"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 27 (2017), No. 1 , 101 - 112

SELECTED OPEN PROBLEMS IN VECTORIAL OPTIMIZATION

VASILE POSTOLICĂ

Abstract. We present some significant open problems in Euclidean spaces $R^n, n \in N^*$ which can be extended, mutatis mutandis, to R^I for every non –empty set I .

1. INTRODUCTION

Multi – objective programming is an important component of the *general optimization* in *the abstract spaces* and a part of it is represented by *the Pareto optimality* as an illustrative finite dimensional example of *the efficiency*. Not even for the market economies there exists no an universal mathematical model. Pareto efficiency or Pareto optimality *(Pareto, Vilfredo, 1909)* is *a central theory* in *economics* with *broad applications in game theory, engineering and the social sciences*. It represents *the actual finite dimensional part of the Multiobjective Programming* in *Vector Optimization*. Thus, whenever a feasible deviation from a genuine solution S of an arbitrary multiobjective programme generates the improvement of at least one of the objectives while some other objectives degrade, any such a solution S as this is called *efficient or nondominated*. *A system in economics and in politics is called Pareto optimal* or *efficient* in the language of *the general efficiency* whenever *"no individual can be made better off without another being made worse off"* that is, *a social state is economically efficient, or Pareto optimal, provided that "no person in society can become better off without anyone else becoming worse off"*.

Keywords and phrases: *multi-objective optimization, Pareto optimality strong and vectorial optimal solution, multifunction, efficiency, Isac's cone.*
Mathematics Subject Classification (2000): 90C29, 90C46 .

This characteristic of Pareto type efficiency has been pointed out in *(Arrow, K. J., 1963, Arrow, K. J., Hahn, F. M., 1971,Aubin, J. P., 1993, Debreu Gerard, 1959, Hansson, S. O., 2004, Pareto, Vilfredo, 1909, Stewart, D. J., 2004 and the others).* In terms of the alternative allocations this means that given a set of alternative allocations and a set of individuals, any movement from one alternative allocation to another that can make at least one individual better off, without making any other individual worse off is called a Pareto improvement or a Pareto optimization. An allocation of resources is named *Pareto optimal* whenever *no further Pareto improvements can be made*. If the allocation is *strictly preferred* by one person and no *other allocation would be as good for everyone*, then it is called *strongly Pareto optimal*. A *weakly Pareto optimal allocation* is one where *any feasible reallocation would be strictly preferred by all agents* (*Zimmermann, H., 2000).* Consequently, *Pareto type efficiency* is *an important approximate criterion* for *evaluating the economic systems* and *the political policies*, with *minimum assumptions on the interpersonal comparability*. We said *"approximate criterion"* because it asks *"the ideal"* which may not reflect, for example, *the workings of real economics*, thanks to the folowing *restrictive assumptions* necessary for *the existence* of Pareto efficient outcomes*: the markets exist for all possible goods*, being *perfectly competitive* and *the transaction costs are negligible*. In *the political policies, not every Pareto efficient outcome* is regarded as *"desirable"* (*see, for instance, the strategies based on the unilateral benefits*). For these reasons, Pareto optimality was accepted with *some or much uncertainty and controversy*, but, by the Arrow's renowned impossibility theorem given in 1951 according to which *"no social preference ordering based on individual orderings only could satisfy a small set of very reasonable conditions"- the Pareto criterion* being one of them, it remains *"plausible and uncontroversial" (Rosencrantz, Holger, 2005)*. Usually, *the concept of efficiency* replaces the notion of *optimality* in *multiple criteria optimization* because *whenever the solutions of a multiple – objectives program exist in Pareto'sense, they cannot be improved following the ordering induced by the cone*. Seemingly, the concept of *efficiency* is equivalent to *the optimality* but, in *reality, the optimality represents a particular case of the efficiency*, that is, *"the best approximation" of all the efficient points*. The next section is dedicated to the main kind of optimization problems in R^n . We also notice that *the finite dimensionality* of *the background space is essential* in contrast with *the general ordered vector spaces*, since *it controls* the results at least thanks to *the approximate numerical methods* for *the efficient points*. The bibliography of this research work represents a synthesis of all the references given in [1] - [6], being quoted from them.

