"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 27(2017), No. 2, 37-58

UNIFIED STUDY OF CERTAIN GENERALIZATIONS OF g-CONTINUITY FOR MULTIFUNCTIONS

TAKASHI NOIRI AND VALERIU POPA

Abstract. In this paper, by using gm-closed sets [32], we obtain the unified definitions and properties of g-continuity, gs-continuity, gp-continuity, αg-continuity, γg-continuity and gsp-continuity for multifunctions.

Dedicated by the first author to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. Introduction

The concept of generalized closed (briefly g-closed) sets in topological spaces was introduced by Levine [25] in 1970. These sets also considered by Dunham [18] and Dunham and Levine [19]. In 1981, Munshy and Bassan [29] introduced the notion of generalized continuous (briefly g-continuous) functions. The notion of g-continuity is also studied in [10], [11], [13] and other papers. The notions of generalized semi-closed sets and generalized semi-continuity are introduced and studied in [15].

Keywords and phrases: m-space, g-closed, gm-closed, g-continuous, gm-continuous, multifunction.
(2010) Mathematics Subject Classification: 54A05, 54C10, 54C60.

The notions of generalized preclosed sets and generalized precontinuity are introduced and investigated in [6]. The notions of α-generalized closed sets and α-generalized continuity are studied in [16]. The notions of generalized semi-preclosed sets and *gsp*-continuity are introduced in [17]. The notions of γg-closed sets and γg-continuity [21] are introduced and investigated. Quite recently, in [3], [12] and [22] two forms of g-continuity for multifunctions are introduced.

Recently, the present authors [42], [44] have introduced the notions of m-structures, m-spaces and m-continuity. In [32], the first author introduced the notion of generalized m-closed (briefly gm-closed) sets and tried to unify certain types of modifications of g-closed sets such as stated above. In this paper, by using gm-closed sets, we introduce the unified definitions of g-continuity, gs-continuity, gp-continuity, αg-continuity, γg-continuity and gsp-continuity for multifunctions and obtain several unified properties.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be
(1) *α-open* [31] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [24] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [27] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [1] or *semi-preopen* [4] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(5) *γ-open* [21] or *b-open* [5] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all α-open (resp. semi-open, preopen, β-open, γ-open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$, $\gamma(X)$).

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α-closed* [28] (resp. *semi-closed* [14], *preclosed* [27], *β-closed* [1] or *semi-preclosed* [4], *γ-closed* [21] or *b-closed* [5]) if the complement of A is α-open (resp. semi-open, preopen, β-open, γ-open).

Definition 2.3. Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed, β-closed, γ-closed) sets of X containing A is called the *α-closure* [28] (resp. *semi-closure* [14], *preclosure* [20], *β-closure* [2] or *semi-preclosure* [4],

γ-closure [21] or *b-closure* [5]) of A and is denoted by $\alpha\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$), $\mathrm{Cl}_\gamma(A)$ or $\mathrm{bCl}(A)$).

Definition 2.4. Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, β-open, γ-open) sets of X contained in A is called the *α-interior* [28] (resp. *semi-interior* [14], *preinterior* [20], *β-interior* [2] or *semi-preinterior* [4], *γ-interior* [21] or *b-interior* [5]) of A and is denoted by $\alpha\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$), $\mathrm{Int}_\gamma(A)$ or $\mathrm{bInt}(A)$).

A point $x \in X$ is called a *θ-cluster* point of a subset A of X [46] if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed [46]. The complement of a θ-closed set is said to be θ-open. The union of all θ-open sets contained in A is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$. It is shown in [46] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of X and $\mathrm{Cl}_\theta(A)$ is closed in X for each subset A of X.

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ (or simply $F : X \to Y$) presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper continuous* (resp. *upper quasi-continuous* or *upper semi-continuous* [38], [39] *upper precontinuous* [43], *upper α-continuous* [30], *upper β-continuous* [41]) at a point $x \in X$ if for each open set V containing $F(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open) set $U \subset X$ containing x such that $F(U) \subset V$,

(2) *lower continuous* (resp. *lower quasi-continuous* or *lower semi-continuous* [38], [39] *lower precontinuous* [43], *lower α-continuous* [30], *lower β-continuous* [41]) at a point $x \in X$ if for each open set V of Y meeting $F(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open) set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper (lower) continuous* (resp. *upper (lower) quasi-continuous*

or *semi-continuous*, *upper (lower) precontinuous*, *upper(lower) α-continuous*, *upper (lower) β-continuous*) in X if it has this property at every point of X.

3. m-structures and m-continuity for multifunctions

Definition 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [42], [44] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed*.

Remark 3.1. Let (X, τ) be a topological space. The families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, and $\gamma(X)$ are all m-structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [26] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{Cl}_\gamma(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{Int}_\gamma(A)$).

Lemma 3.1. (Maki et al. [26]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,
(2) If $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2. (Popa and Noiri [42]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. An m-structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [26] if the union of any family of subsets belong to m_X belongs to m_X.

Remark 3.3. If (X, τ) is a topological space, then $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ and $\gamma(X)$ have property $\mathcal{B}$,

Lemma 3.3. (Popa and Noiri [42]). *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,
(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ and $\mathrm{mCl}(A)$ is m_X-closed.

Definition 3.4. Let (X, m_X) be an m-space and and (Y, σ) be a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be
(1) *upper m-continuous* if for each $x \in X$ and each $V \in \sigma$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,
(2) *lower m-continuous* if for each $x \in X$ and each $V \in \sigma$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,
(3) *upper/lower m-continuous* if it has this property at each point $x \in X$.

Theorem 3.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^+(V) = \mathrm{mInt}(F^+(V))$ for every open set V of Y;
(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ for every closed set K of Y;
(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ for every subset B of Y.

Proof. The proof follows from Theorem 3.1 of [34] and Lemma 4.1 of [45].

Corollary 3.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^+(V)$ is m_X-open for every open set V of Y;
(3) $F^-(K)$ is m_X-closed for every closed set K of Y.

