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PROPERTIES OF IDEAL BITOPOLOGICAL α-OPEN SETS

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Abstract. The aim of this paper is to introduce and characterize some concepts of α-open sets and their related notions in ideal bitopological spaces.

Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. Introduction

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [10] and Vaidyanathasamy [11]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a bitopological space (X, τ_1, τ_2) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)_i^*$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [11] of A with respect to τ_i and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A_i^*(\tau_i, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I} \text{ for every } U \in \tau_i(x)\}$, where $\tau_i(x) = \{U \in \tau_i | x \in U\}$.

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For every ideal topological space $(X,\tau,\mathcal{I})$, there exists topology $\tau^*(I)$, finer than τ, generated by the base $\beta(\mathcal{I},\tau) = \{U\backslash I \mid U \in \tau$ and $I \in \mathcal{I}\}$, but in general $\beta(\mathcal{I},\tau)$ is not always a topology [7]. Observe additionally that τ_i-Cl$^*(A) = A \cup A_i^*(\tau_i,\mathcal{I})$ defines a Kuratowski closure operator for $\tau^*(\mathcal{I})$, when there is no chance of confusion, $A_i^*(\mathcal{I})$ is denoted by A_i^* and τ_i-Int$^*(A)$ denotes the interior of A in $\tau_i^*(\mathcal{I})$. The aim of this paper is to introduced and characterized the concepts of α-open sets and their related notions in ideal bitopological spaces.

2. Preliminaries

Let A be a subset of a bitopological space (X,τ_1,τ_2). We denote the closure of A and the interior of A with respect to τ_i by τ_i-Cl(A) and τ_i-Int(A), respectively.

Definition 2.1. *A subset A of a bitopological space (X,τ_1,τ_2) is said to be (i,j)-α-open* [9] *if $A \subset \tau_i$-Int$(\tau_j$-Cl$(\tau_i$-Int$(A)))$, where $i,j = 1,2$ and $i \neq j$.*

Definition 2.2. *A subset S of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be α-$\mathcal{I}$-open* [8] *if $S \subset$ Int(Cl*(Int(S))). The family of all α-$\mathcal{I}$-open sets of $(X,\tau,\mathcal{I})$ is denoted by $\alpha\mathcal{I}O(X,\tau)$.*

Definition 2.3. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-α-continuous* [9] *if the inverse image of every σ_j-open set in (Y,σ_1,σ_2) is (i,j)-α-open in $(X,\tau_1,\tau_2,\mathcal{I})$, where $i \neq j$, i,j=1, 2.*

Definition 2.4. *A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is said to be*

(i) *(i,j)-R-$\mathcal{I}$-open* [1] *if $A = \tau_i$-Int$(\tau_j$-Cl$^*(A))$.*
(ii) *(i,j)-semi-$\mathcal{I}$-open* [3] *if $A \subset \tau_j$-Cl$^*(\tau_i$-Int$(A))$.*
(iii) *(i,j)-pre-$\mathcal{I}$-open* [2] *if $A \subset \tau_i$-Int$(\tau_j$-Cl$^*(A))$.*
(iv) *(i,j)-b-$\mathcal{I}$-open* [4] *if $A \subset \tau_i$-Int$(\tau_j$-Cl$^*(A)) \cup \tau_j$-Cl$^*(\tau_i$-Int$(A))$.*
(v) *(i,j)-β-$\mathcal{I}$-open* [5] *if $A \subset \tau_j$-Cl$(\tau_i$-Int$(\tau_j$-Cl$^*(A)))$.*
(vi) *(i,j)-δ-$\mathcal{I}$-open* [1] *if τ_i-Int$(\tau_j$-Cl$^*(A) \subset \tau_j$-Cl$^*(\tau_i$-Int$(A))$.*

The complemenet of an (i,j)-pre-$\mathcal{I}$-open (resp. (i,j)-β-$\mathcal{I}$-open) set is called an (i,j)-pre-$\mathcal{I}$-closed (resp. (i,j)-β-$\mathcal{I}$-closed) set.

Lemma 2.5. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space. Then*

(i) *A subset A is (i,j)-pre-$\mathcal{I}$-closed if and only if τ_i-Cl$(\tau_j$-Int$^*(A)) \subset A$* [2];
(i) *A subset A is (i,j)-β-$\mathcal{I}$-closed if and only if τ_j-Int$(\tau_i$-Cl$(\tau_j$-Int$^*(A))) \subset A$* [5].

Definition 2.6. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(i) *pairwise pre-$\mathcal{I}$-continuous* [2] *if the inverse image of every σ_i-open set of Y is (i, j)-pre-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(i) *pairwise semi-$\mathcal{I}$-continuous* [3] *if the inverse image of every σ_i-open set of Y is (i, j)-semi-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(i) *pairwise b-$\mathcal{I}$-continuous* [4] *if the inverse image of every σ_i-open set of Y is (i, j)-b-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(i) *pairwise β-$\mathcal{I}$-continuous* [5] *if the inverse image of every σ_i-open set of Y is (i, j)-β-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(i) *pairwise δ-$\mathcal{I}$-continuous* [3] *if the inverse image of every σ_i-open set of Y is (i, j)-δ-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

(i) *pairwise strongly β-$\mathcal{I}$-continuous* [5] *if the inverse image of every σ_i-open set of Y is strongly (i, j)-β-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

3. (i, j)-α-$\mathcal{I}$-open sets

Definition 3.1. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be (i, j)-α-$\mathcal{I}$-open if and only if $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))$, where $i, j = 1, 2$ and $i \neq j$.*
The family of all (i, j)-α-$\mathcal{I}$-open sets of $(X, \tau_1, \tau_2, \mathcal{I})$ is denoted by $\alpha\mathcal{I}O(X, \tau_1, \tau_2)$ or (i, j)-$\alpha\mathcal{I}O(X)$. Also, The family of all (i, j)-α-$\mathcal{I}$-open sets of $(X, \tau_1, \tau_2, \mathcal{I})$ containing x is denoted by (i, j)-$\alpha\mathcal{I}O(X, x)$.

Remark 3.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on (X, τ_1, τ_2). If $\mathcal{I} \subset \mathcal{J}$, then $\alpha\mathcal{J}O(X, \tau_1, \tau_2) \subset \alpha\mathcal{I}O(X, \tau_1, \tau_2)$.*

Proposition 3.3. (i) *Every (i, j)-α-$\mathcal{I}$-open set is (i, j)-semi-$\mathcal{I}$-open.*
(ii) *Every (i, j)-α-$\mathcal{I}$-open set is (i, j)-α-open.*
(iii) *Every (i, j)-α-$\mathcal{I}$-open set is (i, j)-pre-$\mathcal{I}$-open.*
(iv) *Every (i, j)-α-$\mathcal{I}$-open set is (i, j)-b-$\mathcal{I}$-open.*
(v) *Every (i, j)-α-$\mathcal{I}$-open set is (i, j)-β-$\mathcal{I}$-open.*

Proof. The proof follows from the definitions. □

The following example show that the converses of Proposition 3.3 is not true in general.

Example 3.4. *Let* $X = \{a, b, c\}$ $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then the set* $\{a, c\}$ *is* (i, j)*-b-*$\mathcal{I}$*-open but not* (i, j)*-*α*-*$\mathcal{I}$*-open. Also, the set* $\{b, c\}$ *is* (i, j)*-semi-*$\mathcal{I}$*-open but not* (i, j)*-*α*-*$\mathcal{I}$*-open and the set* $\{a, c\}$ *is* (i, j)*-pre-*$\mathcal{I}$*-open and* (i, j)*-*α*-open but not* (i, j)*-*α*-*$\mathcal{I}$*-open.*

Proposition 3.5. *For an ideal bitopological space* $(X, \tau_1, \tau_2, \mathcal{I})$ *and* $A \subset X$ *we have:*

(i) *If* $\mathcal{I} = \{\emptyset\}$, *then* A *is* (i, j)*-*α*-*$\mathcal{I}$*-open if and only if* A *is* (i, j)*-*α*-open.*

(ii) *If* $\mathcal{I} = \mathcal{P}(X)$, *then* A *is* (i, j)*-*α*-*$\mathcal{I}$*-open if and only if* A *is* τ_i*-open.*

Proof. The proof follows from the fact that

(i) If $\mathcal{I} = \{\emptyset\}$, then $A^* = \mathrm{Cl}(A)$.