2. MULTI – OBJECTIVE OPTIMIZATION PROGRAMS

Let E be an arbitrary non – empty set and let
$f = (f_1, f_2, ..., f_n) : E \to R^n (n \in N^*, n \geq 2)$ a vector - valued function. The main *general strong* (P_1), (P_2) and *vectorial* (P_3), (P_4) respectively, *optimization problems* that can be formulated in this context are the following:

$$(P_1) \ \ SMIN\left\{f(x) : x \in E\right\}$$

where $x_0 \in E$ is a *strong minimum solution* if $f(x_0) \leq f(x), \forall x \in E$;

$$(P_2) \ \ SMAX\left\{f(x) : x \in E\right\}$$

with $y_0 \in E$ *strong maximum solution* if $f(y_0) \geq f(x), \forall x \in E$;

$$(P_3) \ \ VMIN\left\{f(x) : x \in E\right\};$$

$$(P_4) \ \ VMAX\left\{f(x) : x \in E\right\}.$$

In order to define *the vectorial solutions,* we need to know the possibility of *"comparison"* in R^n. Let $R_+^n = \left\{(x_i) \in R^n : x_i \geq 0, \forall i = \overline{1,n}\right\}$ be the usual *ordering cone*. If $a = (a_i), b = (b_i) \in R^n$, then it is possible to have:

1) $a \leq b \Leftrightarrow a_i \leq b_i, \forall i = \overline{1,n} \Leftrightarrow b - a \in R_+^n$;

2) $a < b \Leftrightarrow a_i \leq b_i, \forall i = \overline{1,n}$ *with at least one of these inequalities being strict, that is*

$$b - a \in R_+^n \setminus \left\{\theta\right\}, \text{ where } \theta = (0, 0, ..., 0) \in R_+^n;$$

3)

$$a \ll b \Leftrightarrow a_i < b_i, \forall i = \overline{1,n} \Leftrightarrow b - a \in \operatorname{int}(R_+^n) =$$
$$= \left\{x_0 \in R_+^n : \exists B(x_0, \varepsilon > 0) = \left\{x \in R^n : \|x - x_0\| < \varepsilon\right\} \subset R_+^n\right\},$$

where $\|.\| : R^n \to R_+$ is the usual norm in R^n given by $\|u\| = \sqrt{\sum_{i=1}^{n} |u_i|^2}$,
$\forall u = (u_i) \in R^n$.

Thus, we are led to say that x_0 is *a vectorial minimum solution* of (P_3) if *there exists no* $x \in E$ such that $f(x) < f(x_0)$, which is equivalent with each of the next properties:

$$(i) \ \ f(E) \cap [f(x_0) - R_+^n] = \left\{f(x_0)\right\};$$

$$(ii)\ \ f(E)\bigcap[f(x_0)-R_+^n\setminus\{\theta\}]=\varnothing;$$
$$(iii)\ \ R_+^n\bigcap[f(x_0)-f(E)]=\{\theta\};$$
$$(iv)\ \ (R_+^n\setminus\{\theta\}]\bigcap[f(x_0)-f(E)]=\varnothing;$$
$$(v)\ \ [f(E)+R_+^n]\cap[f(x_0)-R_+^n\setminus\{\theta\}]=\varnothing.$$

These relations show that $f(x_0)$ is *a fixed point* for each of the following *multifunctions*:

$$F_1: f(E)\to f(E), F_1(t)=\left\{\alpha\in f(E): f(E)\bigcap\left(\alpha-R_+^n\right)\subseteq t+R_+^n\right\},$$
$$F_2: f(E)\to f(E), F_2(t)=\left\{\alpha\in f(E): f(E)\bigcap\left(t-R_+^n\right)\subseteq \alpha+R_+^n\right\},$$
$$F_3: f(E)\to f(E), F_3(t)=\left\{\alpha\in f(E):\left(f(E)+R_+^n\right)\bigcap\left(\alpha-R_+^n\right)\subseteq t+R_+^n\right\},$$
$$F_4: f(E)\to f(E), F_4(t)=\left\{\alpha\in f(E):\left(f(E)+R_+^n\right)\bigcap\left(t-R_+^n\right)\subseteq \alpha+R_+^n\right\},$$

that is, $f(x_0)\in F_i[f(x_0)]$ for every $i=\overline{1,4}$. By replacing R_+^n with $(-R_+^n)$, one obtains the concept of *vectorial maximum solution* for (P_4). Clearly, any *vector optimization program* (which has its origin in the usual ordered Euclidean spaces programs thanks to the Pareto type optimality of the vector - valued real functions) includes the corresponding *strong optimization program*. Moreover, it is possible to replace the usual convex cone R_+^n by any other convenient convex cone in R^n. In this way, we are sent to examine for a set $\varnothing\neq A\subseteq R^n$ the following subsets

