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SOME RESULTS ON QUARTER-SYMMETRIC METRIC CONNECTION ON A PARA-SASAKIAN MANIFOLDS

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Abstract. We classify para-Sasakian manifolds with respect to quarter-symmetric metric connection. Among others it is proved that ϕ-concircularly flat para-Sasakian manifold is an η-Einstein manifold and a non-semisymmetric Ricci-generalized pseudosymmetric para-Sasakian manifold has constant curvature if and only if the space like vector field ξ is harmonic. Para-Sasakian manifolds admitting certain conditions on the concircular curvature tensor and Ricci tensor are studied and several new results are obtained.

1. INTRODUCTION

In 1992, Hyden [9] introduced the idea of metric connection with torsion in a Riemannian manifold. Further, some properties of semisymmetric metric connection have been studied by Yano [22].

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In [8], Golab defined and studied quarter-symmetric connection on a differentiable manifold with affinity connection, which generalizes the idea of semi symmetric connection. Various properties of quarter-symmetric metric connection has been studied by many geometers like De and Sengupta [6], Mishra and Pandey [12], Narain and Yadav [14], Rastogi [15, 16], Yadav and Suthar [18], Yadav et al. [19] and Yano and Imai [23].

A linear connection $\tilde{\nabla}$ on an n-dimensional differentiation manifold is said to be a quarter-symmetric connection [9] if its torsion tensor T is of the form

$$T(X,Y)=\tilde{\nabla}_XY-\tilde{\nabla}_YX-[X,Y]=\eta(Y)\phi X-\eta(X)\phi Y, \tag{1.1}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In particular, if we replace ϕX by X and ϕY by Y, then the quarter-symmetric connection reduces to the semisymmetric connection [7]. Thus, the notion of quarter-symmetric connection generalizes the idea of the semisymmetric connection. If quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$\left(\tilde{\nabla}_X g\right)(Y,Z)=0, \tag{1.2}$$

for all $X,Y,Z\in\aleph(M)$, where $\aleph(M)$ is the Lie algebra of the vector field on the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection.

2. PRELIMINARIES

An n-dimensional differentiable manifold M is called an almost papa-contact manifold if it admits an almost paracontact structure (ϕ,ξ,η) consisting of a (1, 1)-tensor field ϕ, a vector field ξ, and a 1-form η satisfying

$$\phi^2X=X-\eta(X)\xi, \tag{2.1}$$

$$\eta(\xi)=1,\qquad \phi\circ\xi=0,\qquad \eta\circ\phi=0, \tag{2.2}$$

If g is a compatible Riemannian metric with (ϕ,ξ,η), that is,

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\ \ g(X,\xi)=\eta(X), \tag{2.3}$$

$$g(X,\phi Y)=g(\phi X,Y), \tag{2.4}$$

for all vector fields X and Y on M, then M becomes almost para-contact Riemannian manifold equipped with an almost paracontact

Riemannian structure $(\phi,\ \xi,\ \eta,\ g)$. An almost paracontact Riemannian manifold is called a para-Sasakian manifold if it satisfies

$$(\nabla_X \phi)(Y) = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation. From the above equation it follows that

$$\nabla_X \xi = \phi X, \quad (\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X. \tag{2.6}$$

In an n-dimensional para-Sasakian manifold, the following relations hold [1, 17]:

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$S(X,\xi) = -(n-1)(Y)X, \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.10}$$

for any vector fields X, Y and Z, where R and S are the Riemannian curvature tensor and the Ricci tensor of M, respectively.

Definition 2.1. A para-Sasakian manifold is said to be an η-Einstein manifold if its Ricci operator Q satisfies

$$Q = \lambda_1\, Id + \lambda_2\, \eta \otimes \xi,$$

where λ_1 and λ_2 are smooth function on the manifold. In particular, if $\lambda_2 = 0$, then M is an Einstein manifold.

A transformation of n-dimensional Riemannian manifold M which transform every geodesic circle of M in to geodesic circle, is called a concircular transformation [21]. A concircular transformation is always a conformal transformation [21]. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, i.e. the concircular geometry is the generalization of inverse geometry in the sense that the change of metric is more general then the induced by circle preserving diffeomorphism [3]. An interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{Z}$. It is defined by [21, 22]).

$$\tilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}\left\{g(Y,W)X - g(X,W)Y\right\}, \tag{2.11}$$

From (2.11), it follows that

$$\tilde{Z}'(X,Y,W,U) = \tilde{R}(X,Y,W,U) - \frac{r}{n(n-1)}\left[g(Y,W)g(X,U)\right.$$

$$-g(X,W)g(Y,U)], \tag{2.12}$$

for all $X, Y, U \in TM$ and $\tilde{Z}'(X,Y,W,U) = g(Z(X,Y,W),U)$, respectively, where r is the scalar curvature of M.