(ii) If $\mathcal{I} = \mathcal{P}(X)$, then $A^* = \emptyset$ for every subset A of X.

□

Proposition 3.6. *Let* A *be a subset of an ideal bitopological space* $(X, \tau_1, \tau_2, \mathcal{I})$. *If* B *is an* (i, j)*-semi-*$\mathcal{I}$*-open set of* X *such that* $B \subset A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(B))$, *then* A *is an* (i, j)*-*α*-*$\mathcal{I}$*-open set of* X.

Proof. Since B is an (i, j)-semi-$\mathcal{I}$-open set of X, we have $B \subset \tau_j\text{-Cl}^*(\tau_i\text{-Int}(B))$. Thus, $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(B)) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(B)))) = \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(B))) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))$, and so A is an (i, j)-α-$\mathcal{I}$-open set of X. □

Proposition 3.7. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space. Then a subset of*X *is* (i, j)*-*α*-*$\mathcal{I}$*-open if and only if it is both* δ*-*$\mathcal{I}$*-open and pre-*$\mathcal{I}$*-open.*

Proof. Let A be an (i, j)-α-$\mathcal{I}$-open set. Since every (i, j)-α-$\mathcal{I}$-open set is (i, j)-semi-$\mathcal{I}$-open, by Proposition 3.3 A is an (i, j)-δ-$\mathcal{I}$-open. Now we prove that $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$. Since A is an (i, j)-α-$\mathcal{I}$-open, we have $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$. Hence A is (i, j)-pre-$\mathcal{I}$-open. Conversely, let A be an (i, j)-δ-$\mathcal{I}$-open and (i, j)-pre-$\mathcal{I}$-open set. Then we have $\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \subset \tau_j\text{-Cl}^*(\tau_i\text{-Int}(A))$ and hence $\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A)) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))$. Since A is (i, j)-pre-$\mathcal{I}$-open, we have $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$. Therefore, we obtain that $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A)))$; hence A is (i, j)-α-$\mathcal{I}$-open. □

Lemma 3.8. *A subset* A *is* (i, j)*-*α*-*$\mathcal{I}$*-open if and only if* (i, j)*-semi-*$\mathcal{I}$*-open and* (i, j)*-pre-*$\mathcal{I}$*-open.*

Proof. Let A be (i,j)-semi-$\mathcal{I}$-open and (i,j)-pre-$\mathcal{I}$-open. Then, $A \subset \tau_i$-Int$(\tau_j$-Cl$^*(A)) \subset \tau_i$-Int$(\tau_j$-Cl$^*(\tau_j$-Cl$^*(\tau_i$-Int$(A)))) = \tau_i$-Int$(\tau_j$-Cl$^*(\tau_i$-Int$(A)))$. This shows that A is (i,j)-α-$\mathcal{I}$-open. The converse is obvious. $\square$

Corollary 3.9. *The following properties are equivalent for subsets of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$:*

(i) *Every (i,j)-pre-$\mathcal{I}$-open set is (i,j)-semi-$\mathcal{I}$-open.*
(ii) *A subset A of X is (i,j)-α-$\mathcal{I}$-open if and only if it is (i,j)-pre-$\mathcal{I}$-open.*

Corollary 3.10. *The following properties are equivalent for subsets of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$:*

(i) *Every (i,j)-semi-$\mathcal{I}$-open set is (i,j)-pre-$\mathcal{I}$-open.*
(ii) *A subset A of X is (i,j)-α-$\mathcal{I}$-open if and only if it is (i,j)-semi-$\mathcal{I}$-open.*

Proposition 3.11. *Let A be a subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$. If A is (i,j)-pre-$\mathcal{I}$-closed and (i,j)-α-$\mathcal{I}$-open, then it is τ_i-open.*

Proof. Suppose A is (i,j)-pre-$\mathcal{I}$-closed and (i,j)-α-$\mathcal{I}$-open. Then by Lemma 2.5 τ_i-Cl$(\tau_j$-Int$^*(A)) \subset A$ and $A \subset \tau_i$-Int$(\tau_j$-Cl$^*(\tau_i$-Int$(A)))$. Now τ_i-Cl$(\tau_i$-Int$(A)) \subset \tau_i$-Cl$(\tau_i$-Int$(A)) \subset \tau_i$-Cl$(\tau_i$-Int$^*(A)) \subset A$ and so $A \subset \tau_i$-Int$(\tau_i$-Cl$^*(tau_i - \text{Int}(A)) \subset A \subset \tau_i$-Int$(A)$. Therefore, A is τ_i-open. $\square$

Lemma 3.12. [1] *If A is any subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, then τ_i-*Int*$(\tau_j$-*Cl*$^*(A))$ is (i,j)-R-$\mathcal{I}$-open.*

Proposition 3.13. *Let A be a subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$. If A is (i,j)-α-$\mathcal{I}$-open and (i,j)-β-$\mathcal{I}$-closed, then it is (i,j)-R-$\mathcal{I}$-open.*

Proof. Let A be (i,j)-α-$\mathcal{I}$-open and (i,j)-β-$\mathcal{I}$-closed. We have by Lemma 2.5, $A \subset \tau_j$-Int$(\tau_i$-Cl$^*(\tau_j$-Int$(A)))$ and τ_j-Int$(\tau_i$-Cl$^*(\tau_j$-Int$(A))) \subset \tau_j$-Int$(\tau_i$-Cl$(\tau_j$-Int$^*(A))) \subset A$; hence $A = \tau_j$-Int$(\tau_i$-Cl$^*(\tau_j$-Int$(A)))$. Thus, by Lemma 3.12, A is (i,j)-R-$\mathcal{I}$-open. $\square$

An ideal bitopological space is said to satisfy the condition $(\mathcal{A})$ if $U \cap \tau_j$-Cl$^*(A) \subset \tau_j$-Cl$^*(U \cap A)$ for every $U \in \tau_i$.

Theorem 3.14. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space that satisfies the condition $(\mathcal{A})$. Then we have the following*

(i) *If* $V \in (i,j)$-$\alpha\mathcal{I}O(X)$ *and* $A \in (i,j)$-$\alpha\mathcal{I}O(X)$, *then* $V \cap A \in (i,j)$-$\alpha\mathcal{I}O(X)$.

(ii) *If* $V \in (i,j)$-$\alpha\mathcal{I}O(X)$ *and* $A \in (i,j)$-$\alpha\mathcal{I}O(X)$, *then* $V \cap A \in (i,j)$-$\alpha\mathcal{I}O(X)$.

Proof. (i). Let $V \in (i,j)$-$\alpha\mathcal{I}O(X)$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)$. Then $V \cap A \subset \tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(V)) \cap \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}\, A))) \subset \tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(V) \cap \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A)))) \subset \tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(V) \cap \tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A))) \subset \tau_j$-$\mathrm{Cl}^*(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(V) \cap \tau_i$-$\mathrm{Int}(A))) \subset \tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(V))$. This shows that $V \cap A \in (i,j)$-$\alpha\mathcal{I}O(X)$.
(ii). Let $V \in (i,j)$-$P\mathcal{I}O(X)$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)$. Then $V \cap A \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A)) \cap \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A)))) = \tau_i$-$\mathrm{Int}(\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(V)) \cap \tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A))) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(V)) \cap \tau_i$-$\mathrm{Int}(A))) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_j$-$\mathrm{Cl}^*(V) \cap \tau_i$-$\mathrm{Int}(A))) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_j$-$\mathrm{Cl}^*(V \cap \tau_i$-$\mathrm{Int}(A)))) \subset \tau_i$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}^*(V \cap V))$. This shows that $V \cap A \in (i,j)$-$P\mathcal{I}O(X)$. $\square$

Remark 3.15. *The intersection of two* (i,j)-α-$\mathcal{I}$*-open sets need not be* (i,j)-α-$\mathcal{I}$*-open as it can be seen from the following example.*

Example 3.16. *Let* $X = \{a,b,c,d\}$, $\tau_1 = \{\varnothing,\ \{a\},\ \{b\},\ \{a,b\},\ \{a,b,c\},\ X\}$, $\tau_2 = \{\varnothing,\ X\}$ *and* $\mathcal{I} = \{\varnothing, \{c\}, \{d\}, \{c,d\}\}$. *Then the sets* $\{a,c\}$ *and* $\{b,c\}$ *are* $(1,2)$-α-$\mathcal{I}$*-open sets of* $(X,\tau_1,\tau_2,\mathcal{I})$ *but their intersection* $\{c\}$ *is not an* $(1,2)$-α-$\mathcal{I}$*-open set of* $(X,\tau_1,\tau_2,\mathcal{I})$.