α) $SMIN(A)=\left\{a_0\in A: a-a_0\in R_+^n, \forall a\in A\right\}$ *(the set of all strong minimum points);*

β) $SMAX(A)=\left\{\overline{a_0}\in A: \overline{a_0}-a\in R_+^n, \forall a\in A\right\}$ *(the set of all strong maximum points);*

γ) $VMIN(A)=\left\{b_0\in A: A\cap(b_0-R_+^n)=\{b_0\}\right\}$ *(the set of all vectorial minimum points);*

δ) $VMAX(A)=\left\{\overline{b_0}\in A: A\cap(\overline{b_0}+R_+^n)=\left\{\overline{b_0}\right\}\right\}$*(the set of all vectorial maximum points).*

All *the strong optimal points* for A are contained in the set $S(A)=SMIN(A)\cup SMAX(A)$ while the set of all *vectorial optimal(efficient)points* is defined by $V(A)=VMIN(A)\cup VMAX(A)=eff(A,R_+^n)$, the last equality being in accordance with (*Postolică, V., 2017)*. Moreover, $a_0 \in eff\left(A, R_+^n\right)$ if and only if it is *a critical (ideal or balance) point* for *the generalized dynamical systems* $\Gamma_{\mp} : A \to 2^A$ defined by $\Gamma_{\mp}(a) = A \cap \left(a \mp R_+^n\right), \forall a \in A$. In this way , $eff\left(A, R_+^n\right)$ describes *the balance extremum moments* for $\Gamma_{\mp}$, which in *the market context*, expresses *the competitive equilibrium/non - equilibrium* consisting of *the general relation price/consumption*. The first main connection between the above mentioned sets is the following, established in the general conditions offered by any real or complex vector space ordered by an arbitrary convex cone.

Theorem 1. (*Postolică, V., 2017)***.** *If* $S(A) \neq \varnothing$, *then* $V(A) = S(A)$.

In fact,

$$SMIN(A) \neq \varnothing (SMAX(A) \neq \varnothing) \Rightarrow VMIN(A) = SMIN(A)(VMAX(A) = SMAX(A))$$

Remark 1. This result is also important for for *the numerical methods* since it says that, if there exist*"strong optimal points"*, then these elements are *the only "vectorial optimal points"*. The next examples are significant.

Remark 2. To establish the optimal points for a non – empty set in any ordered (possible topological) vector space the best direction of study is to establish the main connections between the mathematical qualities of the set for which one looks for the optimums and the properties of the ordering cone. Thus, the largest class ζ of the convex cones ensuring the existence for the efficient points in all non - empty compacts subsets of every Hausdorff separated topological vector space was defined by *Sterna – Karwat, A., 1986* as follows: if *V* is an arbitrary Hausdorff topological vector space, a convex cone *C* belongs to ζ iff for every closed vector subspace *L* of *V*, $C \cap L$ is a vector subspace whenever its closure $\overline{C \cap L}$ is a vector subspace. Consequently, $eff\left(A, K\right)$ is non – empty with respect to each compact set A and any convex cone satisfying the above characterization, in every Hausdorff topological vector space. In order to maximum relax the compactness hypothesis, for example, by the completeness, it was necessary to improve the quality of the ordering cone. Professor George Isac from College of St. Jean, Québec, Canada introduced such as this concept of convex cone in 1981, published it in 1983 like *nuclear or supernormal cone,* called by us and, officially recognized, as *"Isac's Cone"* in 2009, after the acceptance of this last agreed denomination by professor Isac. Thus, let X be a real or complex linear space and let $P = \{p_\alpha : \alpha \in A\}$ be a family of seminorms defined on X . For every $x \in X, \varepsilon > 0$ and $n \in N^*$ let