The paper is organized as follows: After introduction in section 2, 3 we give a brief account of para-Sasakian manifolds and quarter-symmetric metric connection respectively. Section 4 is devoted to the study of ϕ-concircularly flat para-Sasakian manifold and also determine the ϕ-sectional curvature of the plane by two vectors, some conditions on the Ricci tensor and introduce the notion that the Ricci tensor of para-Sasakian manifold to be η-recurrent, ϕ-parallel, with respect to quater-symmetric metric connection which generalize the notion of η-parallel Ricci tensor. Then we obtain some necessary and sufficient conditions for the Ricci tensor of para-Sasakian manifold to be η-recurrent and ϕ-parallel. However, in such a manifold the scalar curvature is constant if and only if $\Psi = \frac{-\mu}{n-1}$, where $\mu = Tr.(Q\phi)$ and $\Psi = Tr.\phi$. Also it is proved that in para-Sasakian manifold with respect to such connection with η-recurrent or ϕ-recurrent Ricci tensor, $\frac{1}{2}\{r+(n-1)\}$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector $\phi\rho$, ρ being the associate vector field of the 1-form given by $A(X) = g(X,\rho)$. Section 5, 6, deals with $\tilde{Z}\cdot\tilde{S} = 0$, $\tilde{Z}\cdot\tilde{Z} = 0$ in para-Sasakian manifold with respect to quarter-symmetric metric connection. Finally, in section 7, we study Ricci-generalized pseudosymmetric para-Sasakian manifold under such connection and proved that such manifold is either a space of constant curvature 1 or an η-Einstein manifold.

3. QUARTER-SYMMETRIC METRIC CONNECTION

Let $\tilde{\nabla}$ be a linear connection and ∇ a Riemannian connection of an almost contact metric manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y), \tag{3.1}$$

where U is tensor of type $(1,1)$. For $\tilde{\nabla}$ to e a quarter-symmetric metric connection in M, then we have [8]

$$U(X,Y) = \frac{1}{2}\left[T(X,Y) + T(X,Y) + T'(Y,X)\right], \tag{3.2}$$

$$g\left(T'(X,Y),Z\right) = g\left(T(Z,X),Y\right). \tag{3.3}$$

From (1.1) and (3.3), we get

$$T'(X,Y) = \eta(X)\,\phi Y - g(\phi X,Y)\xi, \tag{3.4}$$

Using (1.1) and (3.4) in (3.2), we obtain

$$U(X,Y) = \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{3.5}$$

Thus a quarter symmetric metric connection $\tilde{\nabla}$ in a para-Sasakian manifold given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{3.6}$$

Hence (3.6) is the relation between Riemannian connection and quarter-symmetric metric connection on a para-Sasakian manifold. A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by

$$\begin{aligned} \tilde{R}(X,Y)Z = {} & R(X,Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X \\ & + \eta(Z)\left[\eta(X)Y - \eta(Y)X\right] - \left[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\right]\xi, \end{aligned} \tag{3.7}$$

where $\tilde{R}$ and R denotes the Riemannian curvatures of the connections $\tilde{\nabla}$ and ∇, respectively. From (3.7), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) - (n+1)\,\eta(Y)\,\eta(Z), \tag{3.8}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇, respectively.
Contracting (3.8) over Y and Z, we get

$$\tilde{r} = r + (n-1), \tag{3.9}$$

where $\tilde{r}$ and r are the scalar curvature of the connections $\tilde{\nabla}$ and ∇, respectively.

4. ϕ-CONCIRCULARLY FLAT PARA-SASAKIAN MANIFOLD ADMITTING QUARET-SYMMETRIC METRIC CONNECTION

Let C be the Weyl-conformal curvature tensor of a n-dimensional manifold M. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into the direct sum $\chi_p(M) = \phi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is an 1-dimensional linear subspace of $\chi_p(M)$ generated by ξ_p. Therefore, we have a map:

$$C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to \phi(\chi_p(M) \oplus L(\xi_p).$$

It may be considered as the following particular case
(i) $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to L(\xi_p)$., i.e. the projection of the image of C in $\phi(\chi_p(M))$ is zero.