Theorem 3.17. *If* $\{A_\alpha\}_{\alpha\in\Omega}$ *be a family of* (i,j)-α-$\mathcal{I}$*-open sets in* $(X,\tau_1,\tau_2,\mathcal{I})$, *then* $\bigcup\limits_{\alpha\in\Omega} A_\alpha$ *is* (i,j)-α-$\mathcal{I}$*-open in* $(X,\tau_1,\tau_2,\mathcal{I})$.

Proof. Since $\{A_\alpha : \alpha \in \Omega\} \subset (i,j)$-$\alpha\mathcal{I}O(X)$, then $A_\alpha \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A_\alpha)))$ for every $\alpha \in \Omega$. Thus, $\underset{\alpha\in\Omega}{\cup} A_\alpha \subset \underset{\alpha\in\Omega}{\cup} \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A_\alpha))) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\underset{\alpha\in\Omega}{\cup} \tau_i$-$\mathrm{Int}(A_\alpha)) = \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(\underset{\alpha\in\Omega}{\cup} A_\alpha))$. Therefore, we obtain $\underset{\alpha\in\Omega}{\cup} A_\alpha \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(\underset{\alpha\in\Omega}{\cup} A_\alpha))$. Hence any union of (i,j)-α-$\mathcal{I}$-open sets is (i,j)-α-$\mathcal{I}$-open. $\square$

If $(X,\tau,\mathcal{I})$ is an ideal topological space and A is a subset of X, we denote by $\tau_{|A}$, the relative topology on A and $\mathcal{I}_{|A} = \{A \cap I \in \mathcal{I}\}$ is obviously an ideal on A.

Theorem 3.18. *Let* $(X,\tau_1,\tau_2,\mathcal{I})$ *be an ideal bitopological space satisfies the condition* $(\mathcal{A})$. *If* $A \in (i,j)$-$\alpha\mathcal{I}O(X)$ *and* $A \subset B \in (i,j)$-$\alpha\mathcal{I}O(X)$, *Then* $A \in (i,j)$-$\alpha\mathcal{I}_{|B}O(B)$.

Proof. By definition, $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(A \cap B))) \cap B = \tau_i\text{-Int}_B(\tau_i\text{-Int}(\tau_j\text{-Cl}^*(A \cap B))) \cap B) \subset \tau_i\text{-Int}_B(\tau_j\text{-Cl}^*(A \cap B)) \cap B) = \tau_i\text{-Int}_B(\tau_j\text{-Cl}^*_B(\tau_i\text{-Int}(A) \cap \tau_i\text{-Int}(B))) \subset \tau_i\text{-Int}_B(\tau_j\text{-Cl}^*_B(\tau_i\text{-Int}(A) \cap B)) = \tau_i\text{-Int}_B(\tau_i\text{-Cl}^*_B(\tau_i\text{-Int}_B(\tau_i\text{-Int}(A) \cap B))) \subset \tau_i\text{-Int}_B(\tau_j\text{-Cl}^*_B(\tau_i\text{-Int}_B(A)))$. This shows that $A \in (i,j)\text{-}\alpha\mathcal{I}_{|B}O(B)$. □

Definition 3.19. *In an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, $A \subset X$ is said to be (i,j)-α-$\mathcal{I}$-closed if $X \backslash A$ is (i,j)-α-$\mathcal{I}$-open in X, $i, j = 1, 2$ and $i \neq j$.*

Theorem 3.20. *If A is an (i,j)-α-$\mathcal{I}$-closed set in an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ if and only if $\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(A))) \subset A$.*

Proof. The proof follows from the definitions. □

Theorem 3.21. *If A is an (i,j)-α-$\mathcal{I}$-closed set in an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, then $\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}^*(A))) \subset A$.*

Proof. Since $A \in (i,j)\text{-}\alpha\mathcal{I}C(X)$, $X \backslash A \in (i,j)\text{-}\alpha\mathcal{I}O(X)$. Hence, $X \backslash A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(X \backslash A)) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}(X \backslash A))) = X \backslash \tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}(A))) \subset X \backslash (\tau_i\text{-Cl}(\tau_j\text{-Int}(\tau_i\text{-Cl}^*(A)))$. Therefore, we obtain $\tau_i\text{-Cl}(\tau_i\text{-Int}(\tau_i\text{-Cl}^*(A)) \subset A$. □

Proposition 3.22. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space. If a subset of X is (i,j)-β-$\mathcal{I}$-closed and (i,j)-δ-$\mathcal{I}$-open, then it is (i,j)-α-$\mathcal{I}$-closed.*

Proof. The proof follows from the definitions. □

Theorem 3.23. *Arbitrary intersection of (i,j)-α-$\mathcal{I}$-closed sets is always (i,j)-α-$\mathcal{I}$-closed.*

Proof. Follows from Theorems 3.17 and 3.21. □

Definition 3.24. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(i) *x is called an (i,j)-α-$\mathcal{I}$-interior point of S if there exists $V \in (i,j)\text{-}\alpha\mathcal{I}O(X, \tau_1, \tau_2)$ such that $x \in V \subset S$.*

ii) *the set of all (i,j)-α-$\mathcal{I}$-interior points of S is called (i,j)-α-$\mathcal{I}$-interior of S and is denoted by $(i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(S)$.*

Theorem 3.25. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

(i) *$(i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(A) = \cup\{T : T \subset A \text{ and } A \in (i,j)\text{-}\alpha\mathcal{I}O(X)\}$.*

(ii) *$(i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(A)$ is the largest (i,j)-α-$\mathcal{I}$-open subset of X contained in A.*

(iii) *A is (i,j)-α-$\mathcal{I}$-open if and only if $A = (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$.*
(iv) (i,j)-$\alpha\mathcal{I}\operatorname{Int}((i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)) = (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$.
(v) *If $A \subset B$, then (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B)$.*
(vi) (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A \cap B) = (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A) \cap (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B)$.
(vii) (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A \cup B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A) \cup (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B)$.