$V(x; p_1, p_2,..., p_n; \varepsilon) = \{y \in X : p_\alpha(y-x) < \varepsilon, \forall \alpha = \overline{1,n}\}$ be the family $\triangle_0(x) = \left\{V(x; p_1, p_2,..., p_n; \varepsilon) : n \in N^*, p_\alpha \in P, \alpha = \overline{1,n}, \varepsilon > 0\}\right\}$, with the following properties :

(V_1) $x \in V, \forall V \in \triangle_0(x)$;

(V_2) $\forall V_1, V_2 \in \triangle_0(x), \exists V_3 \in \triangle_0(x) : V_3 \subseteq V_1 \cap V_2$;

(V_3) $\forall V \in \triangle_0(x), \exists U \in \triangle_0(x), U \subseteq V$ such that $\forall y \in U, \exists W \in \triangle_0(y)$ *with* $W \subseteq V$.

Therefore, $\triangle_0$(x) is *a base of neighborhoods* for x and taking $\triangle(x) = \{V \subseteq X : \exists U \in \triangle_0(x)$ with $U \subseteq V\}$, the set $\tau = \{D \subseteq X : D \in \triangle(x), \forall x \in D\} \cup \{\varnothing\}$ is the locally convex topology generated by the family P . Obviously, the usual operations which induce the structure of linear space on X are continuous with respect to this topology. The corresponding topological space (X, τ) is a Hausdorff locally convex space iff the family P is sufficient, that is, $\forall\ x_0 \in X \setminus \{\theta\}, \exists p_\alpha \in P$ with $p_\alpha(x_0) \neq 0$. In this context, a convex cone $K \subset X$ is an Isac's cone iff

$$\forall p_\alpha \in P, \exists f_\alpha \in X^* : p_\alpha(x) \leq f_\alpha(x), \forall x \in K .$$

Clearly, any pointed, convex cone, in every Euclidean space R^k ($k \in N^*$) is an Isac's cone. Many examples together with pertinent comments of such as these important convex cones were given in *(Isac, G., Postolică, V., 1993)* and in (*Postolică, V., 2017).*

Open Problem 1. *We introduce Isac's sets in the following area: let M be a non-empty subset of X and let $cone(M)$ be the convex cone generated by M . Obviously, the quality of Isac's cone for $cone(M)$ is equivalent with the fact that each seminorm is majorized by at least a linear and continuous functional on M . Therefore, it is natural to call it "Isac set". An open problem is to discover the properties of these sets.*

Remark 3. In our opinion, following the context of the ordered linear spaces, any optimization programme can be generally formulated as follows: *let E be a non – empty set, let X be a vector space ordered by a convex cone K and let $f : E \to (X, K)$ be an objective function. The problem consists to look for the optimums of the set E following the properties of its image $f(E) = \{f(x) : x \in E\}$ in X by K and conversely.*

Example 1. For each real number c let us consider the sets
$A_{0c} = \left\{ (x_i) \in R^n : \sum_{i=1}^{n} x_i = c \right\}$, $A_{1c} = \left\{ (x_i) \in R^n : \sum_{i=1}^{n} x_i \geq c \right\}$,
$A_{2c} = \left\{ (x_i) \in R^n : \sum_{i=1}^{n} x_i \leq c \right\}$ then it is clear that $S(A_{0c}) = S(A_{1c}) = S(A_{2c}) = \varnothing$,
$V(A_{0c}) = VMIN(A_{0c}) = VMAX(A_{0c}) = A_{0c}$, $VMIN(A_{1c}) = VMAX(A_{2c}) = A_{0c}$ so, by extrapolation, $eff(A_{1c}) = A_{0c} \cup \{(+\infty, +\infty, ..., +\infty)\}$ and
$eff\left(A_{2c}\right) = A_{0c} \cup \{(-\infty, -\infty, ..., -\infty)\}$.