(ii) $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to \phi(\chi_p(M))$, i.e. the projection of the image of C in $L(\xi_p)$ is zero.

$$C(X,Y)\xi = 0. \tag{4.1}$$

(iii) $C : \phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M)) \to L(\xi_p)$, i.e. when C is restricted to $\phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M))$, the projection of the image of C in $\phi(\chi_p(M))$ is zero. The condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{4.2}$$

Here the case (i), (ii) and (iii) are conformally symmetric, ξ-conformally flat and ϕ-conformally flat respectively. Then the case (i) and (ii) were considered in ([24, 5]) respectively. The case (iii) was considered in [4] for the case M is a K-contact manifold. Moreover in [2] the authors studied contact metric manifolds satisfying (iii). Analogous to the definition of ξ-conformally flat and ϕ-conformally flat, we give the following definitions.

Definition 4.1. A para-Sasakian manifold is said to be ϕ-concircularly flat if

$$g(\tilde{Z}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.3}$$

where $X, Y, Z, W \in \chi(M)$.

Theorem 4.1. *If a para-Sasakian manifold M is ϕ-concircularly flat with respect to quarter-symmetric metric connection, then the manifold is an η-Einstein manifold.*

Proof. Substituting $X = \phi X, Y = \phi Y, U = \phi U$ and $W = \phi W$ in (2.12) and using (2.7)and (3.7), it follows that

$$\begin{aligned} g(\tilde{Z}(\phi X, \phi Y, \phi W, \phi U)) &= \acute{R}(\phi X, \phi Y, \phi W, \phi U) + 3g(X, \phi W)g(Y, \phi U) \\ &-3g(Y, \phi W)g(X, \phi U) - \tfrac{\bar{r}}{n(n-1)} \begin{bmatrix} g(\phi Y, \phi W)g(\phi X, \phi U) \\ -g(\phi X, \phi W)g(\phi Y, \phi U) \end{bmatrix}, \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we get

$$\begin{aligned} \acute{R}(\phi X, \phi Y, \phi W, \phi U) &= -3g(X, \phi W)g(Y, \phi U) + 3g(Y, \phi W)g(X, \phi U) \\ &+\tfrac{\bar{r}}{n(n-1)} \left[g(\phi Y, \phi W)g(\phi X, \phi U) - g(\phi X, \phi W)g(\phi Y, \phi U) \right], \end{aligned} \tag{4.5}$$

Let $(e_{1,}e_2, ..., e_{n-1}, \xi)$ be a local orthonormal basis of the vector fields in M, and then $(\phi\, e_{1,}\phi\, e_2, ..., \phi\, e_{n-1}, \xi)$ is also a local orthonormal basis. Putting $X = U = e_i$ in (4.5) and summing over i, $1 \le i \le n-1$, we get

$$\sum_{i=1}^{n-1} R'(\phi\, e_i, \phi\, Y, \phi\, W, \phi\, e_i)$$

$$= -3\sum_{i=1}^{n-1}\{\, g(e_i, \phi\, W)g(Y, \phi\, e_i) - g(Y, \phi\, W)\}g(e_i, \phi\, e_i)$$

$$+\frac{\bar{r}}{n(n-1)}\sum_{i=1}^{n-1}\left[g(\phi Y, \phi W)g(\phi\, e_i, \phi\, e_i) - g(\phi\, e_i, \phi W)g(\phi Y, \phi\, e_i)\right],$$

which on simplification, we get

$$S(\phi Y, \phi W) = \frac{\tilde{r}(n-2)}{n(n-1)} g(\phi Y, \phi W), \tag{4.6}$$

In view of (2.3), (2.10), (3.9) and (4.6), it follows that

$$S(Y, W) = \left\{\frac{n^2 - 3n + 2 + r(n-2)}{n(n-1)}\right\} g(Y, W)$$
$$+\left\{\frac{-n^3 + n^2 + 3n - 2 - r(n-2)}{n(n-1)}\right\} \eta(Y)\eta(W).$$

This proves the Theorem 4.1.

Definition 4.2. A plane section in $\chi(M)$ is called a ϕ-sectional if their exists a unit vector X in $\chi(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is orthogonal basis of the plane section. Then the sectional curvature $^{/}K(X, \phi X)$ is called ϕ-sectional curvature.

Theorem 4.2. *If a para-Sasakian manifold M admits a quarter-symmetric metric connection, then the ϕ-sectional curvature of the plane determined by two vectors $X, \phi X \in \xi^{\perp}$ is* 2.

Proof. Let $\xi^{\perp}$denotes the n-dimensional distribution orthogonal to ξ in para-Sasakian manifold admitting a quarter-symmetric metric connection. Then for any $X \in \xi^{\perp}, g(X, \xi) = 0$. Now we shall determine the ϕ-sectional curvature $^{/}K$ at the plane determine by the vectors $X, \phi X \in \xi^{\perp}$. Putting $Y = \phi X, Z = \phi X$ and $U = X$ in (3.7) and using (2.7), it is seen that

$$R'(X, \phi X, \phi X, X) = 2\left\{-g(X, \phi X)^2 - g(\phi X, \phi X)g(X, X)\right\}, \tag{4.7}$$

Therefore

$$^{/}K(X, \phi X) = \frac{R'(X, \phi X, \phi X, X)}{-g(\phi X, \phi X)g(X, X) - g(X, \phi X)^2} = 2.$$

This proves the Theorem 4.2.

