int_0^{p(Ty,By)} h(t)dt = \psi(p(Ty, By)), \\ \int_0^{p(Sx,By)} h(t)dt &= \psi(p(Sx, By)), \int_0^{p(Ty,Ax)} h(t)dt = \psi(p(Ty, Ax)). \end{aligned}$$

By (5.1) we obtain

$$\begin{array}{c} F(\psi(p(Ax, By)), \psi(p(Sx, Ty)), \psi(p(Sx, Ax)), \\ \psi(p(Ty, By)), \psi(p(Sx, By)), \psi(p(Ty, Ax))) \leq 0, \end{array}$$

which is inequality (4.1). Hence the conditions of Theorem 4.1 are satisfied and Theorem 5.1 follows by Theorem 4.1. □

By Theorem 5.3 and Example 3.2 we obtain

Theorem 5.4. *Let (X,p) be a partial metric space and A,B,S and T be self mappings of X such that*

$$\begin{aligned}\int_0^{p(Ax,By)} h(t)dt \le k\max\{&\int_0^{p(Sx,Ty)} h(t)dt, \int_0^{p(Sx,Ax)} h(t)dt,\\ &\int_0^{p(Ty,By)} h(t)dt, \int_0^{p(Sx,By)} h(t)dt, \int_0^{p(Ty,Ax)} h(t)dt\}\end{aligned} \tag{5.2}$$

for all $x,y\in X$, $k\in[0,1)$ and $h(t)$ is as in Theorem 5.1.

If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A,S)\neq\emptyset$,

2) $\mathcal{C}(B,T)\neq\emptyset$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Remark 5.5. *By Theorem 5.3 and Examples 3.3 - 3.9 we obtain new particular results.*

Similarly, by Theorems 4.5 and 5.3 we obtain

Theorem 5.6. *Let S,T and $\{A_i\}_{i\in\mathbf{N}^*}$ be self mappings of a partial metric space (X,p) satisfying*

$$\begin{aligned}F(&\int_0^{p(A_ix,A_{i+1}y)} h(t)dt, \int_0^{p(Sx,Ty)} h(t)dt, \int_0^{p(Sx,A_ix)} h(t)dt,\\ &\int_0^{p(Ty,A_{i+1}y)} h(t)dt, \int_0^{p(Sx,A_{i+1}y)} h(t)dt, \int_0^{p(Ty,A_ix)} h(t)dt)\le 0,\end{aligned}$$

for all $x,y\in X$, $F\in\mathcal{F}_p$ and $h(t)$ is as in Theorem 5.1, $i\in\mathbf{N}^*$.

If (A_1,S) and T satisfy $CLR_{(A,S)T}$ - property and (A_1,S) and (A_2,T) are weakly compatible, then S,T and $\{A_i\}_{i\in\mathbf{N}^*}$ have a unique fixed point z with $p(z,z)=0$.

Remark 5.7. *By Theorem 5.6 and Examples 3.2 - 3.9 we obtain new particular results.*

5.2. **Fixed points for mappings satisfying a φ - contractive condition.** As in [21], let Φ be the set of all real nondecreasing lower semi - continuous functions $\varphi:[0,\infty)\to[0,\infty)$ with $\lim_{n\to\infty}\varphi^n(t)=0$ for all $t\in[0,\infty)$. If $\varphi\in\Phi$, then

1) $\varphi(t)<t$ for all $t\in(0,\infty)$,

2) $\varphi(0)=0$.

The following functions $F(t_1,...,t_6):\mathbf{R}_+^6\to\mathbf{R}$ satisfy conditions (F_1), (F_2), (F_3), (F_4), (F_5).

Example 5.8. $F(t_1,...,t_6)=t_1-\varphi(\max\{t_2,t_3,t_4,t_5,t_6\})$.

Example 5.9. $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)$.

Example 5.10. $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)$.

Example 5.11. $F(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, \sqrt{t_3t_4}, \sqrt{t_5t_6}, \sqrt{t_4t_6}\right\}\right)$.

Example 5.12. $F(t_1, ..., t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + d + e \leq 1$.

Example 5.13.

$$F(t_1, ..., t_6) = t_1 - \varphi(at_2 + b\max\{t_3, t_4\} + c\max\{t_5, t_6\}),$$

where $a, b, c \geq 0$ *and* $a + b + c \leq 1$.

By Theorem 4.1 and Example 5.8 we obtain

Theorem 5.14. *Let* (X, p) *be a partial metric space and* A, B, S, T *be self mappings of* X *such that*

$$\begin{aligned}\psi(p(Ax, By)) \leq \varphi(\max\{&\psi(p(Sx, Ty)), \psi(p(Sx, Ax)),\\ &\psi(p(Ty, By)), \psi(p(Sx, By)), \psi(p(Ty, Ax))\}),\end{aligned}$$

for all $x, y \in X$, $\varphi \in \Phi$ *and* ψ *is an altering distance.*

If (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property, then*

1) $\mathcal{C}(A, S) \neq \emptyset$,

2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

By Example 5.8 and Theorem 5.14 we obtain

Theorem 5.15. *Let* A, B, S *and* T *be self mappings of partial metric space* (X, p) *such that*

$$\begin{aligned}\int_0^{p(Ax,By)} h(t)dt \leq \varphi(\max\{&\int_0^{p(Sx,Ty)} h(t)dt, \int_0^{p(Sx,Ax)} h(t)dt,\\ &\int_0^{p(Ty,By)} h(t)dt, \int_0^{p(Sx,By)} h(t)dt, \int_0^{p(Ty,Ax)} h(t)dt\}),\end{aligned}$$

for all $x, y \in X$, $\varphi \in \Phi$ *and* $h(t)$ *is as in Theorem 5.1.*

If (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property, then*

1) $\mathcal{C}(A, S) \neq \emptyset$,

2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Remark 5.16. *Similarly, by Examples 5.9 - 5.13 and Theorem 5.14 we obtain new particular results.*

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“Vasile Alecsandri” University of Bacău,
157 Calea Mărăşeşti, Bacău, 600115, ROMANIA
E-mail address: vpopa@ub.ro

“Dunărea de Jos” University of Galaţi,
Faculty of Sciences and Environment,
Department of Mathematics and Computer Sciences,
111 Domnească Street, Galaţi, 800201, ROMANIA
E-mail address: Alina.Patriciu@ugal.ro