Proof. The proof follows from Theorem 3.1 and Lemma 3.3.

Theorem 3.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower m-continuous;
(2) $F^-(V) = \mathrm{mInt}(F^-(V))$ for every open set V of Y;
(3) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every closed set K of Y;
(4) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}(F(A))$ for every subset A of X;
(6) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(B))$ for every subset B of Y.

Proof. The proof follows from Theorem 3.2 of [34] and Lemma 4.1 of [45].

Corollary 3.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^-(V)$ is m_X-open for every open set V of Y;
(3) $F^+(K)$ is m_X-closed for every closed set K of Y.

Proof. The proof follows from Theorem 3.2 and Lemma 3.3.

4. m-CONTINUITY AND gm-CONTINUITY FOR MULTIFUNCTIONS

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be
(1) *g-closed* [25] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
(2) *αg-closed* [16] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
(3) *gs-closed* [15] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
(4) *gp-closed* [6], [33] if $\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
(5) *gsp-closed* [17] if $\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
(6) *γg-closed* [21] or *gb-closed* if $\mathrm{Cl}_\gamma(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

Definition 4.2. A subset A of a topological space is said to be *g-open* (resp. *gs-open*, *gp-open*, *αg-open*, *gsp-open*, γg-open) if $X - A$ is g-closed (resp. gs-closed, gp-closed αg-closed, gsp-closed, γg-closed).

The family of all g-open (resp. gs-open, gp-open, αg-open, gsp-open, γg-open) sets of X is denoted by $\mathrm{GO}(X)$ (resp. $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\mathrm{GSPO}(X)$, $\gamma\mathrm{GO}(X)$).

Definition 4.3. Let (X, τ) be a topological space and A a subset of X. The intersection of all g-closed (resp. αg-closed, gs-closed, gp-closed, gsp-closed, γg-closed) sets of X containing A is called the *g-closure* [18] (resp. *αg-closure*, *gs-closure*, *gp-closure*, *gsp-closure*, *γg-closure*) of A and is denoted by $\mathrm{Cl}_g(A)$ (resp. $\alpha\mathrm{Cl}_g(A)$, $\mathrm{sCl}_g(A)$, $\mathrm{pCl}_g(A)$, $\mathrm{spCl}_g(A)$, $\gamma\mathrm{Cl}_g(A)$).

Definition 4.4. Let (X, τ) be a topological space and A a subset of X. The union of all g-open (resp. αg-open, gs-open, gp-open, gsp-open, γg-open) sets of X contained in A is called the *g-interior* [13] (resp. *αg-interior*, *gs-interior*, *gp-interior*, *gsp-interior*, *γg-interior*) of A and is denoted by $\mathrm{Int}_g(A)$ (resp. $\alpha\mathrm{Int}_g(A)$), $\mathrm{sInt}_g(A)$, $\mathrm{pInt}_g(A)$, $\mathrm{spInt}_g(A)$, $\gamma\mathrm{Int}_g(A)$).

Remark 4.1. Let (X, τ) be a topological space and A a subset of X.

(1) Then, $\mathrm{GO}(X)$, $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\mathrm{GSPO}(X)$ and $\gamma\mathrm{GO}(X)$ are all m-structures on X. Hence, if $m_X = \mathrm{GO}(X)$ (resp. $\alpha\mathrm{GO}(X)$, $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\mathrm{GSPO}(X)$, $\gamma\mathrm{GO}(X)$), then we have

(i) $\mathrm{mCl}_g(A) = \mathrm{Cl}_g(A)$ (resp. $\alpha\mathrm{Cl}_g(A)$, $\mathrm{sCl}_g(A)$, $\mathrm{pCl}_g(A)$, $\mathrm{spCl}_g(A)$), $\gamma\mathrm{Cl}_g(A)$),

(ii) $\mathrm{mInt}_g(A) = \mathrm{Int}_g(A)$ (resp. $\alpha\mathrm{Int}_g(A)$), $\mathrm{sInt}_g(A)$, $\mathrm{pInt}_g(A)$, $\mathrm{spInt}_g(A)$, $\gamma\mathrm{Int}_g(A)$).

(2) If $m_X = \mathrm{GO}(X)$, then by Lemma 3.1 we obtain the results established in Theorem 2.1 (3), (5) and Theorem 2.8 (2), (3), (5), (6), (7) in [13]. By Lemma 3.2, we obtain the result established in Theorem 2.1 (4) in [13]. By Lemma 3.3, we obtain Lemma 3.1 of [13].

(3) The m-structures $\mathrm{GO}(X)$, $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\mathrm{GSPO}(X)$ and $\gamma\mathrm{GO}(X)$ do not have, in general, property $\mathcal{B}$.

Definition 4.5. Let (X, τ) be a topological space and m_X an m-structure on X. A subset A of X is said to be *generalized m-closed* (briefly *gm-closed*) [32] if $\mathrm{mCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

The complement of a gm-closed set is said to be *gm-open*. The family of all gm-open sets is denoted by $\mathrm{GMO}(X)$. Obviously, $\mathrm{GMO}(X)$ is an m-structure on X and is called a *gm-structure* on X.

Remark 4.2. Let (X, τ) be a topological space and m_X an m-structure on X. We put $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\gamma(X)$). Then, a gm-closed set is a g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) set and $\mathrm{GMO}(X) = \mathrm{GO}(X)$ (resp. $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\mathrm{GSPO}(X)$, $\gamma\mathrm{GO}(X)$).

Definition 4.6. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper generalized continuous* (briefly *upper g-continuoous*) [3] if $F^+(K)$ is g-closed in X for every closed set K of Y,

(2) *lower generalized continuous* (briefly *lower g-continuous*) [3] if $F^-(K)$ is g-closed in X for every closed set K of Y.