Theorem 4.3. *If a para-Sasakian manifold M admits a quarter-symmetric metric connection, whose curvature tensor vanishes then the scalar curvature of the manifold is constant if and only if the space like vector filed ξ is harmonic.*

Proof. In view of (3.7), we get

$$S(Y,Z) = -2g(Y,Z) + (n+1)\eta(Y)\eta(Z), \tag{4.8}$$

Differentiating (4.8) covariantly along X, we obtain

$$(\nabla_X S)(Y,Z) = (n+1)\{(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y)\}$$

Taking an orthomormal frame field and contracting above over X and Z, we have

$$dr(Y) = 2(n+1)\Psi\eta(Y),$$

where $\Psi = Tr.\phi$ and r is the scalar curvature of the manifold. From above relation, it follows that $dr(Y) = 0$ if and only if $\Psi = 0$, which implies that ξ is harmonic. This proves the Theorem 4.3.

Definition 4.3. The Ricci tensor of a para-Sasakian manifold is said to be η-recurrent if its Ricci tensor S satisfying the following

$$(\nabla_X S)(\phi Y, \phi Z) = A(X)S(\phi Y, \phi Z)$$

for all X, Y, Z. Where $A(X) = g(X, \rho)$, ρ is the associate vector field of the 1-form A. In particular, if the 1-form A vanishes then the Ricci tensor of para-Sasakianmanifold is said to be η-parallel and this notion for Sasakian manifolds was first introduced by Kon [11].

Theorem 4.4. *If a para-Sasakian manifold M admits a quarter-symmetric metric connection, whose Ricci tensor is η-recurrent if and only if the relations (4.11) hold.*

Proof. From (2.9)and (3.8), we get

$$\tilde{S}(\phi Y, \phi Z) = S(Y,Z) + 2g(Y,Z) + (n-3)\eta(Y)\eta(Z), \tag{4.9}$$

Differentiating (4.9) covariantly along X and using (2.6), we obtain

$$\begin{aligned}&(\nabla_X\tilde{S})(\phi Y, \phi Z) = (\nabla_X S)(Y,Z) - S(X,\phi Z)\eta(Y) - S(X,\phi Y)\eta(Z)\\ &+(n-3)\{g(X,\phi Y)\eta(Z) + g(X,\phi Z)\eta(Y)\},\end{aligned} \tag{4.10}$$

In view of definition 4.3 and equation (4.10), It follows that

$$\begin{aligned}(\nabla_X S)(Y,Z) &= A(X)\{S(Y,Z)+2g(Y,Z)+(n-3)\eta(Y)\eta(Z)\}\\ &+S(X,\phi Z)\eta(Y)+S(X,\phi Y)\eta(Z)\\ &-(n-3)\{g(X,\phi Y)\eta(Z)+g(X,\phi Z)\eta(Y)\}.\end{aligned} \tag{4.11}$$

This proves the Theorem 4.4.

Theorem 4.5. *If a para-Sasakian manifold M admitting a quarter-symmetric metric connection, whose Ricci tensor is η-recurrent then its Ricci tensor along with associate vector field of the 1-form is given by (4.16).*

Proof. Let $\{e_i, i=1,2,3,...n\}$ be an orthonormal frame at any point of the manifold. Then contracting over Y and Z in (4.11), we get

$$dr(X) = A(X)\ \{r+(n-1)\} \tag{4.12}$$

Again contraction over X and Z in (2.2), it follows that

$$\frac{1}{2}dr(X) = \{\mu+(n-1)\Psi\}\,\eta(Y)+S(Y,\rho)+(n-1)\eta(Y)\eta(\rho), \tag{4.13}$$

where $\mu = Tr.(Q\phi) = \sum_{i=1}^{n}\varepsilon_i S(\phi e_i, e_i)$, $\Psi = Tr.\phi = \sum_{i=1}^{n}\varepsilon_i\, g(\phi e_i, e_i)$ and $\varepsilon_i = g(e_i,e_i)$.
In view of (4.12) and(4.13), it is seen that

$$\frac{A(Y)(r+(n-1))}{2} = \{\mu+(n-1)\psi\}\,\eta(Y)+S(Y,\rho)+(n-1)\eta(Y)\eta(\rho), \tag{4.14}$$

The relation (4.14) yields for $Y=\xi$.

$$\frac{1}{2}\left[r+(n-1)\right]\eta(\rho) = \left[\mu+(n-1)\Psi\right]. \tag{4.15}$$

From (4.14) and (4.15), we yields

$$S(Y,\rho) = \frac{1}{2}\{r+(n-1)\}\,g(Y,\rho)+\frac{1}{2}\{r+(3(n-1)\}\,\eta(Y)\eta(\rho). \tag{4.16}$$

This proves the Theorem 4.5.

