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EQUILIBRIA BY FIXED POINTS

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Abstract. Our study is devoted to the projections of the general efficiency as fixed points of the multifunctions, with applications to the balance extremum moments in the framework of the generalized dynamical systems.

Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. INTRODUCTION

Until now, the basis on which was developed our general concept of the Efficiency in the Ordered Linear Spaces and its Applications, supplied by the Ordered Locally Convex Spaces, is represented about the vastness class of the Convex Cones introduced by Professor Isac in 1981, published in 1983, called by us and, officially recognized, as *"Isac's Cone"* in 2009, after the acceptance of this last agreed denomination by Professor Isac. To Professor Isac is devoted this scientific research work.We think that it is not possible to obtain more than these results concerning the General Efficiency in the world of the separated Locally Convex Spaces.

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They represent the most development background. All the elements concerning the ordered topological vector spaces used here are in acordance with *Nachbin, L., 1965* and *Peressini, A., L., 1967.* The main bibliography of this research work containing all the examined references was indicated in [2].

2. GENERAL EFFICIENCY

Let E be any real or complex vector space with the origin element θ, ordered by a convex cone K, that is, $K + K \subseteq K$ and $\chi K \subseteq K$ for all $\chi \in R_+$, K_1 a non - void subset of K and A a non - empty subset of E. The following definition introduces a new concept of *the approximate efficiency*, which expresses *the general efficiency*, generalizes the well known notion of Pareto type efficiency in every Euclidean space and not only.

Definition 1. *(Postolică, V., 2002, 2008)*. *We say that $a_0 \in A$ is a K_1-minimal efficient point of A, in notation, $a_0 \in MIN(A, K, K_1)$ if it satisfies one of the following equivalent conditions:*

(i) $A \bigcap (a_0 - K - K_1) \subseteq a_0 + K + K_1$;

(ii) $(K + K_1) \bigcap (a_0 - A) \subseteq -K - K_1$;

In a similar manner one defines the K_1-*maximal efficient points* by replacing $K + K_1$ with $-(K + K_1)$. The set of such as these points will be denoted by $MAX(A, K, K_1)$. Whenever$K_1 = \{\theta\}$ we consider $MIN(A, K, K_1) = MIN(A, K)$ and $MAX(A, K, K_1) = MAX(A, K)$, respectively. *All the efficient points of the set A with respect to the convex cone Kfollowing K_1 are represented by the set $eff\,(A, K, K_1) = MIN(A, K, K_1) \cup MAX(A, K, K_1)$*. Consequently, $eff(A, K) = MIN(A, K) \cup MAX(A, K)$. It is obvious that

$A \bigcap (a_0 \mp K) \subseteq a_0 \pm K_1 \Rightarrow A \bigcap (a_0 \pm K \pm K_1) \subseteq a_0 \mp K \mp K_1 \Rightarrow A \bigcap (a_0 \mp K_1) \subseteq a_0 \pm K$,

which suggests other concepts for the *efficiency and its approximate term* in *the general ordered linear spaces.*

Remark 1. We notice that $a_0 \in eff\,(A, K, K_1)$ iff it is *a fixed point* for at least one of the multifunctions $F_\mp : A \to 2^A$ defined by $F_\mp\,(t) = \{a \in A : A \bigcap (a \mp K \mp K_1) \subseteq t \pm K \pm K_1\}$, that is, $a_0 \in F_\mp(a_0)$, with the corresponding signs. Consequently, for *the existence* of these *general efficient points* we can also apply *appropriate fixed points theorems* concerning *the multifunctions* (see, for instance, *Cardinali, T., Papalini, F., 1994*, *Park, S.,1992*, *Patriche, M.,2013, Zhang Cong – Jun, 2005 and any other proper scientific papers mentioned in [2]*).

Remark 2. *Németh, A.B., 1989*, proved that, whenever $K_1 \subseteq K\setminus\{\theta\}$, the existence of *this new type of the efficient points* for *the lower bounded sets characterizes the*

semi - Archimedian Ordered Vector Spaces and *the regular Ordered Locally Convex Spaces.*