Example 2. *(Isac, G., Postolică, V., 1993*. Let *(X, $\mathcal{P}$ ={$p_\alpha : \alpha \in I$}) be a H-locally convex space* with *each seminorm p_α being induced by a scalar semiproduct $(.,.)_\alpha$ ($\alpha \in I$)* and *M a closed linear subspace* of *X for which there exist a H-locally convex space (Y,$\mathcal{Q}$ ={ $q_\alpha : \alpha \in I$})* with *each seminorm $q_\alpha \in \mathcal{Q}$ generated by a scalar semiproduct $<.,.>_\alpha$ ($\alpha \in I$)* and *a linear possible continuous operator $U:X \to Y$ such that*

$$M=\{x \in X: (x,y)_\alpha = <Ux,Uy>_\alpha, \forall \ \alpha \in I\}.$$

The space of all spline functions with respect to U was defined by Postolică, V. (1981) as the U-orthogonal of M, that is,

$$M^\perp=\{x \in X: \ <Ux,U\zeta>_\alpha=0, \forall \ \zeta \in M, \alpha \in I\}.$$

Clearly, $M^\perp$ *is the orthogonal of M in the H - locally convex sense.* Let $x_0 \in X$ and let *G be a non-empty subset of X*. Following the previous notations, we have

Definition 1. (*Isac, G., Postolică, V.,1993) $g_0 \in$G is said to be a best simultaneous approximation for* x_0 *by the elements of* G *with respect the family* $\mathcal{P}$ *(abbreviated g_0 is a* $\mathcal{P}$ *- b.s.a. of x_0 if*

$$(p_\alpha (x_0 - g_0)) \in \text{SMIN}(\{(p_\alpha(x_0 - g)): g \in G, p_\alpha \in \mathcal{P}\}) .$$

Whenever each element $x \in X$ *possesses at least one* $\mathcal{P}$*- b.s.a. in* G *the set* G *is called* $\mathcal{P}$ *- simultaneous proximinal.*

Definition 2. *(Isac, G., Postolică, V., 1993) $g_0 \in$G is said to be a best vectorial approximation of* x_0 *by* G *with respect to* $\mathcal{P}$ *(abbreviated g_0 is a* $\mathcal{P}$ *- b.v.a. of x_0) if*

$$(p_\alpha (x_0 - g_0)) \in \text{VMIN}(\{(p_\alpha(x_0 - g)): g \in G, p_\alpha \in \mathcal{P}\}).$$

When, in addition, each element $x \in X$ *possesses at least one* $\mathcal{P}$ *- b.v.a. in* G, *the set* G *is called* $\mathcal{P}$ *- vectorial proximinal.*

Let us consider *the direct sum* $X'=M\oplus M^{\perp}$ and *for every* $x\in X'$*, we denote its projection onto* $M^{\perp}$ *by* s_x. Then, taking into account *the Theorem 4* obtained by *Postolică, V.* in *1981*, it follows that *this spline is a best simultaneous U-approximation of x with respect to* $M^{\perp}$ since *it satisfies all the next conditions* : $p_\alpha(x - s_x) \leq p_\alpha(x - y)\ \forall\ y\in M^{\perp}, p_\alpha\in \mathcal{P}$.
Moreover, *following the definition of the approximate efficiency, the results given in Chapter 3* of *(Isac, G., Postolică, V., 1993)* and *the conclusions obtained* by *(Postolică, V., 1984, 2017)*, we have

Theorem 2.

(i) for every $x\in X'$ *the only elements of best simultaneous and vectorial approximation with respect to any family of seminorms which generates the* H-*locally convex topology on* X *by the linear subspace of splines are the spline functions* s_x. *Moreover, if* M *and* $M^{\perp}$ *supply an orthogonal decomposiion for* X, *that is* $X=M\oplus M^{\perp}$, *then* $M^{\perp}$ *is simultaneous and vectorial proximinal;*

(ii)For each $s\in M^{\perp}$, *every* $\sigma\in M^{\perp}$ *is the only solution of following optimization problem:* VMIN($\{(q_\alpha(U(\eta\text{-}s))): \eta\in X'$ *and* $\eta-\sigma\in M\}$);

(iii) for every $x\in X'$ *its spline function* s_x *is the only solution for the next vectorial optimization problems:*

$$\mathrm{VMIN}(\{(q_\alpha(U(\eta\text{-}x))): \eta\in M^{\perp}\}),\ \mathrm{VMIN}(\{(p_\alpha(x\text{-}y)): y\in M^{\perp}\}),$$
$$\mathrm{VMIN}(\{(q_\alpha(Uy)): y\text{-}x\in M\}).$$

Finally, let us consider *two numerical examples* in which, following *Postolică,V.,(1981), Isac, G., Postolică, V.,(1993) and Postolică, V., (2017),* we specify *the expressions of splines* and *M and* $M^{\perp}$ *realize orthogonal decompositions*.