Corollary 4.1. *If the Ricci tensor of para-Sasakian manifold M admitting a quarter-symmetric metric connection, is η-recurrent then $\alpha = \frac{1}{2}\{r+(n-1)\}$ is eigenvalue of the Ricci tensor corresponding the eigenvector $\phi\rho$ defined by $g(X,L) = T(X) = g(X,\phi\rho)$.*

Definition 4.4. The Ricci tensor of para-Sasakian manifold is said to be ϕ-parallel if its Ricci tensor satisfies the following

$$(\nabla_{\phi X} S)(\phi Y, \phi Z) = 0,$$

for all X, Y, Z, the condition of ϕ-parallelity is more weaker then η-parallelity. Then proceeding in the same manner as before we state the results.

Theorem 4.6. *If the Ricci tensor of a para-Sasakian manifold M admitting a quarter-symmetric metric connection, is ϕ-parallel if and only if the relation (4.17) holds.*

$$\begin{aligned}(\nabla_{\phi X} S)(Y, Z) &= S(X, Z)\eta(Y) + S(X, Y)\eta(Z) \\ &+(4n-8)\eta(X)\eta(Y)\eta(Z) - (n-3)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y)\},\end{aligned} \tag{4.17}$$

for all X, Y, Z.

Theorem 4.7. *Let the Ricci tensor of para-Sasakian manifold M admitting a quarter-symmetric metric connection, is ϕ-parallel, then the scalar curvature of the manifold is constant if and only if $\Psi = -\frac{\mu}{n-1}$, where $\mu = Tr.(Q\phi)$. and $\Psi = Tr.(\phi)$.*

Proof. We consider para-Sasakian manifold whose Ricci tensor is ϕ-parallel with respect to quarter-symmetric metric connection. Then (4.17) holds. Substituting $X = \phi X$ in (4.17), we get

$$\begin{aligned}(\nabla_X S)(Y, Z) - \eta(X)(\nabla_\xi S)(Y, Z) &= S(\phi X, Z)\eta(Y) + S(\phi X, Y)\eta(Z) \\ &-(n-3)\{g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)\},\end{aligned} \tag{4.18}$$

Taking an orthomormal frame field and contracting above over X and Z in (4.18), we have

$$dr(Y) = 2\{\mu + (n-1)\Psi\}(Y)$$

where $\mu = Tr.(Q\phi)$. This proves the Theorem 4.7.

5. A PARA-SASAKIAN MANIFOLDS ADMITTING QUARET SYMMETRIC-METRIC CONNECTION SATISFYING $\tilde{Z} \cdot \tilde{S} = 0$.

Theorem 5.1. *If a para-Sasakian manifold M admitting a quarter-symmetric metric connection satisfies the condition $\tilde{Z} \cdot \tilde{S} = 0$, if and only if either the scalar curvature r of M is $r = 2n(1-n)$ or M is an η-Einstein manifold.*

Proof. We consider that para-Sasakian manifolds with respect to quarter symmetric metric connection satisfying the condition

$$(\tilde{Z}(U,Y)\cdot\tilde{S})\,(W,X)=0.$$

Therefore, we have

$$\tilde{S}(\tilde{Z}(U,Y)W,X)+\tilde{S}(W,\tilde{Z}(U,Y)X)=0, \tag{5.1}$$

Substituting $U=\xi$ in (5.1), it follows that

$$\tilde{S}(\tilde{Z}(\xi,Y)W,X)+\tilde{S}(W,\tilde{Z}(\xi,Y)X)=0. \tag{5.2}$$

In view of (2.7), (2.11) and (3.7), we get

$$\tilde{Z}(\xi,Y,W)=\left\{2+\frac{r}{n(n-1)}\right\}\,[\eta(W)Y-g(Y,W)\xi]\,, \tag{5.3}$$

Using (5.3) in (5.2), we have

$$\left\{2+\frac{r}{n(n-1)}\right\}$$

$$\times\begin{bmatrix} S(X,Y)\eta(W)+2g(X,Y)\eta(W)\\ -(n+1)\eta(X)\eta(Y)\eta(W)+2(n-1)g(Y,W)\eta(X)\\ -2\eta(Y)g(Y,W)+S(Y,W)\eta(X)+2g(Y,W)\eta(X)\\ -(n+1)\eta(X)\eta(Y)\eta(W)+2(n-1)g(Y,X)\eta(W)\\ -2\eta(W)g(Y,X)\end{bmatrix}=0$$