Definition 4.7. Let (X, τ) be a topological space and $\mathrm{GMO}(X)$ a gm-structure on X. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be *upper/lower gm-continuous* at a point $x \in X$ (resp. on X) if $F : (X, \mathrm{GMO}(X)) \to (Y, \sigma)$ is upper/lower m-continuous at $x \in X$ (resp. on X).

Remark 4.3. By Definition 4.7, for a multifunction $F : (X, \tau) \to (Y, \sigma)$, we have

(1) F is upper gm-continuous at $x \in X$ if for each $V \in \sigma$ containing $F(x)$, there exists a gm-open set U containing x such that $F(U) \subset V$,

(2) F is lower gm-continuous at $x \in X$ if for each $V \in \sigma$ such that $F(x) \cap V \neq \emptyset$, there exists a gm-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

By Definition 4.7, Remark 4.3, Theorems 3.1 and 3.2, Corollaries 3.1 and 3.2, we obtain the following theorems:

Theorem 4.1. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper gm-continuous;
(2) $F^+(V) = \mathrm{mInt}_g(F^+(V))$ for every open set V of Y;
(3) $F^-(K) = \mathrm{mCl}_g(F^-(K))$ for every closed set K of Y;
(4) $\mathrm{mCl}_g(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}_g(F^+(B))$ for every subset B of Y.

Corollary 4.1. *Let (X, τ) be a topological space and m_X an m-structure on X such that GMO(X) has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper gm-continuous;
(2) $F^+(V)$ is gm-open for every open set V of Y;
(3) $F^-(K)$ is gm-closed for every closed set K of Y.

Theorem 4.2. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower gm-continuous;
(2) $F^-(V) = \mathrm{mInt}_g(F^-(V))$ for every open set V of Y;
(3) $F^+(K) = \mathrm{mCl}_g(F^+(K))$ for every closed set K of Y;
(4) $\mathrm{mCl}_g(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F(\mathrm{mCl}_g(A)) \subset \mathrm{Cl}(F(A))$ for every subset A of X;
(6) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}_g(F^-(B))$ for every subset B of Y.

Corollary 4.2. *Let (X, τ) be a topological space and m_X an m-structure on X such that GMO(X) has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower gm-continuous;

(2) $F^{-}(V)$ is gm-open for every open set V of Y;
(3) $F^{+}(K)$ is gm-closed for every closed set K of Y.

Remark 4.4. (1) Every upper/lower m-continuous multifunction is obviously upper/lower gm-continuous. However, by Example 1 of [3], the converse is not true.

(2) If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha\mathrm{O}(X)$, $\mathrm{BO}(X)$, $\mathrm{SPO}(X)$) and F is upper/lower gm-continuous, then F is upper/lower g-continuous (resp. gs-continuous, gp-continuous, αg-continuous, gb-continuous, gsp-continuous).

5. Some properties of gm-continuity

In this section, useing gm-open sets and gm-closed sets we define the notions of gm-compact spaces and gm-connected spaces and obtain their preservation theorems.

Definition 5.1. A subset B of a topological space (Y, σ) is said to be

(1) *α-regular* [23] if for each $b \in B$ and each open set V of Y containing b, there exists an open set G of Y such that $b \in G \subset \mathrm{Cl}(G) \subset V$,

(2) *α-paracompact* [47] if every σ-open cover of B has a σ-open refinement which covers B and is locally finite for each point of Y.

For a multifunction $F : X \to Y$, a multifunction $\mathrm{Cl}(F) : X \to Y$ is defined in [8] as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, $\mathrm{sCl}(F)$, $\mathrm{pCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$ and $\mathrm{spCl}(F)$ are defined. The following theorems are proved in [36].

Theorem 5.1. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper m-continuous if and only if G(F) is upper m-continuous, where G(F) denotes* $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$ *or* $\mathrm{spCl}(F)$.

Theorem 5.2. *A multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower m-continuous if and only if G(F) is lower m-continuous, where G(F) denotes* $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$ *or* $\mathrm{spCl}(F)$.

Corollary 5.1. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$ and m_X a minimal structure on X. Then F is upper gm-continuous if and only if G(F) is upper gm-continuous, where G(F) denotes* $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$ *or* $\mathrm{spCl}(F)$.

Proof. The proof follows from Definition 4.7 and Theorem 5.1.

Corollary 5.2. *A multifunction $F : (X, m_X) \to (Y, \sigma)$, where m_X is an m-structure on X, is lower gm-continuous if and only if G(F) is lower gm-continuous, where G(F) denotes* $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$ *or* $\mathrm{spCl}(F)$.

Proof. The proof follows from Definition 4.7 and Theorem 5.2.

The following theorems are proved in [37].

Theorem 5.3. *Let (Y, σ) be a regular space. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^-(\mathrm{Cl}_\theta(B)) = \mathrm{mCl}(F^-(\mathrm{Cl}_\theta(B)))$ for every subset B of Y;
(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ for every θ-closed set K of Y;
(4) $F^+(V) = \mathrm{mInt}(F^+(V))$ for every θ-open set V of Y.

Theorem 5.4. *Let (Y, σ) be a regular space. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^+(\mathrm{Cl}_\theta(B)) = \mathrm{mCl}(F^+(\mathrm{Cl}_\theta(B)))$ for every subset B of Y;
(3) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every θ-closed set K of Y;
(4) $F^-(V) = \mathrm{mInt}(F^-(V))$ for every θ-open set V of Y.

Corollary 5.3. *Let (Y, σ) be a regular space and m_X have property $\mathcal{B}$. Then for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^-(\mathrm{Cl}_\theta(B))$ is m_X-closed for every subset B of Y;
(3) $F^-(K)$ is m_X-closed for every θ-closed set K of Y;
(4) $F^+(V)$ is m_X-open for every θ-open set V of Y.

Corollary 5.4. *Let (Y, σ) be a regular space and m_X have property ($\mathcal{B}$). Then for a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^+(\mathrm{Cl}_\theta(B))$ is m_X-closed for every subset B of Y;
(3) $F^+(K)$ is m_X-closed for every θ-closed set K of Y;
(4) $F^-(V)$ is m_X-open for every θ-open set V of Y.