Proof. (i). Let $x \in \cup\{T : T \subset A$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)\}$. Then, there exists $T \in (i,j)$-$\alpha\mathcal{I}O(X,x)$ such that $x \in T \subset A$ and hence $x \in (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)\} \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$. For the reverse inclusion, let $x \in (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$. Then there exists $T \in (i,j)$-$\alpha\mathcal{I}O(X,x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)\}$. This shows that (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \subset \cup\{T : T \subset A$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)\}$. Therefore, we obtain (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) = \cup\{T : T \subset A$ and $A \in (i,j)$-$\alpha\mathcal{I}O(X)\}$.
The proof of $(ii) - (v)$ are obvious.
(vi). By (v), we have (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A \cup B)$ and (i,j)-$\alpha\mathcal{I}\operatorname{Int}(B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A \cup B)$. Then we obtain (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A)$ $\cup$ (i,j)-$\alpha\mathcal{I}\operatorname{Int}(B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A \cup B)$ Since (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \subset A$ and (i,j)-$\alpha\mathcal{I}\operatorname{Int}(B) \subset B$, we obtain (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A \cup B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A) \cup (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B)$. It follows that (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A \cap B) = (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A) \cap (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B)$.
(vii). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (v), we have (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A \cap B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$ and (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A \cap B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B)$. Therefore, (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \cup (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(B) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A \cap B)$. ☐

Theorem 3.26. *If $(X, \tau_1, \tau_2, \mathcal{I})$ is an ideal bitopological space satisfying the condition $(\mathcal{A})$, then (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) = A \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)))$ holds for every subset A of X.*

Proof. Since $A \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A))) \subset \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A))) = \tau_i$-$\operatorname{Int}(\tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)))) = \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)) \cap (\tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)))) \subset \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A) \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A))))) = \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A))))))$, $A \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)))$ is an (i,j)-α-$\mathcal{I}$-open set contained in A and so $A \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A))) \subset (i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)$. Since (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A)$ is (i,j)-α-$\mathcal{I}$-open, (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \subset \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}((i,j)$-$\alpha\mathcal{I}\operatorname{Int}(A)))) \subset \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)))$ and so (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) \subset A \cap \tau_i$-$\operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\operatorname{Int}(A)))$. Hence (i,j)-$\alpha\mathcal{I}\operatorname{Int}(A) = A \cap \operatorname{Int}(\tau_j$-$\operatorname{Cl}^*(\tau_i$-$\operatorname{Int}(A)))$. ☐

Definition 3.27. *The union of all (i,j)-pre-$\mathcal{I}$-open sets of $(X,\tau_1,\tau_2,\mathcal{I})$ containing A is called the (i,j)-pre-$\mathcal{I}$-interior of A and is denote by (i,j)-$p\mathcal{I}\,\mathrm{Int}(A)$.*

Lemma 3.28. *If $(X,\tau_1,\tau_2,\mathcal{I})$ is an ideal bitopological space satisfies the condition $(\mathcal{A})$, then (i,j)-$p\mathcal{I}\,\mathrm{Int}(A) = A\cap\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A))$ holds for every subset A of X.*

Theorem 3.29. *If $(X,\tau_1,\tau_2,\mathcal{I})$ is an ideal bitopological space satisfies the condition $(\mathcal{A})$, then (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(A) = (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$ holds for every (i,j)-δ-$\mathcal{I}$-open subset A of X.*

Proof. Since every (i,j)-α-$\mathcal{I}$-open set is (i,j)-pre-$\mathcal{I}$-open, (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(A)\subset(i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$. By Theorem 3.26, $\alpha\mathcal{I}\,\mathrm{Int}(A) = A\cap\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(\tau_i$-$\mathrm{Int}(A)))$. Since A is (i,j)-δ-$\mathcal{I}$-open, (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(A)\supset A\cap\tau_i$-$\mathrm{Int}(\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A))) = A\cap\tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}^*(A)) = (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$ by Lemma 3.28 and so (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(A)\supset(i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$. Therefore, (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(A) = (i,j)$-$p\mathcal{I}\,\mathrm{Int}(A)$. □

Definition 3.30. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(i) *x is called an (i,j)-α-$\mathcal{I}$-cluster point of S if $V\cap S\neq\emptyset$ for every $V\in(i,j)$-$\alpha\mathcal{I}O(X,x)$.*
(ii) *the set of all (i,j)-α-$\mathcal{I}$-cluster points of S is called (i,j)-α-$\mathcal{I}$-closure of S and is denoted by (i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(S)$.*

Theorem 3.31. *Let A and B be subsets of $(X,\tau_1,\tau_2,\mathcal{I})$. Then the following properties hold:*

(i) *(i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A\subset F$ and $F\in(i,j)$-$\alpha\mathcal{I}C(X)\}$.*
(ii) *(i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest (i,j)-α-$\mathcal{I}$-closed subset of X containing A.*
(iii) *A is (i,j)-α-$\mathcal{I}$-closed if and only if $A = (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$.*
(iv) *(i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A) = (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$.*
(v) *If $A\subset B$, then (i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A)\subset(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(B)$.*
(vi) *(i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A\cup B) = (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)\cup(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(B)$.*
(vii) *(i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A\cap B)\subset(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)\cap(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(B)$.*

Proof. (*i*). Suppose that $x\notin(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $F\in(i,j)$-$\alpha\mathcal{I}O(X)\}$ such that $V\cap S\neq\emptyset$. Since $X\backslash V$ is (i,j)-α-$\mathcal{I}$-closed set containing A and $x\notin X\backslash V$, we obtain $x\notin\cap\{F : A\subset F$ and $F\in(i,j)$-$\alpha\mathcal{I}C(X)\}$. Then there exists $F\in(i,j)$-$\alpha\mathcal{I}C(X)$ such that $A\subset F$ and $x\notin F$. Since $X\backslash V$ is (i,j)-α-$\mathcal{I}$-closed set containing x, we obtain $(X\backslash F)\cap A = \emptyset$. This shows that $x\notin(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$.

Therefore, we obtain (i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i,j)$-$\alpha\mathcal{I}C(X)\}$.
The other proofs are obvious. □

Theorem 3.32. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. A point $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\alpha\mathcal{I}O(X, x)$.*

Proof. Suppose that $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\alpha\mathcal{I}O(X, x)$. Suppose that there exists $U \in (i,j)$-$\alpha\mathcal{I}O(X, x)$ such that $U \cap A = \emptyset$. Then $A \subset X\backslash U$ and $X\backslash U$ is (i,j)-α-$\mathcal{I}$-closed. since $A \subset X\backslash U$, (i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(X\backslash U)$. Since $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$, we have $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(X\backslash U)$. Since $X\backslash U$ is (i,j)-α-$\mathcal{I}$-closed, we have $x \in X\backslash U$; hence $x \notin U$, which is a contradicition that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\alpha\mathcal{I}O(X, x)$. We shall show that $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Suppose that $x \notin (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $U \in (i,j)$-$\alpha\mathcal{I}O(X, x)$ such that $U \cap A = \emptyset$. This is a contradicition to $U \cap A \neq \emptyset$; hence $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. □

Theorem 3.33. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. Then the following propeties hold:*

(i) (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$;
(i) (i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(X\backslash A) = X\backslash(i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(A)$.

Proof. (i). Let $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Since $x \notin (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$, there exists $V \in (i,j)$-$\alpha\mathcal{I}O(X, x)$ such that $V \cap A \neq \emptyset$; hence we obtain $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Let $x \in (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Since (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(X\backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X\backslash(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(A)$.
(ii). Follows from (i). □

Theorem 3.34. *If $(X, \tau_1, \tau_2, \mathcal{I})$ is an ideal bitopological space satisfies the condition $(\mathcal{A})$, then (i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(A) = A \cup \tau_i$-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}^*(\tau_i$-$\mathrm{Cl}(A)))$ holds for every subset A of X.*

Proof. The proof follows from the definitions. □

Definition 3.35. *A subset B_x of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be an (i,j)-α-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an (i,j)-α-$\mathcal{I}$-open set U such that $x \in U \subset B_x$.*

Theorem 3.36. *A subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-α-$\mathcal{I}$-open if and only if it is an (i,j)-α-$\mathcal{I}$-neighbourhood of each of its points.*

Proof. Let G be an (i,j)-α-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is an (i,j)-α-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is (i,j)-α-$\mathcal{I}$-open. Conversely, suppose G is an (i,j)-α-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in (i,j)$-$\alpha\mathcal{I}O(X)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is (i,j)-α-$\mathcal{I}$-open, G is (i,j)-α-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$. □

Proposition 3.37. *The product of two (i,j)-α-$\mathcal{I}$-open sets is (i,j)-α-$\mathcal{I}$-open.*

Proof. The proof follows from Lemma 3.3 of [12]. □

4. PAIRWISE α-$\mathcal{I}$-CONTINUOUS FUNCTIONS

Definition 4.1. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-α-$\mathcal{I}$-continuous if the inverse image of every σ_i-open set of Y is (i,j)-α-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

