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SUBSPACES IN RHEONOMIC FINSLER SPACES

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Abstract. The purpose of the present paper is to investigate some properties of subspaces in Rheonomic Finsler Spaces. We describe the induced connections of the subspaces.

1. Preliminaries. Rheonomic Finsler Spaces

Let M be an n-dimensional real manifold and (TM, π, M) the tangent bundle of M with $\pi : TM \to M$. We denote $E = TM \times R$ a $2n+1$- dimensional real manifold. In E we consider a local chart $U \times (a,b)$. The points $u = (x,y,t) \in E$ have the local coordinates (x^i, y^i, t) .

Definition 1.1. A rheonomic Finsler space is a pair $RF^n = (M, F(x,y,t))$, where $F : TM \times R \to R$ satisfy the following axioms:

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a) F is a positive scalar function on E called the fundamental function of RF^n

b) F is a positive 1-homogeneous with respect to y^i;

c) F is diffferentiable on $(TM\backslash\{0\}) \times R$ and continuous on the nul section of the projection π

d) The Hessian of $F, g_{ij} = \frac{1}{2}\frac{\partial^2 F}{\partial y^i \partial y^j}$ is positive defined and is called the fundamental tensor of RF^n.

The canonical spray S of RF^n is

$$S = y^i \frac{\partial}{\partial x^i} - \left(N_0^i(x,y,t) + N_k^i(x,y,t)\, y^k\right)\frac{\partial}{\partial y^i} + \frac{\partial}{\partial t},$$

with

$$N_j^i(x,y,t) = \frac{1}{2}\frac{\partial}{\partial y^j}\left(\gamma_{rs}^i(x,y,t)\, y^r y^s\right)$$

and

$$N_j^0(x,y,t) = \frac{1}{2}\frac{\partial g_{jk}}{\partial t} y^k$$

where γ_{rs}^i are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y,t)$.

The Cartan nonlinear connection N has the coefficients $\left(N_j^i(x,y,t), N_j^0(x,y,t)\right)$ .N determines the horizontal distribution on E which is supplementary to the vertical distribution. The adapted basis to these distribution is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right)$with

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y,t)\frac{\partial}{\partial y^j} - N_i^0(x,y,t)\frac{\partial}{\partial t}$$

The dual adapted basis is $(dx^i, \delta y^i, \delta t)$ where

$$\begin{aligned} \delta y^i &= dy^i + N_j^i(x,y,t)\, dx^j \\ \delta t &= dt + N_j^0(x,y,t)\, dx^j. \end{aligned}$$

Theorem 1.1. The canonical metrical N-connection has the coefficients expressed by the generalized Christoffel symbols:

$$\begin{cases} F_{jk}^i = \frac{1}{2} g^{is}\left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta xs}\right) \\ C_{jk}^i = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right) \\ C_{j0}^i = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj0}}{\partial y^k} + \frac{\partial g_{sj}}{\partial t} - \frac{\partial g_{j0}}{\partial y^s}\right) \end{cases}$$

2. Subspaces in Rheonomic Finsler Spaces

Let M^m be a submanifold of M, $m < n$. It is represented parametrically by the equations

$$x^i = x^i\left(u^\alpha\right), \alpha = 1, 2, ..., m.$$

We denote

$$B_\alpha^i\left(u\right) = \frac{\partial x^i}{\partial u^\alpha}$$

and we assume that $rang\left(B_\alpha^i\right) = m$. We consider the supporting element y^i at a point $\left(u^\alpha\right)$ of M^m to be tangent to M^m. So

$$y^i = B_\alpha^i\left(u\right) v^\alpha$$

The function $\widetilde{F}\left(u, v, t\right) = F\left(x\left(u\right), y\left(u, v\right), t\right)$ is a metric Finsler on M^m.Consequently we get m-dimensional Rheonomic Finsler space $R\widetilde{F}^m$ which is a subspace of RF^n. The fundamental function $\widetilde{F}$ is called the induced metric on $R\widetilde{F}^m$.We have

$$g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 \widetilde{F}^2}{\partial v^\alpha \partial v^\beta}$$

We denote

$$B_{\alpha\beta}^i = \frac{\partial}{\partial u^\beta}\left(B_\alpha^i\right)$$

so,

$$B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$$

We also employ

$$B_{0\beta}^i = v^\alpha B_{\alpha\beta}^i = v^\alpha \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial u^\alpha} &= B_\alpha^i \frac{\partial}{\partial x^i} + B_{0\alpha}^i \frac{\partial}{\partial y^i} \\ \frac{\partial}{\partial v^\alpha} &= B_\alpha^i \frac{\partial}{\partial y^i} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial u^\alpha} &= \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i} + v^\beta \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{\partial}{\partial y^i} \\ \frac{\partial}{\partial v^\alpha} &= \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial y^i} \end{aligned}$$

We get

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^i_\beta$$

that is non-degenerate and positive definite and called the induced metric tensor of $R\widetilde{F}^m$.