By Definition 4.7, Theorems 5.3 and 5.4, and Corollaries 5.3 and 5.4 we have the following theorems and corollaries:

Theorem 5.5. *Let (Y, σ) be a regular space and m_X an m-structure on (X, τ). For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following* *properties are equivalent:*

(1) F is upper gm-continuous;

(2) $F^-(\mathrm{Cl}_\theta(B)) = \mathrm{mCl}_g(F^-(\mathrm{Cl}_\theta(B)))$ *for every subset B of Y;*

(3) $F^-(K) = \mathrm{mCl}_g(F^-(K))$ *for every θ-closed set K of Y;*

(4) $F^+(V) = \mathrm{mInt}_g(F^+(V))$ *for every θ-open set V of Y.*

Theorem 5.6. *Let (Y, σ) be a regular space and m_X an m-structure on (X, τ). For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower gm-continuous;

(2) $F^+(\mathrm{Cl}_\theta(B)) = \mathrm{mCl}_g(F^+(\mathrm{Cl}_\theta(B)))$ *for every subset B of Y;*

(3) $F^+(K) = \mathrm{mCl}_g(F^+(K))$ *for every θ-closed set K of Y;*

(4) $F^-(V) = \mathrm{mInt}_g(F^-(V))$ *for every θ-open set V of Y.*

Corollary 5.5. *Let (Y, σ) be a regular space and m_X an m-structure on (X, τ) such that* GMO*(X) has property $\mathcal{B}$. Then for a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper gm-continuous;

(2) $F^-(\mathrm{Cl}_\theta(B))$ *is gm-closed for every subset B of Y;*

(3) $F^-(K)$ *is gm-closed for every θ-closed set K of Y;*

(4) $F^+(V)$ *is gm-open for every θ-open set V of Y.*

Corollary 5.6. *Let (Y, σ) be a regular space and m_X an m-structure on (X, τ) such that* $\mathrm{GMO}(X)$ *has property $\mathcal{B}$. Then for a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower gm-continuous;

(2) $F^+(\mathrm{Cl}_\theta(B))$ *is gm-closed for every subset B of Y;*

(3) $F^+(K)$ *is gm-closed for every θ-closed set K of Y;*

(4) $F^-(V)$ *is gm-open for every θ-open set V of Y.*

Definition 5.2. An m-space (X, m_X) is said to be *m-compact* [44] if every cover of X by sets of m_X has a finite subcover.

A subset K of an m-space (X, m_X) is said to be *m-compact* [44] if every cover of K by sets of m_X has a finite subcover.

Remark 5.1. (1) If (X, τ) is a topological space and $(X, \mathrm{GMO}(X))$ is m-compact, then (X, τ) is said to be *gm-compact*. A subset K of X is said to be *gm-compact* if every cover of K by *gm*-open sets of X has a finite subcover.

(2) If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$), then we obtain the definition of *GO*-compactness [7] (resp. *GSO*-compactness [15], *GPO*-compactness [6], α*GO*-compactness [16]).

Theorem 5.7. *Let $F : (X, m_X) \to (Y, \sigma)$ be upper m-continuous and F(x) compact for each $x \in X$. If K is m-compact, then F(K) is compact.*

Proof. The proof follows from Theorem 4 of [34].

Theorem 5.8. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and m_X an m-structure on X. If $F : (X, \tau) \to (Y, \sigma)$ is upper gm-continuous, F(x) is compact for each $x \in X$ and K is a gm-compact set of X, then F(K) is compact.*

Proof. The proof follows from Definition 4.7 and Theorem 5.7.

Corollary 5.7. *If $F : (X, \tau) \to (Y, \sigma)$ is an upper g-continuous surjection such that F(x) is compact for each $x \in X$ and X is GO-compact, then Y is compact.*

Remark 5.2. This is a corrected form of Theorem 13 of [3].

Definition 5.3. An m-space (X, m_X) is said to be *m-connected* [44] if X cannot be written as the union of two nonempty disjoint m-open sets.

Remark 5.3. Let (X, τ) be a topological space and m_X an m-structure on X, then

(1) (X, τ) is said to be *gm-connected* if X cannot be written as the union of two nonempty disjoint gm-open sets.

(2) If $m_X = \tau$ (resp. $\alpha(X)$), then by Definition 5.3 we obtain the definition of GO-connected spaces [7] (resp. αGO-connected spaces [16]).

Theorem 5.9. *Let (X, m_X) be an m-space, where m_X has property $\mathcal{B}$. If $F : (X, m_X) \to (Y, \sigma)$ is an upper/lower m-continuous surjection, F(x) is connected for each $x \in X$ and (X, m_X) is m-connected, then (Y, σ) is connected.*

Proof. The proof follows from Theorem 5.1 of [34].

Theorem 5.10. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and m_X an m-structure on X such that* $\mathrm{GMO}(X)$ *has property $\mathcal{B}$. If $F : (X, \tau) \to (Y, \sigma)$ is an upper/lower gm-continuous surjective multifunction such that F(x) is connected for each $x \in X$ and (X, τ) is gm-connected, then (Y, σ) is connected.*

Proof. The proof follows from Definition 4.7 and Theorem 5.3.

Corollary 5.8. *Let $F : (X, \tau) \to (Y, \sigma)$ is an upper/lower g-continuous surjection such that F(x) is connected for each $x \in X$ and (X, τ) is g-connected, then (Y, σ) is connected.*

Remark 5.4. This is a corrected form of Theorem 14 of [3].

Definition 5.4. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the graph $\mathrm{G}(F) = \{(x, F(x)) : x \in X\}$ is said to be *strongly m-closed* [43] if for each $(x, y) \in (X \times Y) - \mathrm{G}(F)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(F) = \emptyset$.

Remark 5.5. Let (X, τ) be a topological space and m_X an m-structure on X. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to have a *strongly gm-closed graph* if for each $(x, y) \in (X \times Y) - \mathrm{G}(F)$, there exist a gm-open set containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(F) = \emptyset$.