Proposition 4.2.
(i) *Every (i,j)-α-$\mathcal{I}$-continuous function is (i,j)-semi-$\mathcal{I}$-continuous but not conversely.*
(ii) *Every (i,j)-α-$\mathcal{I}$-continuous function is (i,j)-α-continuous but not conversely.*
(iii) *Every (i,j)-α-$\mathcal{I}$-continuous function is (i,j)-pre-$\mathcal{I}$-continuous but not conversely.*
(iv) *Every (i,j)-α-$\mathcal{I}$-continuous function is (i,j)-b-$\mathcal{I}$-continuous but not conversely.*
(v) *Every (i,j)-α-$\mathcal{I}$-continuous function is (i,j)-β-$\mathcal{I}$-continuous but not conversely.*

Proof. The proof follows from Proposition 3.3 and Example 3.4. □

Theorem 4.3. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)-α-$\mathcal{I}$-continuous if and only if it is (i,j)-semi-$\mathcal{I}$-continuous and (i,j)-pre-$\mathcal{I}$-continuous.*

Proof. This is an immediate consequence of Lemma 3.8. □

Theorem 4.4. *For a function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:*

(i) *f is pairwise α-$\mathcal{I}$-continuous;*
(ii) *For each point x in X and each σ_i-open set F in Y such that $f(x) \in F$, there is a (i,j)-α-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;*

(iii) *The inverse image of each σ_i-closed set in Y is (i,j)-α-$\mathcal{I}$-closed in X;*
(iv) *For each subset A of X, $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_i\text{-}\mathrm{Cl}(f(A))$;*
(v) *For each subset B of Y, $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))$;*
(vi) *For each subset C of Y, $f^{-1}(\sigma_i\text{-}\mathrm{Int}(C)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(C))$.*
(vii) $\tau_i\text{-}\mathrm{Cl}(\tau_j\text{-}\mathrm{Int}^*(\tau_i\text{-}\mathrm{Cl}(f^{-1}(B)))) \subset f^{-1}(\tau_i\text{-}\mathrm{Cl}(B))$ *for each subset B of Y.*
(viii) $f(\tau_i\text{-}\mathrm{Cl}(\tau_j\text{-}\mathrm{Int}^*(\tau_i\text{-}\mathrm{Cl}(A)))) \subset \tau_i\text{-}\mathrm{Cl}(f(A))$ *for each subset A of X.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Let $x \in X$ and F be a σ_j-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is (i,j)-α-$\mathcal{I}$-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.
(ii) $\Rightarrow$ *(i)*: Let F be σ_j-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is an (i,j)-α-$\mathcal{I}$-open set U_x in X such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is (i,j)-α-$\mathcal{I}$-open in X.
(i) $\Leftrightarrow$ *(iii)*: This follows due to the fact that for any subset B of Y, $f^{-1}(Y\backslash B) = X\backslash f^{-1}(B)$.
(iii) $\Rightarrow$ *(iv)*: Let A be a subset of X. Since $A \subset f^{-1}(f(A))$ we have $A \subset f^{-1}(\sigma_j\text{-}\mathrm{Cl}(f(A)))$. Now, $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f(A))$ is σ_j-closed in Y and hence $f^{-1}(\sigma_j\text{-}\mathrm{Cl}(A)) \subset f^{-1}(\sigma_j\text{-}\mathrm{Cl}(f(A)))$, for $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest (i,j)-α-$\mathcal{I}$-closed set containing A. Then $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_j\text{-}\mathrm{Cl}(f(A))$.
(iv) $\Rightarrow$ *(iii)*: Let F be any (i,j)-α-$\mathcal{I}$-closed subset of Y. Then $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(F))) \subset (i,j)\text{-}\sigma_i\text{-}\mathrm{Cl}(f(f^{-1}(F))) = (i,j)\text{-}\sigma_i\text{-}\mathrm{Cl}(F) = F$. Therefore, $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is (i,j)-α-$\mathcal{I}$-closed in X.
(iv) $\Rightarrow$ *(v)*: Let B be any subset of Y. Now, $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(B))) \subset (i,j)\text{-}\sigma_i\text{-}\mathrm{Cl}(f(f^{-1}(B))) \subset \sigma_i\text{-}\mathrm{Cl}(B)$. Consequently, $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))$.
(v) $\Rightarrow$ *(iv)*: Let $B = f(A)$ where A is a subset of X. Then, $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B)) = f^{-1}(\sigma_i\text{-}\mathrm{Cl}(f(A)))$. This shows that $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_i\text{-}\mathrm{Cl}(f(A))$.
(i) $\Rightarrow$ *(vi)*: Let B be a σ_j-open set in Y. Clearly, $f^{-1}(\sigma_i\text{-}\mathrm{Int}(B)$ is (i,j)-α-$\mathcal{I}$-open and we have $f^{-1}(\sigma_i\text{-}\mathrm{Int}(B)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}\sigma_i\text{-}\mathrm{Int}(B)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}B)$.
(vi) $\Rightarrow$ *(i)*: Let B be a σ_j-open set in Y. Then $\sigma_i\text{-}\mathrm{Int}(B) = B$ and $f^{-1}(B)\backslash f^{-1}(\sigma_i\text{-}\mathrm{Int}(B)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. Hence we have

$f^{-1}(B) = (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is (i,j)-α-$\mathcal{I}$-open in X.
(iii) $\Rightarrow$ *(vii)*: Let B be any subset os Y. Since τ_i-Cl(B) is τ_i-closed in Y, by (iii), $f^{-1}(\tau_i\text{-Cl}(B))$ is α-$\mathcal{I}$-closed and $X\backslash f^{-1}(\tau_i\text{-Cl}(B))$ is α-$\mathcal{I}$-open. Then $X\backslash f^{-1}(\tau_i\text{-Cl}(B)) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(f^{-1}(\tau_i\text{-Cl}(B))))) = X\backslash\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(f^{-1}(\tau_i\text{-Cl}(B)))))$. Hence we obtain $\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(f^{-1}(B)))) \subset f^{-1}(\tau_i\text{-Cl}(B))$.
(vii) $\Rightarrow$ *(viii)*: Let A be any ubset of X. By(iv), we have $\mathrm{Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(A)) \subset \tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(f^{-1}(f(A))))) \subset f^{-1}(\tau_i\text{-Cl}(f(A)))$ and hence $f(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(A)))) \subset \tau_i\text{-Cl}(f(A))$.
(viii) $\Rightarrow$ *(i)*: Let V be any open set of Y. Then by (v), $f(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(f^{-1}(Y\backslash V))))) \subset \tau_i\text{-Cl}(f(f^{-1}(Y\backslash V))) \subset \tau_i\text{-Cl}(Y\backslash V) = Y\backslash V$. Therefore, we have $\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(f^{-1}(Y\backslash V)))) \subset f^{-1}(Y\backslash V) \subset X\backslash f^{-1}(V)$. Consequently, we obtain $f^{-1}(V) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(f^{-1}(V))))$. This shows that $f^{-1}(V)$ is α-$\mathcal{I}$-open. Thus, f is α-$\mathcal{I}$-continuous. □

Corollary 4.5. *Let $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{I})$ be an (i,j)-α-$\mathcal{I}$-continuous function, then*

(i) $f(\tau_j\text{-Cl}^*(U)) \subset \tau_j\text{-Cl}(f(U))$ *for every (i,j)-pre-$\mathcal{I}$-open set U of X,*
(ii) $\tau_j\text{-Cl}^*(f^{-1}(V)) \subset f^{-1}(\tau_j\text{-Cl}(V))$ *for every (i,j)-pre-$\mathcal{I}$-open set V of Y.*