Example 3. *Let* $X=H^m(R)=\{f\in C^{m-1}(R): f^{(m-1)}$ *is locally absolutely continuous and* $f^{(m)}\in L^2_{loc}(R)\}$, $m\in N^*$ *endowed with the H-locally convex topology generated by the scalar semiproducts*

$$(x,y)_k=\sum_{h=0}^{m-1}[x^{(h)}(k)\,y^{(h)}(k)+x^{(h)}(-k)\,y^{(h)}(-k)]+\int_{-k}^{k}x^{(m)}(t)\,y^{(m)}(t)dt,\ k=0,1,2,\ \ldots \text{ and}$$

$Y=L^2_{loc}(R)$ *with the H-locally convex topology induced by the scalar semiproducts* $\langle x,y\rangle_k=\int_{-k}^{k}x(t)\,y(t)dt$, $k=0,1,2,\ \ldots$ *If* $U:X\to Y$ *is the derivation operator of order m, then*

$$M=\{x\in H^m(R): x^{(h)}(\nu)=0,\ \forall\ h=\overline{0,m-1},\ \nu\in Z\}$$

and $M^{\perp}=\{s\in H^m(R): \int_{-k}^{k} s^{(m)}(t)\, x^{(m)}(t)dt,\ \forall x\in M,\ k=0,1,2,\ \ldots\}$

We proved in (Postolică, V., 1981) that

$M^{\perp}=\{s\in H^m(R): s_{/(\nu,\ \nu+1)}$ *is a polynomial function of degree 2m-1 at most}*

and if $y=(y_{\nu}),\ y'=(y'_{\nu}),\ y''=(y''_{\nu}),\ y^{(m-1)}=(y^{(m-1)}{}_{\nu})$ *are m sequences of real numbers, then there exists an unique spline* $S\in M^{\perp}$ *satisfying the following conditions of interpolation:* $S^{(h)}(\nu)=y^{(h)}(\nu)$ *whenever* $h=\overline{0,\ m-1}$ *and* $\nu\in Z$. *Moreover, we observed in the paragraph 3 of (Isac, G., Postolică, V., 1993) that any spline function S such as this is defined by*

$$S(x)=p(x)+\sum_{h=0}^{m-1} c_1^{(h)}(x-1)_+^{2m-1}+\sum_{h=0}^{m-1} c_2^{(h)}(x-2)_+^{2m-1}+\ \ldots\ +\sum_{h=0}^{m-1} c_0^{(h)}(-x)_+^{2m-1}+\ \ldots$$

where $u_+=(|u|+u)/2$ *for every real number u, p is a polynomial function of degree 2m-1 at most perfectly determined by the conditions* $p^{(h)}(0)=y_0^{(h)}$ *and* $p^{(h)}(1)=y_1^{(h)}$ *for all* $h=\overline{0,\ m-1}$ *and the coefficients* $c_{\nu}^{(h)}$ $(h=\overline{0,\ m-1},\ \nu\in Z)$ *are successively given by the general interpolation.Therefore, for every function* $f\in H^m(R)$, *there exists an unique function denoted by* $S_f\in M^{\perp}$ *such that* $S_f^{(h)}(\nu)=f^{(h)}(\nu),\ \forall\ h=\overline{0,\ m-1}$ *and* $\nu\in Z$. *Hence, in this case, M and* $M^{\perp}$ *give an orthogonal decomposition for the space* $H^m(R)$.