Remark 3. Whenever K is pointed, that is, $K\bigcap(-K) = \{\theta\}$, then $a_0 \in MIN(A, K, K_1)\,(a_0 \in MAX(A, K, K_1))$ means that $A\bigcap(a_0 \mp K \mp K_1) = \emptyset$ or, equivalently, $(\pm K + K_1)\bigcap(a_0 - A) = \emptyset$ for $\theta \notin K_1$ and $A\bigcap(a_0 \mp K \mp K_1) = \{a_0\}$, respectively, if $\theta \in K_1$. Clearly, $K \subseteq K + K_1$ if $\theta \in K_1$ and, if the convex cone K is pointed, then $K \subseteq K + K_1 \Leftrightarrow \theta \in K_1$. This inclusion is not always valid as we can see taking $E = R^2$, $K = \{(x; y) \in R^2 : x \in R, y \geq 0\}$ and $K_1 = \{(1, 1)\}$, but, in our considered case, it remains true for $K_1 = \{(0, -1)\}$. Whenever$K_1 = \{\theta\}$, from the **Definition 1** one obtains the usual concept of the efficient (*Pareto minimal, optimal or admissible*) point : $a_0 \in MIN(A, K)(MAX(A, K))$ if it fulfils *(i)*, *(ii)* or any of the next equivalent properties:

(iii) $(A\pm K)\bigcap(a_0\mp K) \subseteq a_0\pm K$;

(iv) $K\bigcap(a_0 - A \mp K) \subseteq \mp K$

This shows that a_0 is *a fixed point* for at least one of the following multifunctions:

$$F_1 : A \to A,\ F_1(t) = \left\{a \in A : A\bigcap(a \mp K) \subseteq t \pm K\right\},$$

$$F_2 : A \to A,\ F_2(t) = \left\{a \in A : A\bigcap(t \mp K) \subseteq a \pm K\right\},$$

$$F_3 : A \to A,\ F_3(t) = \left\{a \in A : (A \pm K)\bigcap(a \mp K) \subseteq t \pm K\right\},$$

$$F_4 : A \to A,\ F_4(t) = \left\{a \in A : (A \pm K)\bigcap(t \mp K) \subseteq a \pm K\right\},$$

that is, $a_0 \in F_i(a_0)$ for some $i = \overline{1, 4}$.If, in addition, K *is pointed, then* $a_0 \in A$ *is an efficient point of* A *with respect to* K *if and only if one of the following equivalent relations holds :*

(v) $A\bigcap(a_0\mp K) = \{a_0\}$;

(vi) $A\bigcap(a_0 \mp K\setminus\{\theta\}) = \emptyset$;

(vii) $(\pm K)\bigcap(a_0 - A) = \{\theta\}$;

(viii) $(\pm K\{\theta\})\bigcap(a_0 - A) = \emptyset$;

(ix) $(A\pm K)\bigcap[a_0\mp(K\{\theta\})] = \emptyset$.

We also notice that $MIN\,(A,K)\,(MAX(A,K)) = \bigcap_{\{0\}\neq K_2\subseteq K} MIN\,(A,K,K_2)\,(\bigcap_{\{0\}\neq K_2\subseteq -K} MAX\,(A,K,K_2))$.
Moreover,$a_0 \in eff\,(A,K)$ *if only if it is a critical (ideal or balance) point (see, for example, Kim, W. K., Tan, K.K., 2001; Isac,G. 1985; Isac, G., Bulavsky, V. A., Kalashnikov, V., V. 2002; Isac,G., Postolică, V.,1993; Postolică, V., Scarelli, A., Venzi,L., 2001; Postolică, V., 2004 and the corresponding references) for the generalized dynamical systems* $\Gamma_{\mp} : A \to 2^A$ *defined by* $\Gamma_{\mp}(a) = A\bigcap(a\mp K)\,,\ a\in A$*.In this way,* $eff\,(A,K)$ *describes the balance extremum moments* for $\Gamma_{\mp}$, which in *the abstract market context, expresses the competitive equilibrium/non - equilibrium consisting of the general relation price/consumption.* By considering $K_1 = \{\varepsilon\}(\varepsilon \in K\setminus\{\theta\})$, one obtains that $a_0 \in eff\,(A,K,K_1)$ iff$A\bigcap(a_0-\varepsilon\mp K) = \emptyset$. In all these cases, for the set $MIN\,(A,K,K_1)$ we used the notation$\varepsilon - MIN\,(A,K)$. It is obvious that $MIN\,(A,K) = \bigcap_{\varepsilon\in K\setminus\{0\}}[\varepsilon - MIN\,(A,K)]$, with the natural projections for $MAX(A,K)$ obtained by replacing the convex cone K with $-K$.When referring to *the existence of the efficient points, the significant properties of the sets containing these points and the applications in the usual or in the extended contexts we mention, as a comprehensive bibliography given in [2] (Aubin, J.P., 1993; Benson, H. P., 1077, 1979,1983;Borwein, J. M., 1977, 1983;Dauer, J. P., Gallagher, R. J., 1990; Dominiak, Cezary, 2006; Ehrgott, Matthias, 2005;Fishburn, P. C.,1984; Gass, Saul.,I., Harris, Carl., M., 2001; Geoffrion, A. M;, 1968; Hartley, R., 1978; Henig, M. I., 1982; Heyne, Paul, 2000; Hyers, D., H., Isac, G., Rassias,T.M.,1997; Isac, G., 1981, 1983, 1985, 1994, 1998,2003, Isac, G., Bahya, A. O., 2002; Isac, G., Postolică, 1989, 1993,1995, 1997, 1999, 2001, 2002, 2009, 2014, 2015, 2016; Isac, G., Tammer,Chr.,2003; Jablonsky, J.,2006; Loridan, P., 1984; Luc, D. T., 1989; Luptáčik, M., Bőhm, B., 2006; Németh, A., B., 1989; Ng., K., F., Zheng, X., Y., 2002; Postolică, V.,1989, 1993, 1994, 1997, 1999, 2001, 2002, 2004, 2008, 2009; Songsak Sriboonchitta, Wing – Keung Wong, Sompong Dhopongsa, Hung, T., Nguyen, 2009; Sontag, Z., Zălinescu, C., 2000; Stewart, D., T., 2004; Truong, X. D. H., 1994; Zimmermann, H., 2000), with widely established results.*