Lemma 5.1. (Popa and Noiri [43]). *A multifunction $F : (X, m_X) \to (Y, \sigma)$ has a strongly m-closed graph if and only if for each $(x, y) \in (X \times Y) - \mathrm{G}(F)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $F(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Theorem 5.11. *Let (Y, σ) be a regular space. If $F : (X, m_X) \to (Y, \sigma)$ is an upper m-continuous multifunction such that F(x) is closed in Y for each $x \in X$, then G(F) is strongly m-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(F)$, then $y \in Y \setminus F(x)$. Since Y is regular, there exist disjoint open sets V_1 and V_2 such that $F(x) \subset V_1$ and $y \in V_2$. Moreover, there exists an open set V such that $y \in V \subset \mathrm{Cl}(V) \subset V_2$. Since F is upper m-continuous and $Y \setminus \mathrm{Cl}(V)$ is an open set containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset Y \setminus \mathrm{Cl}(V)$. Therefore, we have $F(U) \cap \mathrm{Cl}(V) = \emptyset$ and hence by Lemma 5.1 $G(F)$ is strongly m-closed.

Theorem 5.12. *Let (Y, σ) be a regular space and m_X an m-structure on X. If $F : (X, \tau) \to (Y, \sigma)$ is an upper gm-continuous multifunction such that F(x) is closed in Y for each $x \in X$, then G(F) is strongly gm-closed.*

Proof. The proof follows from Theorem 5.11 and Definition 4.7.

Remark 5.6. Let (Y, σ) be a regular space and $F : (X, \tau) \to (Y, \sigma)$ a multifunction such that $F(x)$ is closed in Y for each $x \in X$. If F is upper g-continuous (resp. upper gs-continuous, upper gp-continuous, upper αg-continuous, upper gb-continuous, upper gsp-continuous), then $G(F)$ is strongly g-closed (resp. strongly gs-closed, strongly gp-closed, strongly αg-closed, strongly gb-closed, strongly gsp-closed).

6. Separation axioms and gm-continuity

Definition 6.1. An m-space (X, m_X) is said to be m-T_2 [44] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ containing x, y, respectively, such that $U \cap V = \emptyset$.

Remark 6.1. Let (X, τ) be a topological space and m_X an m-structure on X.

(1) Then (X, τ) is said to be gm-T_2 if for each distinct points $x, y \in X$, there exist gm-open sets U, V containing x, y, respectively, such that $U \cap V = \emptyset$.

(2) If $m_X = \tau$, then X is g-T_2 [9].

Definition 6.2. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be *injective* if for each points $x, y \in X$, $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$.

Theorem 6.1. *If $F : (X, m_X) \to (Y, \sigma)$ is an upper m-continuous injective multifunction such that F(x) is compact for each $x \in X$ and (Y, σ) is Hausdorff, then (X, m_X) is m-T_2.*

Proof. Since F is injective, for any distinct points $x_1, x_2 \in X$ we have $F(x_1) \cap F(x_2) = \emptyset$. Since $F(x_1)$ and $F(x_2)$ are compact sets in a Hausdorff space, there exist open sets V_1, V_2 of Y such that $F(x_1) \subset V_1, F(x_2) \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since F is upper m-continuous, there exist $U_1, U_2 \in m_X$ containing x_1, x_2, respectively, such that $F(U_1) \subset V_1, F(U_2) \subset V_2$. Therefore, $U_1 \cap U_2 = \emptyset$ and hence (X, m_X) is m-T_2.

Theorem 6.2. *Let (X, τ) be a topological space and m_X an m-structure on X. If $F : (X, \tau) \to (Y, \sigma)$ is an upper gm-continuous injective multifunction such that F(x) is compact for each $x \in X$ and (Y, σ) is Hausdorff, then (X, τ) is gm-T_2.*

Proof. The proof follows from Definition 4.7 and Theorem 6.1.

Definition 6.3. A subset A of an m-space (X, m_X) is said to be m-*dense* in X [37] if $\mathrm{mCl}(A) = X$.

Theorem 6.3. *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$ and (Y, σ) be a Hausdorff space.*
If the following four conditions are satisfied,

(1) a multifunction $G : (X, m_X^1) \to (Y, \sigma)$ is upper m-continuous,

(2) a multifunction $F : (X, m_X^2) \to (Y, \sigma)$ is upper m-continuous,

(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,

(4) $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of (X, m_X^2), then $F(x) \cap G(x) \neq \emptyset$ for each point x in X.

Proof. The proof follows from Theorem 8.1 of [37].

Theorem 6.4. *Let (X, τ) be a topological space with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in \mathrm{GM_X^2O}(X)$ whenever $U \in \mathrm{GM_X^1O}(X)$ and $V \in \mathrm{GM_X^2O}(X)$ and (Y, σ) be a Hausdorff space. If the following four conditions are satisfied,*

(1) a multifunction $F : (X, \mathrm{GM_X^1O}(X)) \to (Y, \sigma)$ is upper m-continuous,

(2) a multifunction $G : (X, \mathrm{GM_X^2O}(X)) \to (Y, \sigma)$ is upper m-continuous,

(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,

(4) $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of $(X, \mathrm{GM_X^2O}(X))$,
then $F(x) \cap G(x) \neq \emptyset$ for each point x in X.

Proof. The proof follows from Definition 4.7 and Theorem 6.3.

Theorem 6.5. *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$ and (Y, σ) be a normal space.*
If the following four conditions are satisfied,

(1) a multifunction $F : (X, m_X^1) \to (Y, \sigma)$ is upper m-continuous,

(2) a multifunction $G : (X, m_X^2) \to (Y, \sigma)$ is upper m-continuous,

(3) F(x) and G(x) are closed in (Y, σ) for each $x \in X$,

(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,
then $A = m_X^2\text{-}\mathrm{Cl}(A)$.