Therefore either the scalar curvature r of M is $r=2n(1-n)$, which is of constant or

$$\begin{bmatrix} S(X,Y)\eta(W)+2g(X,Y)\eta(W)-(n+1)\eta(X)\eta(Y)\eta(W)\\ +2(n-1)g(Y,W)\eta(X)-2\eta(Y)g(Y,W)+S(Y,W)\eta(X)\\ +2g(Y,W)\eta(X)-(n+1)\eta(X)\eta(Y)\eta(W)\\ +2(n-1)g(Y,X)\eta(W)-2\eta(W)g(Y,X)\end{bmatrix}=0, \tag{5.4}$$

Putting $W=\xi$ in (5.4) and using (2.2), we get

$$S(X,Y)=-2(n-1)g(X,Y)+(n-3)\eta(X)\eta(Y).$$

This proves the Theorem 5.1.

6. A PARA-SASAKIAN MANIFOLDS ADMITTING QUARET SYMMETRIC-METRIC CONNECTION SATISFYING $\tilde{Z} \cdot \tilde{Z} = 0$.

Theorem 6.1. If *a para-Sasakian manifold M with respect to quarter-symmetric metric connection, satisfying the condition $\tilde{Z} \cdot \tilde{Z} = 0.$, if and only if either the scalar curvature r of M is $r = 2n(n-1)$ or M is locally isometric to the hyperbolic space $H^n(-\lambda), \lambda = 2 - \frac{r}{n(n-1)}$*

Proof. We consider that para-Sasakian manifolds with respect to quarter symmetric metric connection satisfying the condition

$$\tilde{Z}(X,Y) \cdot \tilde{Z})\ (U,V)W = 0.$$

Therefore, we have

$$\tilde{Z}(X,Y) \cdot \tilde{Z}) - \tilde{Z}(U, \tilde{Z}(X,Y)V)W - \tilde{Z}(U,V)\tilde{Z}(X,Y)W = 0. \tag{6.1}$$

Substituting $X = U = \xi$ in (6.1), It follows that

$$\tilde{Z}(\xi,Y) \cdot \tilde{Z})\ (\xi,V)W - \tilde{Z}(\tilde{Z}(\xi,Y)\xi,V)W - \tilde{Z}(\xi,\tilde{Z}(\xi,Y)V)W$$
$$-\tilde{Z}(\xi,V)\tilde{Z}(\xi,Y)W = 0. \tag{6.2}$$

Applying (2.8),(2.11) and (3.7) in (6.2), we get

$$\left\{2 - \frac{r}{n(n-1)}\right\}$$
$$\times \begin{bmatrix} \eta(\tilde{Z}(\xi,V)W)Y - g(Y,\tilde{Z}(\xi,V)W)\xi \\ -\tilde{Z}(Y,V)W + \eta(Y)\tilde{Z}(\xi,V)W - \eta(V)\tilde{Z}(\xi,Y)W \\ +g(Y,V)\tilde{Z}(\xi,\xi)W - \eta(W)\tilde{Z}(\xi,V)Y \\ +g(Y,W)\tilde{Z}(\xi,V,\xi) \end{bmatrix} = 0$$

Therefore either the scalar curvature $r = 2n(n-1)$ or

$$\begin{bmatrix} \eta(\tilde{Z}(\xi,V)W)Y - g(Y,\tilde{Z}(\xi,V)W)\xi - \tilde{Z}(Y,V)W \\ +\eta(Y)\tilde{Z}(\xi,V)W - \eta(V)\tilde{Z}(\xi,Y)W + g(Y,V)\tilde{Z}(\xi,\xi)W \\ -\eta(W)\tilde{Z}(\xi,V)Y + g(Y,W)\tilde{Z}(\xi,V,\xi) \end{bmatrix} = 0 \tag{6.3}$$

In view of (2.8) and (2.11,) equation (6.3) reduces to

$$R(Y,V,W) = \lambda \{g(Y,W)V - g(V,W)V\}$$

where $\lambda = 2 - \frac{2r}{n(n-1)}$.
This proves the Theorem 6.1.