Remark 4. Until now, the basis on which was developed the general concept of the Efficiency in the Ordered Linear Spaces and its Applications, supplied by the Ordered Locally Convex Spaces, is represented about the vastness class of the Convex Cones introduced by Professor Isac in 1981, published in 1983, called by us and, officially recognized, as *"Isac's Cone"* in 2009, after the acceptance of this last agreed denomination by Professor Isac. Concerning *the existence and the corresponding properties as fixed points of the equilibria by Isac's cones,* the next and last result in this field is valid, following the concepts indicated below.

Definition 2. *(Postolică,V., 1995). A nonempty set B of a topological vector space X ordered by a convex cone K is called K - bounded if there exists $B_0 \subseteq X$, bounded, so that $B \subseteq B_0 \pm K$ and it is K - closed if its conical extension $B \pm \bar{K}$ is closed, where $\bar{K}$ signifies the topological closure of K.*

Theorem 1. *(Postolică, V., 1995, 1997, 2008).*

(i) if K is an arbitrary Isac's cone in any Hausdorff locally convex space X, $A, B \subseteq X$ are every non-empty subsets positioned by $A \subseteq B \subseteq A + K (A \subseteq B \subseteq A - K)$ and $B \cap (A_0 - K)(B \cap (A_0 + K))$ is a bounded and complete set for at least one non-empty set $A_0 \subseteq A$, then $MIN(A, K) \neq \emptyset (MAX(A, K) \neq \emptyset)$;

(ii) if X is a quasi - complete Hausdorff separated locally convex space and K is a closed Isac's cone in X, then for any K - bounded and K - closed non - empty set A of X we have $MIN(A, K) \neq \emptyset, A \subseteq MIN(A, K) + K$ ($A \subseteq MAX(A, K) - K$) (the domination property), $MIN(A, K) + K = A + K$ $(MAX(A, K) - K = A - K)$ and $MIN(A, K)(MAX(A, K))$ is K - bounded and K - closed;

(iii) according to the same hypotheses from (ii), $MIN(A, K) \neq \emptyset (MAX(A, K) \neq \emptyset)$ for any non - empty subset A with $B \cap (A_0 - K)(B \cap (A_0 + K))$ K - bounded and

K - closed set whenever $A_0 \subseteq A \subseteq X$ and $A \subseteq B \subseteq A + K (A \subseteq B \subseteq A - K)$, with the corresponding implications for the fixed points which give the equilibria in $eff(A, K)$.

References

[1] Isac, G., Bulavsky, V. A., Kalashnikov, V. V. - *Complementarity, Equilibrium, Efficiency and Economics.* Kluwer Academic Publishers, 2002.

[2] Postolică V. – *Efficiency and Vectorial Optimization.* Session chair and expert speach (Sunday, September 03) at the 5th International Multi – Disciplinary Conference organized by PlaceTypeInstitute of PlaceNameResearch Engineers and Doctors - IRED, placecountry-regionUSA September 02 – 03, 2017, placeCityZurich, country-regionSwitzerland.

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