Proof. Let $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since (Y, σ) is normal and $F(x)$ and $G(x)$ are closed sets, there exist disjoint open sets V and W of (Y, σ) such that $F(x) \subset V$ and $G(x) \subset W$. Since F is upper m-continuous, there exists $U_1 \in m_X^1$ containing x such that $F(U_1) \subset V$. Since G is upper m-continuous, there exists $U_2 \in m_X^2$ containing x such that $F(U_2) \subset W$. Put $U_1 \cap U_2 = U$, then $U \in m_X^2$. Therefore, we have $A \cap U = \emptyset$. By Lemma 3.2, we have $x \in X - m_X^2\text{-}\mathrm{Cl}(A)$ and hence $A = m_X^2\text{-}\mathrm{Cl}(A)$.

Theorem 6.6. *Let X be a topological space with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in \mathrm{GM^2O}(X)$ whenever $U \in \mathrm{GM^1O}(X)$ and $V \in \mathrm{GM^2O}(X)$ and (Y, σ) be a normal space. If the following four conditions are satisfied,* *(1) a multifunction* $F : (X, \mathrm{GM}^1\mathrm{O}(X)) \to (Y, \sigma)$ *is upper m-continuous,*

(2) a multifunction $G : (X, \mathrm{GM}^2\mathrm{O}(X)) \to (Y, \sigma)$ *is upper m-continuous,*

(3) F(x) and G(x) are closed in (Y, σ) *for each* $x \in X$*,*

(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$*,*

then $A = m^2\mathrm{Cl}_g(A)$*.*

Proof. The proof follows from Definition 4.7 and Theorem 6.5.

Corollary 6.1. *If* $F, G : (X, \tau) \to (Y, \sigma)$ *are upper g-continuous and punctually closed and* (Y, σ) *is normal, then* $A = \mathrm{Cl}_g(A)$*, where* $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$*.*

Remark 6.2. This is a corrected form of Theorem 12 of [3]. Because it is not true in general that $\mathrm{Cl}_g(A)$ is g-closed (see Example 2.3 of [13]).

Theorem 6.7. *Let X be a nonempty set with two minimal structures* m^1_X *and* m^2_X *such that* $U \cap V \in m^2_X$ *whenever* $U \in m^1_X$ *and* $V \in m^2_X$ *and* (Y, σ) *be a Hausdorff space.*
If the following four conditions are satisfied,

(1) a multifunction $F : (X, m^1_X) \to (Y, \sigma)$ *is upper m-continuous,*

(2) a multifunction $G : (X, m^2_X) \to (Y, \sigma)$ *is upper m-continuous,*

(3) F(x) and G(x) are compact sets of (Y, σ) *for each* $x \in X$*,*

(4) $F(x) \cap G(x) \neq \emptyset$ *for each point* $x \in X$*,*

then the multifunction $H : (X, m^2_X) \to (Y, \sigma)$*, defined by* $H(x) = F(x) \cap G(x)$ *for each* $x \in X$*, is upper m-continuous.*

Proof. Let $x \in X$ and V be an open set of (Y, σ) such that $H(x) \subset V$. Then $A = F(x) - V$ and $B = G(x) - V$ are disjoint compact sets. Since A and B are compact sets of a Hausdorff space (Y, σ), there exist open sets V_1 and V_2 such that $A \subset V_1, B \subset V_2$ and $V_1 \cap V_2 = \emptyset$. Since F is upper m-continuous, there exists $U_1 \in m^1_X$ containing x such that $F(U_1) \subset V_1 \cup V$. Since G is upper m-continuous, there exists $U_2 \in m^2_X$ containing x such that $G(U_2) \subset V_2 \cup V$. Set $U = U_1 \cap U_2$, then $U \in m^2_X$ containing x. If $y \in H(x_0)$ for any $x_0 \in U$, then $y \in (V_1 \cup V) \cap (V_2 \cup V) = (V_1 \cap V_2) \cup V$. Since $V_1 \cap V_2 = \emptyset$, we have $y \in V$ and hence $H(U) \subset V$. Therefore, H is upper m-continuous.

Theorem 6.8. *Let* (X, τ) *be a topological space with two minimal structures* m^1_X *and* m^2_X *such that* $U \cap V \in \mathrm{GM}^2\mathrm{O}(X)$ *whenever* $U \in \mathrm{GM}^1\mathrm{O}(X)$ *and* $V \in \mathrm{GM}^2\mathrm{O}(X)$ *and* (Y, σ) *be a Hausdorff space.*
If the following four conditions are satisfied,

(1) a multifunction $F : (X, \mathrm{GM}^1\mathrm{O}(X)) \to (Y, \sigma)$ is upper m-continuous,
(2) a multifunction $G : (X, \mathrm{GM}^2\mathrm{O}(X)) \to (Y, \sigma)$ is upper m-continuous,
(3) F(x) and G(x) are compact sets of (Y, σ) for each $x \in X$,
(4) $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$,
then the multifunction $H : (X, \mathrm{GM}^2\mathrm{O}(X)) \to (Y, \sigma)$, defined by $H(x) = F(x) \cap G(x)$ for each $x \in X$, is upper m-continuous.

Proof. The proof follows from Definition 4.7 and Theorem 6.7.

7. The set of points of upper/lower gm-discontinuity

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, we define $D^+_m(F)$ and $D^-_m(F)$ as follows:

$$D^+_m(F) = \{x \in X : F \text{ is not upper } m\text{-continuous at } x\},$$
$$D^-_m(F) = \{x \in X : F \text{ is not lower } m\text{-continuous at } x\}.$$

The following theorems are proved in [35].