Corollary 6.1. If *a para-Sasakian manifold M with respect to quarter-symmetric metric connection, satisfying the condition $\tilde{Z} \cdot \tilde{Z} = 0$, then the manifold is flat if the scalar curvature of the manifold is $r = n(n-1)$.*

7. RICCI-GENERALIZED PSEUDO SYMMETRIC PARA-SASAKIAN MANIFOLD ADMITTING QUARET - SYMMETRIC METRIC CONNECTION

Definition 7.1. A para-Sasakian manifolds M is said to be Ricci-generalized pseudo symmetric if and only if the relation

$$R \cdot R = f(p)\, Q\,(S, R) \tag{7.1}$$

holds on the set $A = \{\, x \in M : Q(S,R) \neq 0 \ at \ \ x \,\}$, where $f \in C^\infty(A)$ for $p \in A$,$R \cdot R$ and $Q(S,R)$ are respectively defined by

$$\begin{aligned}(R(X,Y) \cdot R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad -R(U,R(X,Y)V)W - R(U,V)R(X,Y)W,\end{aligned} \tag{7.2}$$

$$Q(S,R) = (S(X,Y) \cdot R)(U,V)W = ((X \wedge_S Y) \cdot R)(U,V)W \tag{7.3}$$

for all $X, Y, U, V, W \in \chi(M), \chi(M)$ being the Lie algebra of all differentiable vector fields on M. Here the endomorphism $(X \wedge_S .Y)$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{7.4}$$

It is clear that any semisymmetric as well as any Ricci flat manifold is Ricci-generalized pseudosymmetric.

Theorem 7.1. *A Ricci-generalized pseudosymmetric para-Sasakian manifold with respect to quarter-symmetric metric connection, is either a space of constant curvature* 1 *or an η-Einstein manifold.*

Proof. Let us consider a Ricci-generalized pseudosymmetric para-Sasakian manifold with respect to quarter symmetric metric connection. Then in view of (7.1) and (7.3), we have

$$\tilde{R}(X,Y) \cdot \tilde{R})(U,V)W = f(p)\, \left\{ \tilde{S}(X,Y) \cdot \tilde{R})(U,V)W \right\}$$

for all $X, Y, U, V, W \in \chi(M)$. From above it is clear that

$$(\tilde{R}(X,\xi) \cdot \tilde{R})(U,V)W = f(p)\, \left\{ \tilde{S}(X,\xi) \cdot \tilde{R})(U,V)W \right\}. \tag{7.5}$$

In view of (2.7), (3.7) and (7.2), we get

$$\tilde{R}(X,\xi)\cdot\tilde{R})(U,V)W = 2\left\{\begin{array}{l}\tilde{R}'(U,V)W)\xi - \eta(\tilde{R}(U,V)W)X\\ -g(X,U)\tilde{R}(\xi,V)W + \eta(U)\tilde{R}(X,V,W)\\ -g(X,V)\tilde{R}(U,\xi,W) + \eta(V)\tilde{R}(U,X)W\\ -g(X,W)\tilde{R}(U,V)\xi + \eta(W)\tilde{R}(U,V)X\end{array}\right\} \tag{7.6}$$

where $\tilde{R}'(U,V,W,X) = g(\tilde{R}(U,V)W,X)$. Also from the definition (7.1), we have

$$(S(X,\xi)\cdot\tilde{R})(U,V)W = ((X\wedge_S\xi)\cdot\tilde{R})(U,V)W = (X\wedge_S\xi)\tilde{R}(U,V)W$$
$$+\tilde{R}((X\wedge_S\xi)U,V)W + \tilde{R}(U,(X\wedge_S\xi)V)W + \tilde{R}(U,V)(X\wedge_S\xi)W. \tag{7.7}$$

In view of (2.8), (3.9)and (7.4), it is clear from (7.7)

$$(S(X,\xi)\cdot\tilde{R})(U,V)W$$

$$\begin{array}{l}= (-2n+2)\left\{\begin{array}{l}\eta(\tilde{R}(U,V)W)X + \tilde{R}(X,V,W)\eta(U)\\ +\eta(V)\tilde{R}(U,X)W + \eta(W)\tilde{R}(U,V)X\end{array}\right\}\\ +(n+1)\left\{\begin{array}{l}\eta(X)\eta(\tilde{R}(U,V)W)\xi + \eta(X)\eta(V)R(\xi,V,W)\\ \eta(X)\eta(U)\tilde{R}(U,\xi,W) + \eta(X)\eta(W)\tilde{R}(U,V,\xi)\end{array}\right\}\\ -S(X,\tilde{R}(U,V)W)\xi - 2g(X,\tilde{R}(U,V,W)\xi - S(X,U)\tilde{R}(\xi,V,W)\\ -2g(X,V)\tilde{R}(\xi,V,W) - S(X,V)\tilde{R}(U,\xi,W) - 2g(X,V)\tilde{R}(U,\xi,W)\\ -S(X,W)\tilde{R}(U,V,\xi) - 2g(X,W)\tilde{R}(U,V)\xi\end{array} \tag{7.8}$$