Proof. (1). Let U be any (i,j)-pre-$\mathcal{I}$-open set of X, then $U \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(U))$. Therefore, by Theorem 4.4, we have $f(\tau_j\text{-Cl}^*(U)) \subset f(\tau_j\text{-Cl}(U)) \subset f(\tau_j\text{-Cl}(\tau_j\text{-Int}(\tau_j\text{-Cl}^*(U)))) \subset f(\tau_j\text{-Cl}(\tau_j\text{-Int}^*(\tau_j\text{-Cl}(U)))) \subset \tau_j\text{-Cl}(f(U))$.
(2). Let V be any (i,j)-pre-$\mathcal{I}$-open set of Y. By Theorem 4.4, $\tau_j\text{-Cl}^*(f^{-1}(V)) \subset \tau_j\text{-Cl}(f^{-1}(V)) \subset \tau_j\text{-Cl}(f^{-}(\tau_j\text{-Int}(\tau_j\text{-Cl}^*(V)))) \subset \tau_j\text{-Cl}(\tau_j\text{-Int}(\tau_j\text{-Cl}^*(\tau_j\text{-Int}(f^{-1}(\tau_j\text{-Int}(\tau_j\text{-Cl}^((V)))))))) \subset \tau_j\text{-Cl}(\tau_j\text{-Int}^*(\tau_j\text{-Cl}(f^{-1}(\tau_j\text{-Int}(\tau_j\text{-Cl}^((V))))))) \subset f^{-1}(\tau_j\text{-Cl}(\tau_j\text{-Int}(\tau_j\text{-Cl}^(V)))) \subset f^{-1}(\tau_j\text{-Cl}(V))$. □

Theorem 4.6. *Let $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2)$ be a pairwise α-$\mathcal{I}$-continuous function. Then for each subset V of Y, $f^{-1}(\sigma_i\text{-Int}(V)) \subset \tau_j\text{-Cl}^*(f^{-1}(V))$.*

Proof. Let V be any subset of Y. Then σ_i-Int(V) is σ_i-open in Y and so $f^{-1}(\sigma_i\text{-Int}(V))$ is (i,j)-α-$\mathcal{I}$-open in X. Hence $f^{-1}(\sigma_i\text{-Int}(V)) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(\tau_i\text{-Int}(f^{-1}(\sigma_i\text{-Int}(V))))) \subset \tau_j\text{-Cl}^*(f^{-1}(V))$. □

Theorem 4.7. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be a bijective. Then f is pairwise α-$\mathcal{I}$-continuous if and only if σ_i-$\mathrm{Int}(f(U)) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U))$ for each subset U of X.*

Proof. Let U be any subset of X. Then by Theorem 4.4, $f^{-1}(\sigma_i$-$\mathrm{Int}(f(U))) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(f(U)))$. Since f is bijection, σ_i-$\mathrm{Int}(f(U)) = f(f^{-1}(\sigma_i$-$\mathrm{Int}(f(U))) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U))$. Conversely, let V be any subset of Y. Then σ_i-$\mathrm{Int}(f(f^{-1}(V))) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)))$. Since f is bijection, σ_i-$\mathrm{Int}(V) = \sigma_i$-$\mathrm{Int}(f(f^{-1}(V))) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)))$; hence $f^{-1}(\sigma_i$-$\mathrm{Int}(V) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$. Therefore, by Theorem 4.4, f is pairwise α-$\mathcal{I}$-continuous. $\square$

Theorem 4.8. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be a function. If $g : (X, \tau_1, \tau_2, \mathcal{I}) \to (X \times Y, \sigma_1 \times \sigma_2)$ defined by $g(x) = (x, f(x))$ is a pairwise α-$\mathcal{I}$-continuous function, then f is pairwise α-$\mathcal{I}$-continuous.*

Proof. Let V be a σ_i-open set of Y. Then $f^{-1}(V) = X \cap f^{-1}(V) = g^{-1}(X \times V)$. Since g is a pairwise α-$\mathcal{I}$-continuous function and $X \times V$ is a $\tau_i \times \sigma_i$-open set of $X \times Y$, $f^{-1}(V)$ is a (i,j)-α-$\mathcal{I}$-open set of X. Hence f is pairwise α-$\mathcal{I}$-continuous. $\square$

Definition 4.9. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ is said to be:*

(i) *pairwise α-$\mathcal{I}$-open (resp. pairwise semi-$\mathcal{I}$-open* [3], *pairwise pre-$\mathcal{I}$-open* [6]*) if $f(U)$ is a (i,j)-α-$\mathcal{I}$-open (resp. (i,j)-semi-$\mathcal{I}$-open, (i,j)-pre-$\mathcal{I}$-open) set of Y for every τ_i-open set U of X.*

(ii) *pairwise α-$\mathcal{I}$-closed (resp. pairwise semi-$\mathcal{I}$-closed* [3], *pairwise pre-$\mathcal{I}$-closed* [6]*) if $f(U)$ is a (i,j)-α-$\mathcal{I}$-closed set of Y for every τ_i-closed set U of X.*

Theorem 4.10. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ is (i,j)-α-$\mathcal{I}$-open if and only if it is (i,j)-semi-$\mathcal{I}$-open and (i,j)-pre-$\mathcal{I}$-open.*

Proof. This is an immediate consequence of Lemma 3.8. $\square$

Theorem 4.11. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$, the following statements are equivalent:*

(i) *f is pairwise α-$\mathcal{I}$-open;*

(ii) *$f(\tau_i$-$\mathrm{Int}(U)) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f(U))$ for each subset U of X;*

(iii) *τ_i-$\mathrm{Int}(f^{-1}(V)) \subset f^{-1}((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(V))$ for each subset V of Y.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Let U be any subset of X. Then τ_i-$\mathrm{Int}(U)$ is a τ_i-open set of X. Then $f(\tau_i$-$\mathrm{Int}(U))$ is a (i,j)-α-$\mathcal{I}$-open set of

Y. Since $f(\tau_i\text{-Int}(U)) \subset f(U)$, $f(\tau_i\text{-Int}(U)) = (i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(\tau_i\text{-Int}(U))) \subset (i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(U))$.
$(ii) \Rightarrow (iii)$: Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. Hence $f(\tau_i\text{-Int}(f^{-1}(V))) \subset (i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(f^{-1}(V))) \subset (i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(V))$. Then $\tau_i\text{-Int}(f^{-1}(V)) \subset f^{-1}(f(\tau_i\text{-Int}(f^{-1}(V)))) \subset f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(V))$.
$(iii) \Rightarrow (i)$: Let U be any τ_i-open set of X. Then $\tau_i\text{-Int}(U) = U$ and $f(U)$ is a subset of Y. Now, $V = \tau_i\text{-Int}(V) \subset \tau_i\text{-Int}(f^{-1}(f(V))) \subset f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(V)))$. Then $f(V) \subset f(f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(V)))) \subset (i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(V))$ and $(i,j)\text{-}\alpha\mathcal{I}\,\text{Int}(f(V)) \subset f(V)$. Hence $f(V)$ is a (i,j)-α-$\mathcal{I}$-open set of Y; hence f is pairwise α-$\mathcal{I}$-open. □

Theorem 4.12. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then f is a pairwise α-$\mathcal{I}$-closed function if and only if for each subset V of X, $(i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(V)) \subset f(\tau_i\text{-Cl}(V))$.*

Proof. Let f be a pairwise α-$\mathcal{I}$-closed function and V any subset of X. Then $f(V) \subset f(\tau_i\text{-Cl}(V))$ and $f(\tau_i\text{-Cl}(V))$ is a (i,j)-α-$\mathcal{I}$-closed set of Y. We have $(i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(V)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(\tau_i\text{-Cl}(V))) = f(\tau_i\text{-Cl}(V))$. Conversely, let V be a τ_i-open set of X. Then $f(V) \subset (i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(V)) \subset f(\tau_i\text{-Cl}(V)) = f(V)$; hence $f(V)$ is a (i,j)-α-$\mathcal{I}$-closed subset of Y. Therefore, f is a pairwise α-$\mathcal{I}$-closed function. □

Theorem 4.13. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then f is a pairwise α-$\mathcal{I}$-closed function if and only if for each subset V of Y, $f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(V)) \subset \tau_i\text{-Cl}(f^{-1}(V))$.*