Theorem 7.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^+_m(F) &= \textstyle\bigcup_{G\in\sigma}\{F^+(G) - \mathrm{mInt}(F^+(G))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)} \{F^+(\mathrm{Int}(B)) - \mathrm{mInt}(F^+(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{mCl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}} \{\mathrm{mCl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Theorem 7.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^-_m(F) &= \textstyle\bigcup_{G\in\sigma}\{F^-(G) - \mathrm{mInt}(F^-(G))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)} \{F^-(\mathrm{Int}(B)) - \mathrm{mInt}(F^-(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{mCl}(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{A\in\mathcal{P}(X)} \{\mathrm{mCl}(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl}(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Let (X, τ) be a topological space and m_X an m-structure on X. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we denote

$$D^+_{gm}(F) = \{x \in X : F \text{ is not upper } gm\text{-continuous at } x\},$$
$$D^-_{gm}(F) = \{x \in X : F \text{ is not lower } gm\text{-continuous at } x\}.$$

Theorem 7.3. *Let (X, τ) be a topological space and m_X an m-structure on X. Then, for a multifunction $F : (X, \tau) \to (Y, \sigma)$, the* *following properties hold:*

$$\begin{aligned} D^+_{gm}(F) &= \bigcup_{G\in\sigma}\{F^+(G) - \mathrm{mInt}_g(F^+(G))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{F^+(\mathrm{Int}(B)) - \mathrm{mInt}_g(F^+(B))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{mCl}_g(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\mathcal{F}} \{\mathrm{mCl}_g(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y,σ).

Proof. The proof follows from Definition 4.7 and Theorem 7.1.

Theorem 7.4. *Let (X,τ) be a topological space and m_X an m-structure on X. Then, for a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

$$\begin{aligned} D^-_{gm}(F) &= \bigcup_{G\in\sigma}\{F^-(G) - \mathrm{mInt}_g(F^-(G))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{F^-(\mathrm{Int}(B)) - \mathrm{mInt}_g(F^-(B))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{mCl}_g(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in\mathcal{P}(X)} \{\mathrm{mCl}_g(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{H\in\mathcal{F}}\{\mathrm{mCl}_g(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y,σ).

Proof. The proof follows from Definition 4.7 and Theorem 7.3.

For a multifunction $F : (X,\tau) \to (Y,\sigma)$, we denote

$D^+_g(F) = \{x \in X : F$ is not upper g-continuous at $x\}$,

$D^-_g(F) = \{x \in X : F$ is not lower g-continuous at $x\}$.

Then, as corollaries of Theorems 7.3 and 7.4, we obtain the following:

Corollary 7.1. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

$$\begin{aligned} D^+_g(F) &= \bigcup_{G\in\sigma}\{F^+(G) - \mathrm{Int}_g(F^+(G))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{F^+(\mathrm{Int}(B)) - \mathrm{Int}_g(F^+(B))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{Cl}_g(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\mathcal{F}} \{\mathrm{Cl}_g(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y,σ).

Corollary 7.2. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

$$\begin{aligned} D^-_g(F) &= \bigcup_{G\in\sigma}\{F^-(G) - \mathrm{Int}_g(F^-(G))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{F^-(\mathrm{Int}(B)) - \mathrm{Int}_g(F^-(B))\} \\ &= \bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{Cl}_g(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in\mathcal{P}(X)} \{\mathrm{Cl}_g(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \bigcup_{H\in\mathcal{F}}\{\mathrm{Cl}_g(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y,σ).

Definition 7.1. Let (X, m_X) be an m-space and A a subset of X. The m_X*-frontier* of A, $\mathrm{mFr}(A)$, [37] is defined by $\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A)$.

Theorem 7.5. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction. Then $D_m^+(F)$ (resp. $D_m^-(F)$) is identical with the union of the m-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Proof. We shall prove the first case since the proof of the second is similar.
Let $x \in D_m^+(F)$. Then, there exists an open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every m_X-open set U containing x. By Lemma 3.2, we have $x \in \mathrm{mCl}(X - F^+(V))$. On the other hand, since $x \in F^+(V)$, we have $x \in \mathrm{mCl}(F^+(V))$ and hence $x \in \mathrm{mFr}(F^+(V))$.

Conversely, suppose that F is upper m-continuous at $x \in X$. Then, for any open set V of Y containing $F(x)$, there exists $U \in m_X$ such that $F(U) \subset V$; hence $U \subset F^+(V)$. Therefore, we have $x \in U \subset \mathrm{mInt}(F^+(V))$. This contradicts to the fact that $x \in \mathrm{mFr}(F^+(V))$.

Theorem 7.6. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and m_X an m-structure on X. Then $D_{gm}^+(F)$ (resp. $D_{gm}^-(F)$) is identical with the union of the gm-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Proof. The proof follows from Definition 4.7 and Theorem 7.5.

For example, for upper/lower g-continuous multifunctions we obtain the following corollary from Theorem 7.6.

Corollary 7.3. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction. Then $D_g^+(F)$ (resp. $D_g^-(F)$) is identical with the union of the g-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., **12** (1983), 77–90.
[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, *β-closure and β-interior*, J. Fac. Ed. Ain Shams Univ., **10** (1986), 235–245.
[3] M. Akdag, *On generalized continuity for multifunctions*, Kochi J. Math., **2** (2007), 63–69.
[4] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32.