Taking $W=\xi$ in (7.6) and (7.8) and by using it in (7.5), it is seen that

$$2\left\{\begin{array}{l}\tilde{R}'(U,V)\xi)\xi - \eta(\tilde{R}(U,V)\xi)X - g(X,U)\tilde{R}(\xi,V)\xi\\ +\eta(U)\tilde{R}(X,V,\xi) - g(X,V)\tilde{R}(U,\xi,\xi) + \eta(V)\tilde{R}(U,X)\xi\\ -\eta(X)\tilde{R}(U,V)\xi + \tilde{R}(U,V)X\end{array}\right\}$$

$$\begin{array}{l}= f(p)\,(-2n+2)\left\{\begin{array}{l}\eta(\tilde{R}(U,V)\xi)X + \tilde{R}(X,V,\xi)\eta(U)\\ +\eta(V)\tilde{R}(U,X)\xi + \tilde{R}(U,V)X\end{array}\right\}\\ +(n+1)f(p)\left\{\begin{array}{l}\eta(X)\eta(\tilde{R}(U,V)\xi)\xi + \eta(X)\eta(V)\tilde{R}(\xi,V,\xi)\\ \eta(X)\eta(U)\tilde{R}(U,\xi,\xi) + \eta(X)\tilde{R}(U,V,\xi)\end{array}\right\}\\ f(p)\{-S(X,\tilde{R}(U,V)\xi)\xi - 2g(X,\tilde{R}(U,V,\xi)\xi - S(X,U)\tilde{R}(\xi,V,\xi)\\ -2g(X,V)\tilde{R}(\xi,V,\xi) - S(X,V)\tilde{R}(U,\xi,\xi) - 2g(X,V)\tilde{R}(U,\xi,\xi)\\ -S(X,\xi)\tilde{R}(U,V,\xi) - 2g(X,\xi)\tilde{R}(U,V)\xi\}\end{array} \tag{7.9}$$

Taking the inner product on both side of (7.9) by ξ and using (2.8), (3.7) and $\eta(\tilde{R}(U,V)\xi = 0$, we get

$$f(p)\ \Big[\,(\,2(n+1)\,\eta\,(\tilde{R}(U,V)X) + S(X,\tilde{R}(U,V,\xi)\Big] = 0, \tag{7.10}$$

From which it follow that either

$$f(p) = 0,\ \text{or}\ S(X,\tilde{R}(U,V)\xi) = -2(n+1)\eta(\tilde{R}(U,V)X) \tag{7.11}$$

If $f(p) = 0$, then from (8.5), we have

$$\tilde{R}(U,V)W = g(V,W)U - g(U,W)V, \tag{7.12}$$

which means that the manifold is a space of constant curvature 1. Again substituting $U = \xi$ in (7.11) and applying (2.9) and (3.7), we obtain

$$S(X,V) = -2(n+1)g(X,Y) + (n+3)\eta(X)\eta(V) \tag{7.13}$$

for all X, V. This proves the Theorem 7.1.

Theorem 7.2. *Let (M^n, g) be a non-semisymmetric Ricci-generalized pseudosymmetric para-Sasakian manifold with respect to quarter symmetric metric connection, then the scalar curvature of the manifold is constant if and only if the space like vector field ξ is harmonic.*

Proof. Differentiating (7.13) covariantly along Y and using (2.6), we get

$$(\nabla_Y S)\,(X,V) = (n+3)\,\{g(Y,\phi X)\eta(V) + g(Y,\phi V)\eta(X)\} \tag{7.14}$$

Taking an orthomormal frame field and contracting (7.14) over Y and V, we have

$$dr(X) = 2(n+3)\Psi\eta(X),$$

where $\Psi = Tr.\phi$ and r is the scalar curvature of the manifold. From above relation, it follows that $dr(X) = 0$ if and only if $\Psi = 0$, which implies that ξ is harmonic.
This proves the Theorem 7.2.

Corollary 7.1.*Let (M^n, g) be a para-Sasakian manifold with respect to quarter symmetric metric connection satisfying the condition $\tilde{S}(X,\xi)\cdot\tilde{R} = 0$, then the manifold is locally isometric to the hyperbolic space $H^n(-1)$.*

Corollary 7.2.*Let (M^n, g) is a para-Sasakian manifold admitting quarter symmetric metric connection satisfying the condition $\tilde{S}(X,\xi)\cdot\tilde{R} = 0$, then the manifold is concircularly flat.*

In view of Corollary 7.1 and Corollary 7.2, we state the result as corollary.

Corollary 7.3. *A para-Sasakian manifold admitting quarter symmetric metric connection is concircularly flat then the manifold is locally isometric to the hyperbolic space $H^n(-1)$.*

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