Example 4. *Let* $X=F_m=\{f\in C^{m-1}(R): f^{(m-1)}$ *is locally absolutely continuous and* $f^{(m)}\in L^2(R)\}$ *endowed with the H-locally convex topology induced by the scalar semiproducts*

$(x,y)_{\nu}=x(\nu)y(\nu)+\int_R x^{(m)}(t)\, y^{(m)}(t)dt,\ \nu\in Z,\ Y=L^2(R)$ *with the topology*

generated by the inner product $(x,y)_{\nu}=\int_R x(t)\, y(t)dt,\ \nu\in Z$ *and* $U:X\rightarrow Y$ *be the*

derivation operator of order m. Then, $M=\{x\in F_m: x(\nu)=0$ *for all* $\nu\in Z\}$ *and*

$M^{\perp}=\{\ s\in F_m: \int_R x^{(m)}(t)\, y^{(m)}(t)dt=0$ *for every* $x\in M\}$.

In a similar manner as in ***Example1*** *it may be proved that* $M^{\perp}$ *coincides with the class of all piecewise polynomial functions of order 2m (degree 2m-1 at most) having their knots at the integer points. Moreover, for every function f in* F_m *there exists an unique spline function* $S_f\in M^{\perp}$ *which interpolates f on the set Z of all integer numbers, that is,* S_f *satisfies the equalities* $S_f(\nu)=f(\nu)$ *for every* $\nu\in Z$, *being defined by*

$S_f(x)= p(x)+ a_1(x-1)_+^{2m-1} + a_2 (x-2)_+^{2m-1} + \ldots +a_0(-x)_+^{2m-1} + a_{-1}(-x-1)_+^{2m-1} + \ldots$ where u_+ has the same signification as in ***Example1****, the coefficients a_ν ($\nu \in Z$) are successively and completely determined by the interpolation conditions $S_f(\nu)=f(\nu)$, $\nu \in Z\backslash\{0, 1\}$ and p is a polynomial function satisfying the conditions $p(0)=f(0)$ and $p(1)=f(1)$. The uniqueness of S_f is ensured in Theorem 2 given by (Postolică, V., 1981).Thus, M and $M^\perp$ give an orthogonal decomposition of the space F_m and, as in the preceding example, $M^\perp$ is simultaneous and vectorial proximinal with respect to the family of seminorms generated by the above scalar semiproducts.*

Remark 4. Our examples show that the abstract construction of splines can be used to solve also several frequent problems of interpolation and approximation, having the possibility to choose the spaces and the scalar semiproducts. It is obvious that for a given (closed) linear subspace of a H-locally convex space X such a H-locally convex space Y (respectively, a linear (continuous) operator $U:X\to Y$) would not exist. Otherwise, the problem of best vectorial approximation by the corresponding orthogonal space of any (closed) linear subspace M for the elements in the direct sum $M\oplus M^\perp$ might be always reduced to the best simultaneous approximation. But, in general, such a possibility doesn't exist. Even in a H-locally convex space it is possible that there exist best vectorial approximations and the set of all best simultaneous approximations to be empty for some element of the space. We confine ourselves to mention the following simple example.

Example 5. *Let $X= R^N$ endowed with the topology generated by the family $\mathcal{P} =\{p_i : i \in N\}$ of seminorms defined by $p_i(x)= |x_i|$ ($i \in N$) for every $x= (x_i) \in X$ and $G_c=\{(x_i) \in X: x_i \geq 0$ whenever $i \in N$ and $\sum_{i\in N} x_i=c\}$, $c \in R_+$ X is a $\mathcal{P}$-simultaneous strictly convex (Isac, G., Postolică, V., 1993) H-locally convex space. Nevertheless, every element of G_c is $\mathcal{P}$-b.v.a. for the origin while its corresponding set of the best simultaneous approximations with respect to $\mathcal{P}$ is empty.*

However, at least from the numerical methods point of view, we are interested to solve the next open problem.

Open Problem 2. *Characterizations of the equivalence*

$$S(A)=\varnothing \Leftrightarrow V(A)=A.$$

Definition 3.*(Postolică,V.,2008) A real function $f : A \subseteq R^n \to R$ is called R_+^n - increasing(R_+^n - decreasing) if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in A$ and $x_1 \in x_2 + R_+^n$ ($x_1 \in x_2 - R_+^n$).*

The next coincidence of the efficient points sets and the Choquet boundaries is an immediate consequence of the result established in the general context of Hausdorff linear topological ordered vector spaces *(Postolică, V., 2008)*, and can not be obtained as a consequence of the Axiomatic Potential Theory.