[5] D. Andrijević, *On b-open sets*, Mat. Vesnik, **48** (1996), 59–64.
[6] I. Arokiarani, K. Balachandran and J. Dontchev, *Some characterizations of gp-irresolute and gp-continuous maps between topological spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A Math., **20** (1999), 93–104.
[7] K. Balachandran, P. Sundaram and H. Maki, *On generalized continuous maps in topological spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A Math., **12** (1991), 5–13.
[8] T. Bânzaru, *Multifunctions and M-product spaces* (Romanian), Bul. St. Tehn. Inst. Politehn. "Traian. Vuia", Timişoara, Mat. Fiz. Mec. Teor. Appl., **17**(**31**) (1972), 17–23.
[9] C. Boonpok, *Preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces*, Naresuan Univ. J., **11**(3) (2003), 75–77.
[10] M. Caldas, *On g-closed sets and g-continuous mappings*, Kyungpook Math. J., **33** (1993), 205–209.
[11] M. Caldas, *Further results on generalized open mappings in topological spaces*, Bull. Calcutta Math. Soc., **88** (1996), 37–42.
[12] M. Caldas, S. Jafari and S. P. Moshokoa, *On GO-compact spaces*, Filomat, **22** (2008), 47–55.
[13] M. Caldas, S. Jafari and T. Noiri, *Notions via g-open sets*, Kochi J. Math., **2** (2007), 43–50.
[14] S. G. Crossley and S. K. Hildebrand, *Semi-closure*, Texas J. Sci., **22** (1971), 99–112.
[15] R. Devi, K. Balachandran and H. Maki, *Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces*, Indian J. Pure Appl. Math., **26** (1995), 271–284.
[16] R. Devi, K. Balachandran and H. Maki, *On generalized α-continuous maps and α-generalized continuous maps*, Far East J. Math. Sci., Special Volume (1997), Part I, 1–15.
[17] J. Dontchev, *On generalizing semi-preopen sets*, Mem. Fac. Sci. Kochi Univ. Ser. A, Math., **16** (1995), 35–48.
[18] W. Dunham, *A new closure operator for non-T_1 topologies*, Kyungpook Math. J., **22** (1982), 55–60.
[19] W. Dunham and N. Levine, *Further results on generalized closed sets in topology*, Kyungpook Math. J., **20** (1980), 169–175.
[20] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **27(75)** (1983), 311–315.
[21] T. Fukutake, A. A. Nasef and A. I. El-Maghrabi, *Some topological concepts via γ-generalized closed sets*, Bull. Fukuoka Univ. Ed. III, **52** (2003), 1–9.
[22] S. Jafari, S. P. Mashokoa and T. Noiri, *sg-compact sets and multifunctions*, Kochi J. Math., **3** (2008), 191–200.
[23] I. Kovačević, *Subsets and paracompactness*, Univ. u Novom Sadu Zb. Rad. Priod.-Mat. Fak. Ser. Mat., **14** (1984), 79–87.
[24] N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly, **70** (1963), 36–41.
[25] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo (2), **19** (1970), 89–96.

[26] H. Maki, K. C. Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets*, Pure Appl. Math. Sci., **49** (1999), 17–29.

[27] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings,* Proc. Math. Phys. Soc. Egypt, **53** (1982), 47–53.

[28] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar., **41** (1983), 213–218.

[29] B. M. Munshi and D. S. Bassan, *g-continuous mappings* Vidya J. Gujarat Univ. B Sci., **24** (1981), 63–68.

[30] T. Neubrunn, *Strongly quasi-continuous multivalued mappings*, General Topology and its Relations to Modern Analysis and Algebra, VI, Proc. Praque Topological Symposium, Helderman Verlag, Berlin, 1988, pp. 351–360.

[31] O. Njåstad, *On some classes of nearly open sets,* Pacific J. Math., **15** (1965), 961–970.

[32] T. Noiri, *A unified theory for certain modifications of generalized closed sets*, Internat. J. General Topology, **1** (2008), 87–99.

[33] T. Noiri, H. Maki and J. Umehara, *Generalized preclosed functions*, Mem. Fac. Sci. Kochi Univ. Ser. A Math., **19** (1998), 13–30.

[34] T. Noiri and V. Popa, *On upper and lower M-continuous multifunctions*, Filomat (Nis), **14** (2000), 73–86.

[35] T. Noiri and V. Popa, *A unified theory of points set of discontinuity for multifunctions*, Bull. Math. Soc. Sci. Math. Roumanie, **45**(**94**) (2002), 97–106.

[36] T. Noiri and V. Popa, *Minimal structures, m-continuous multifunctions and bitopological spaces*, J. Pure Math., **23** (2006), 37–52.

[37] T. Noiri and V. Popa, *A unified theory of weak continuity for multifunctions*, Stud. Cerc. St. Ser. Mat. Univ. Bacău, **16** (2006), 167–200.

[38] V. Popa, *Multifonctions semi-continues*, Rev. Roumaine Math. Pures Appl., **27** (1982), 807–815.

[39] V. Popa, *Sur certain formes faibles de continuité pour multifonctions*, Rev. Roumaine Math. Pures Appl., **30** (1985), 539–546.

[40] V. Popa, *Some properties of H-almost continuous multifonctions*, Problemy Mat., **10** (1988), 9–26.

[41] V. Popa and T. Noiri, *On upper and lower β-continuous multifunctions*, Real Anal. Exchange, **22** (1996/97), 362–370.

[42] V. Popa and T. Noiri, *On M-continuous functions*, Anal. Univ. "Dunărea de Jos" Galaţi, Ser. Mat. Fiz. Mec. Teor. (2), **18**(**23**) (2000), 31–41.

[43] V. Popa and T. Noiri, *A unified theory of S-continuity for multifunctions*, Istanbul Univ. Fen. Fac. Mat. Der., **59** (2000), 1–15.

[44] V. Popa and T. Noiri, *On the definitions of some generalized forms of continuity under minimal conditions*, Mem. Fac. Sci. Kochi Univ. Ser. A Math., **22** (2001), 9–18.

[45] V. Popa and T. Noiri, *On m-continuous multifunctions*, Bull. St. Politeh. Univ. Timişoara Ser. Mat. Fiz., **46**(**60**) (2001), 1–12.

[46] N. V. Veličko, *H-closed topological spaces,* Amer. Math. Soc. Transl. (2), **78** (1968), 103–118.

[47] J. D. Wine, *Locally paracompact spaces*, Glasnik Mat., **10**(**30**) (1975), 351–357.

2949-1 SHIOKITA-CHO, HINAGU
YATSUSHIRO-SHI, KUMAMOTO-KEN,
869-5142 JAPAN
e-mail:t.noiri@nifty.com

"Vasile Alecsandri" University of Bacău,
157 Calea Mărăşeşti , 600115 Bacău, ROMANIA,
e-mail:vpopa@ub.ro