Proof. Let V be any subset of Y. Then by Theorem 4.12, $(i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(V) \subset f(\tau_i\text{-Cl}(f^{-1}(V)))$. Since f is bijection, $f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(V)) = f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(f^{-1}(V)))) \subset f^{-1}(f(\tau_i\text{-Cl}(f^{-1}(V)))) = \tau_i\text{-Cl}(f^{-1}(V))$. Conversely, let U be any subset of X. Since f is bijection, $(i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(U)) = f(f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\text{Cl}(f(U))) \subset f(\tau_i\text{-Cl}(f^{-1}(f(U)))) = f(\tau_i\text{-Cl}(U))$. Therefore, by Theorem 4.12, f is a pairwise α-$\mathcal{I}$-closed function. □

Theorem 4.14. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise α-$\mathcal{I}$-open function. If V is a subset of Y and U is a τ_i-closed subset of X containing $f^{-1}(V)$, then there exists a (i,j)-α-$\mathcal{I}$-closed set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. Let V be any subset of Y and U a τ_i-closed subset of X containing $f^{-1}(V)$, and let $F = Y\backslash(f(X\backslash V))$. Then $f(X\backslash V) \subset f(f^{-1}(X\backslash V)) \subset X\backslash V$ and $X\backslash U$ is a τ_i-open set of X. Since f is pairwise α-$\mathcal{I}$-open, $f(X\backslash U)$ is a (i,j)-α-$\mathcal{I}$-open set of Y. Hence F is an (i,j)-α-$\mathcal{I}$-closed set of Y and $f^{-1}(F) = f^{-1}(Y\backslash(f(X\backslash U)) \subset U$. □

Theorem 4.15. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise α-$\mathcal{I}$-closed function. If V is a subset of Y and U is a open subset of X containing $f^{-1}(V)$, then there exists (i,j)-α-$\mathcal{I}$-open set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. The proof is similar to the Theorem 4.14. □

Theorem 4.16. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a pairwise α-$\mathcal{I}$-open function. Then for each subset V of Y, $f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(V))) \subset \tau_i\text{-Cl}(f^{-1}(V)$.*

Proof. Let V be any subset of Y. Then τ_i-Cl$(f^{-1}(V)$ is a τ_i-closed set of X. Then by Theorem 4.14, there exists an (i,j)-α-$\mathcal{I}$-closed set F of Y containing V such that $f^{-1}(F) \subset \tau_i\text{-Cl}(f^{-1}(V)$. Since $Y\backslash F$ is (i,j)-α-$\mathcal{I}$-open, $f^{-1}(Y\backslash F) \subset f^{-1}(\tau_j\text{-Int}(\tau_i\text{-Cl}^*(\tau_j\text{-Int}(Y\backslash F))))$ and $X\backslash f^{-1}(F) \subset f^{-1}(Y\backslash(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(F))))) = X\backslash f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(F))))$. Thus we obtain that $f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(V)))) \subset \tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(F))) \subset f^{-1}(F) \subset \tau_i\text{-Cl}(f^{-1}(V))$. Therefore, we have $f^{-1}(\tau_i\text{-Cl}(\tau_j\text{-Int}^*(\tau_i\text{-Cl}(V))) \subset \tau_i\text{-Cl}(f^{-1}(V))$. □

Definition 4.17. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ is said to be:*

(i) *pairwise α-$(\mathcal{I}, \mathcal{J})$-open if $f(U)$ is a (i,j)-α-$\mathcal{J}$-open set of Y for every (i,j)-α-$\mathcal{I}$-open set U of X.*
(ii) *pairwise α-$(\mathcal{I}, \mathcal{J})$-closed if $f(U)$ is a (i,j)-α-$\mathcal{J}$-closed set of Y for every (i,j)-α-$\mathcal{I}$-closed set U of X.*

It is clear that every pairwise α-$(\mathcal{I}, \mathcal{J})$-open (resp. pairwise α-$(\mathcal{I}, \mathcal{J})$-closed) function is pairwise α-$\mathcal{J}$-open (resp. pairwise α-$\mathcal{J}$-closed) function. But the converse is not true in general.

Example 4.18. *Let $X = \{a, b, c\}$ $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \tau_1, \tau_2, \mathcal{I})$ is pairwise α-$\mathcal{J}$-open but not pairwise α-$(\mathcal{I}, \mathcal{I})$-open.*

Theorem 4.19. *For a function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$, the following statements are equivalent:*

(i) *f is pairwise α-$(\mathcal{I}, \mathcal{J})$-open;*
(ii) *$f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(U)) \subset (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(f(U))$ for each subset U of X;*

(iii) (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)) \subset f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Int}(V))$ *for each subset* V *of* Y.

Proof. $(i) \Rightarrow (ii)$: Let U be any subset of X. Then (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(U)$ is a (i,j)-α-$\mathcal{I}$-open set of X. Then $f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U))$ is a (i,j)-α-$\mathcal{I}$-open set of Y. Since $f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U)) \subset f(U)$, $f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U)) = (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U))) \subset (i,j)$-$s\mathcal{J}\,\mathrm{Int}(f(U))$.
$(ii) \Rightarrow (iii)$: Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. Hence $f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))) \subset (i,j)$-$\alpha\mathcal{J}\,\mathrm{Int}(f(f^{-1}(V))) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(V))$. Then (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)) \subset f^{-1}(f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)))) \subset f^{-1}((i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(V))$.
$(iii) \Rightarrow (i)$: Let U be any (i,j)-α-$\mathcal{I}$-open set of X. Then (i,j)-$\alpha\mathcal{I}\,\mathrm{Int}(U) = U$ and $f(U)$ is a subset of Y. Now, $U = (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(U) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(f(U))) \subset f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Int}(f(U)))$. Then $f(U) \subset f(f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Int}(f(U))) \subset (i,j)$-$\alpha\mathcal{J}\,\mathrm{Int}(f(U))$ and (i,j)-$\alpha\mathcal{J}\,\mathrm{Int}(f(U)) \subset f(U)$. Hence $f(U)$ is a (i,j)-α-$\mathcal{J}$-closed set of Y; hence f is pairwise α-$(\mathcal{I},\mathcal{J})$-open. □

Theorem 4.20. *Let* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{J})$ *be a function. Then* f *is a pairwise* α-$(\mathcal{I},\mathcal{J})$*-closed function if and only if for each subset* U *of* X, (i,j)-$\alpha\mathcal{J}\,\mathrm{Cl}(f(U)) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(U))$.

Proof. Let f be a pairwise α-$(\mathcal{I},\mathcal{J})$-closed function and U any subset of X. Then $f(U) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(U))$ and $f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(U))$ is a (i,j)-α-$\mathcal{J}$-closed set of Y. We have (i,j)-$\alpha\mathcal{J}\,\mathrm{Cl}(f(U)) \subset (i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(U))) = f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(U))$. Conversely, let U be a (i,j)-α-$\mathcal{I}$-open set of X. Then $f(U) \subset (i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(f(U)) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(U)) = f(U)$; hence $f(U)$ α-$\mathcal{J}$-closed subset of Y. Therefore, f is a pairwise α-$(\mathcal{I},\mathcal{J})$-closed function. □

Theorem 4.21. *Let* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{J})$ *be a function. Then* f *is a pairwise* α-$(\mathcal{I},\mathcal{J})$*-closed function if and only if for each subset* V *of* Y, $f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(V)) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V))$.