Theorem 3. *If $\emptyset \neq A \subseteq R^n$ is any non-void, compact subset, then, $VMIN(A)$ ($VMAX(A)$) coincides with the Choquet's boundary $\partial_{S_1} A (\partial_{S_2} A)$ of A with respect to the convex cone S_1 (S_2) of all R_+^n - increasing(R_+^n - decreasing) real continuous functions on A. Consequently, each of these sets, endowed with the corresponding trace topology, is a Baire space and a G_δ - subset. Moreover, $eff(A, R_+^n) = \partial_{S_1} A \cup \partial_{S_2} A)$.*

Corollary 3.1. *(i) If for every $f \in C(A)$ one denotes $\overline{f}(a) = \sup\{f(a'): a' \in A \cap (a + R_+^n)\}$ and $\tilde{f}(a) = \sup\{f(a''): a'' \in A \cap (a - R_+^n)\}$, then*

$$eff(A, R_+^n) = \bigcup_{a \in A} \{a \in A: f(a) = \overline{f}(a)(\tilde{f}(a)), \forall f \in C(A)\};$$

(ii) $VMIN(A)(VMAX(A), VMIN(A)((VMAX(A)) \cap \{a \in A: s(a) \leq 0\}(s \in S_1)(s \in S_2)$ are compact sets with respect to Choquet's topology and as usual compact subsets of A.

Open Problem 3. *Generalize the concept of Choquet's boundary to non-empty complete sets and extend to non-compact sets the established strong connection by coincidence between the general efficient points sets and Choquet's boundaries.*

Open Problem 4. *If A is any non-empty, compact and convex subset of R^n $S = \{f : A \to R / f$ is continuous and concave}, then its Choquet's boundary with respect to S coincides with the set $ex(A)$ of all extreme points x in A, that is, if $y, z \in A$ and there exists $\alpha \in (0,1)$ with $\alpha y + (1-\alpha) z = x$, then $y = z = x$. The hypothesis of concavity imposed on the functions is essential for the validity of this result. To establish the main links between the extreme points of a non-empty set and its efficient points.*

SELECTIVE REFERENCES

1. Isac, G., Postolică, V.- *The Best Approximation and Optimization in Locally Convex Spaces,* Verlag Peter Lang GmbH, Frankfurt am Main, Germany, 1993.
2. Postolică, V. – *Spline Functions in H-locally Convex Spaces*. An. Şt. Univ. "Al. I. Cuza", Iaşi, 27, 1981, p. 333 - 338.

3. Postolică, V. – *A Survey on Pareto Efficiency in Infinite Dimensional Vector Spaces*. Foundations of Computing and Decision Sciences, Vol. 33, No. 3, 2008, p. 271 - 284, ISSN 0867-6356, Index 35830,The Publishing House of Poznań University of Technology, Poland.
4. Postolică, V. – *Choquet Boundaries and Efficiency*. Computer and Mathematics with Applications, 55, 2008, p. 381 - 391.
5. Postolică, V. – *Approximate Efficiency in Infinite Dimensional Ordered Vector Spaces*. International Journal of Applied Management Science (IJAMS), Vol. 1, No. 3, 2009,
p. 300 – 314.
6. Postolică, V. - *General Efficiency*. Invited research work at the 7th European Congress of Mathematics, July 18 – 22, 2016, TU Berlin, Germany. Presented in the Contributed Session 16, CS-16-B: "*Optimization and Control"* on Tuesday, July 19, 2016, 10:40. The Abstract published in the Book of Abstracts, p. 606. Published as Research Paper in IOSR Journal of Mathematics (IOSR-JM) e-ISSN:2278-5728, p-ISSN: 2319-765X, Volume 13, Issue 2 Ver I (Mar.-Apr. 2017), p. 79 - 94, DOI: 10.9790/5728-1302017994 provided by Cross Ref, USA, www. iosrjournals.org.

Romanian Academy of Scientists,
Vasile Alecsandri University of Bacău,
Faculty of Sciences,
Department of Mathematics, Informatics and Educational Sciences
ROMANIA,
E-mail : vpostolica@ambra.ro .