At each point (u^α) of $R\widetilde{F}^m$ we have $n-m$ unit vectors $B^i_a(u,v)$ normal to $R\widetilde{F}^m$ such that

$$\begin{aligned} g_{ij}(x(u), y(u,v), t) B^i_a(u) B^j_a(u,v) &= 0, \\ g_{ij}(x(u), y(u,v), t) B^j_a(u,v) B^j_b(u,v) &= \delta_{ab}, \end{aligned}$$

where $a, b, c = m+1, m+2, ..., n$.

We have

$$B^i_a B^\beta_i = \delta^\beta_\alpha, B^i_a B^a_i = 0.$$

3. Induced connections

The induced tangent connection on the subspace $R\widetilde{F}^m$ is $T\Gamma^m\left(N^\alpha_\beta, F^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma}\right)$ where

$$\begin{aligned} N^\alpha_\beta &= B^\alpha_i \left(B^i_{0\beta} + N^i_j B^j_\beta\right) \\ F^\alpha_{\beta\gamma} &= B^\alpha_i \left(B^i_{\beta\gamma} + B^j_\beta F^i_{j\gamma}\right) \\ C^\alpha_{\beta\gamma} &= B^\alpha_i B^j_\beta C^i_{j\gamma}, \end{aligned}$$

with

$$\begin{aligned} F^i_{j\gamma} &= F^i_{jk} B^k_\gamma + C^i_{jk} B^k_a H^a_\gamma \\ C^i_{j\gamma} &= C^i_{jk} B^k_\gamma \\ H^a_\gamma &= B^\alpha_i \left(B^i_{0\gamma} + N^i_j B^j_\gamma\right). \end{aligned}$$

Now, by a direct calculus we can state the following theorem:

Theorem 3.1. The induced tangent connection $T\Gamma^m\left(N^\alpha_\beta, F^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma}\right)$ on the subspace $R\widetilde{F}^m$ of the Rheonomic Finsler space RF^n has the coefficients expressed by

$$\begin{aligned} N^\alpha_\beta &= B^\alpha_i v^\alpha \frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} + \frac{1}{2} B^\alpha_i B^j_\beta \frac{\partial \gamma^i_{rs}(x,y,t)}{\partial y^j} y^r y^s; \\ F^\alpha_{\beta\gamma} &= B^\alpha_i \frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} + B^\alpha_i \frac{\partial x^j}{\partial u^\beta}\left(F^i_{jk}\frac{\partial x^k}{\partial u^\gamma} + C^i_{jk}\frac{\partial x^k}{\partial u^\alpha} H^a_\gamma\right); \\ C^\alpha_{\beta\gamma} &= B^\alpha_i C^i_{jk} \frac{\partial x^j}{\partial u^\beta}\frac{\partial x^k}{\partial u^\gamma}. \end{aligned}$$

Theorem 3.2. The induced normal connection $N\Gamma^m\left(N^\alpha_\beta, F^a_{b\alpha}, C^a_{b\alpha}\right)$ on the subspace $R\widetilde{F}^m$ of the Rheonomic Finsler space RF^n has the coefficients expressed by

$$\begin{aligned} N^\alpha_\beta &= B^\alpha_i v^\alpha \frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} + \frac{1}{2} B^\alpha_i B^j_\beta \frac{\partial \gamma^i_{rs}(x,y,t)}{\partial y^j} y^r y^s; \\ F^a_{b\alpha} &= B^a_i \frac{\partial B^i_b}{\partial u^\alpha} - B^a_i N^\beta_\alpha \frac{\partial B^i_b}{\partial v^\beta} + B^a_i B^j_b F^i_{j\alpha}; \\ C^a_{b\alpha} &= B^a_i \frac{\partial B^i_b}{\partial v^\alpha} + B^a_i B^j_b C^i_{j\alpha}. \end{aligned}$$

References

[1] Bucataru I., Miron R., **Finsler-Lagrange Geometry-Applications to dynamical systems**, Ed. Academiei Romane, Bucuresti, (2007)

[2] Frigioiu C., **On the geometrization of the rheonomic Finslerian Mechanical Systems**, Analele Stiintifice ale Univ. "l. I. Cuza", Iasi, vol. III, 145-155, (2007)

[3] Frigioiu C., Hedrich K., Birsan I.G., **Lagrangian geometrical model of the rheonomic mechanical systems**, World Academy of Science, Engineering and Technology, vol.5, 283-289, (2001)

[4] Lungu O. , V.Nimineţ, **Some properties of a R-Randers quartic** , Scientific Studies and Research, Vol 20, Nr.1, 2010, 133-140.

[5] Sakaguchi T., **Subspaces in Finsler Space**, Memoirs of the National Defense Academy, Japan, vol.29, no.1, 1-9 (1989)

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