Proof. Let V be any subset of Y. Then by Theorem 4.20, (i,j)-$\alpha\mathcal{J}\,\mathrm{Cl}(f(f^{-1}(V))) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)))$. Since f is bijection, $f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(V)) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V))$. Conversely, let U be any subset of X. Then $f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(f(U))) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(f(U)))$. Hence (i,j)-$\alpha\mathcal{J}\,\mathrm{Cl}(f(U)) \subset f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(f(U))))$. Therefore, by Theorem 4.20 f is a pairwise α-$(\mathcal{I},\mathcal{J})$-closed function. □

Theorem 4.22. *Let* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{J})$ *be a pairwise* α-$(\mathcal{I},\mathcal{J})$*-open function. If* V *is a subset of* Y *and* U *is a* (i,j)-α-$\mathcal{I}$*-closed* *subset of X containing $f^{-1}(V)$, then there exists (i,j)-α-$\mathcal{I}$-closed set F of Y containing V such that $f^{-1}(F) \subset V$.*

Proof. The proof is similar to the Theorem 4.14. □

Theorem 4.23. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ be a pairwise α-$(\mathcal{I}, \mathcal{J})$-closed function. If V is a subset of Y and U is a (i,j)-α-$\mathcal{I}$-open subset of X containing $f^{-1}(V)$, then there exists (i,j)-α-$\mathcal{J}$-open set F of Y containing V such that $f^{-1}(F) \subset V$.*

Proof. The proof is similar to the Theorem 4.14. □

Theorem 4.24. *For a bijective function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$, the following statements are equivalent:*

(i) *f is pairwise α-$(\mathcal{I}, \mathcal{J})$-closed;*
(ii) *f is pairwise α-$(\mathcal{I}, \mathcal{J})$-open.*

Proof. The proof is clear. □

5. PAIRWISE α-$\mathcal{I}$-IRRESOLUTE FUNCTIONS

Definition 5.1. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ is said to be (i,j)-α-$\mathcal{I}$-irresolute if the inverse image of every (i,j)-α-$\mathcal{J}$-open set of Y is (i,j)-α-$\mathcal{I}$-open in X, where $i \neq j$, i, j=1, 2.*

Proposition 5.2. *Every pairwise α-$\mathcal{I}$-irresolute function is pairwise α-$\mathcal{I}$-continuous but not conversely.*

Proof. Straigtforward. □

Theorem 5.3. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ be a function, then*

(1) *f is pairwise α-$\mathcal{I}$-irresolute;*
(2) *the inverse image of each (i,j)-α-$\mathcal{J}$-closed subset of Y is (i,j)-α-$\mathcal{I}$-closed in X;*
(3) *for each $x \in X$ and each $V \in S\mathcal{J}O(Y)$ containing $f(x)$, there exists $U \in \alpha\mathcal{I}O(X)$ containing x such that $f(U) \subset V$.*

Proof. The proof is obvious from that fact that the arbitrary union of (i,j)-α-$\mathcal{I}$-open subsets is (i,j)-α-$\mathcal{I}$-open. □

Theorem 5.4. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ be a function, then*

(i) *f is pairwise α-$\mathcal{I}$-irresolute;*
(ii) *(i,j)-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(V))$ for each subset V of Y;*

(iii) $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(U) \subset (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(U))$ *for each subset* U *of* X.

Proof. $(i) \Rightarrow (ii)$: Let V be any subset of Y. Then $V \subset (i,j)$-$\alpha\mathcal{J}\,\mathrm{Cl}(V)$ and $f^{-1}(V) \subset f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(V))$. Since f is pairwisr α-$\mathcal{I}$-irresolute, $f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(V))$ is a (i,j)-α-$\mathcal{I}$-closed subset of X. Hence $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(V))) = f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(V))$.
$(ii) \Rightarrow (iii)$: Let U be any subset of X. Then $f(U) \subset (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(U))$ and $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(U) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(f(U))) \subset f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(U)))$. This implies that $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(U)) \subset f(f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(U)))) \subset (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(U))$.
$(iii) \Rightarrow (i)$: Let V be a (i,j)-α-$\mathcal{J}$-closed subset of Y. Then $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(f(V))) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(V) = V$. This implies that $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(f((i,j)$-$\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)))) \subset f^{-1}(V)$. Therefore, $f^{-1}(V)$ is a (i,j)-α-$\mathcal{I}$-closed subset of X and consequently f is a pairwise α-$\mathcal{I}$-irresolute function. □

Theorem 5.5. *A function* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{J})$ *is a pairwise* α-$\mathcal{I}$-*irresolute if and only if* $f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V)) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$ *for each subset* V *of* Y.

Proof. Let V be any subset of Y. Then $(i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V) \subset V$. Since f is pairwise α-$\mathcal{I}$-irresolute, $f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V))$ is a (i,j)-α-$\mathcal{I}$-open subset of X. Hence $f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V)) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}((i,j)$-$\alpha\mathcal{J}\,\mathrm{Int}(V))) \subset (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$. Conversely, let V be a (i,j)-α-$\mathcal{J}$-open subset of Y. Then $f^{-1}(V) = f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V)) \subset (i,j)$-$\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is a (i,j)-α-$\mathcal{I}$-open subset of X and consequently f is a pairwisr α-$\mathcal{I}$-irresolute function. □

Corollary 5.6. *Let* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{J})$ *be a function. Then* f *is pairwise* α-$\mathcal{I}$-*closed and pairwise* α-$\mathcal{I}$-*irresolute if and only if* $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(V)) = (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(V)))$ *for evey subset* V *of* X.

Definition 5.7. *An ideal bitopological space* $(X,\tau_1,\tau_2,\mathcal{I})$ *is called pairwise* α-$\mathcal{I}$-*Hausdorff if for each two distinct points* $x \neq y$, *there exist disjoint* (i,j)-α-$\mathcal{I}$-*open sets* U *and* V *containing* x *and* y, *respectively.*

Theorem 5.8. *Let* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2,\mathcal{J})$ *be a pairwise* α-$\mathcal{I}$-*irresolute function. If* Y *is pairwise* α-$\mathcal{J}$-*Hausdorff, then* X *is pairwise* α-$\mathcal{I}$-*Hausdorff.*

Proof. The proof is clear. □

Corollary 5.9. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ be a function. Then f is pairwise α-$\mathcal{I}$-open and pairwise α-$\mathcal{I}$-irresolute if and only if $f^{-1}(((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(V))) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}((V)))$ for evey subset V of Y.*

Definition 5.10. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ is said to be pairwise α-$\mathcal{I}$-homeomorphism if f and f^{-1} are pairwise α-$\mathcal{I}$-irresolute.*

Theorem 5.11. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2, \mathcal{J})$ be a bijection. Then the following statements are equivalent:*

(i) *f is pairwise α-$\mathcal{I}$-homeomorphism;*
(ii) *f^{-1} is pairwise α-$\mathcal{I}$-homeomorphism;*
(iii) *f and f^{-1} are pairwise α-$(\mathcal{I}, \mathcal{J})$-open (pairwise α-$(\mathcal{J}, \mathcal{I})$-closed);*
(1) *f is pairwise α-$\mathcal{I}$-irresolute and pairwise α-$(\mathcal{I}, \mathcal{J})$-open (pairwise α-$(\mathcal{J}, \mathcal{I})$-closed);*
(2) *$f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(V)) = (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(f(V))$ for each subset V of X;*
(3) *$f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(V)) = (i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(f(V))$ for each subset V of X;*
(4) *$f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V)) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$ for each subset V of Y;*
(5) *$(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) = f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Cl}(V))$ for each subset V of Y;*

Proof. (1) $\Rightarrow$ (2): It follows immediately from the definition of a pairwise α-$\mathcal{I}$-homeomorphism.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4): It follows from Theorem 4.24.
(4) $\Rightarrow$ (5): It follows from Theorem 4.21 and Corollary 5.6.
(5) $\Rightarrow$ (6): Let U be a subset of X. Then by Theorem 3.33, $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(U)) = X\backslash f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(X\backslash U)) = X\backslash(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f(X\backslash U)) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f(U))$.
(6) $\Rightarrow$ (7): Let V be a subset of Y. Then $f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f(f^{-1}(V))) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f(V))$. Hence $f^{-1}(f((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V)))) = f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(V))$. Therefore, $f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(V)) = (i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}(V))$.
(7) $\Rightarrow$ (8): Let V be a subset of Y. Then by Theorem 3.33, $(i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) = X\backslash(f^{-1}((i,j)\text{-}\alpha\mathcal{J}\,\mathrm{Int}(Y\backslash V))) = X\backslash((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Int}(f^{-1}((X\backslash V)))) = f^{-1}((i,j)\text{-}\alpha\mathcal{I}\,\mathrm{Cl}(V))$.
(8) $\Rightarrow$ (1): It follows from Theorem 4.21 and Corollary 5.9. $\square$

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