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ROUGH VARIABLES OF CONVERGENCE

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Abstract. Let (x_{mnk}) be a triple sequence of rough variables. This paper discusses some convergence concepts of rough triple sequence : convergence almost surely (a.s), trust of the rough convergence (Tr), convergence in mean, and convergence in distribution.

1. Introduction

The idea of rough convergence was first introduced by Phu [12-14] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex;and introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex.

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Also, Aytar [2] studied that the $r-$limit set of the sequence is equal to intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. E. Dündar and C. Çakan [8,9] and Ölmez and S. Aytar [10,11] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence.

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}^3$, and let us denote the set $\{(m,n,k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{uvw \to \infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, $\mathbb{R}^3$ denotes the real three dimensional space with metric (X, d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3, m, n, k \in \mathbb{N}^3$.

A triple sequence spaces $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}^3$, written as $st - lim\, x = 0$, provided that the set

$$\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}, \bar{0}| \geq \epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence spaces x.

If a triple sequence spaces is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$ neighbour hood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence spaces $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then,we say that the triple sequence spaces x satisfies P for "almost all (m, n, k)" and we abbreviate this by "a.a. (m, n, k)".

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \{(m_i n_j k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence spaces x.

$c \in \mathbb{R}^3$ is called a statistical cluster point of a triple sequence spaces $x = (x_{mnk})$ provided that the natural density of the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sub sequence x by Γ_x.

A triple sequence spaces $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0.$$

First applied the concept of $(p, q)-$calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators. Later, based on (p, q)-integers, some approximation results for Bernstein-Stancu operators,Bernstein- Kantorovich operators, $(p, q)-$Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers,real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [15,16]*,*Esi et al. [3-5]*, *Dutta et al. [6]*,*Subramanian et al. [17]*, *Debnath et al. [7]*.

2. Definitions and Preliminaries

2.1. **Definition.** Let (x_{mnk}) is a triple sequence of rough variables. The triple sequence (x_{mnk}) is said to be convergent almost surely to the rough variable x if and only if there exists a set A with $Tr(A) = 1$ such that

$$lim_{rst\to\infty} |x_{mnk}(\lambda) - x(\lambda)| = 0 \tag{2.1}$$

for every $\lambda \in A$. In that case we write $x_{mnk} \to x$ almost surely.

2.2. **Definition.** Let (x_{mnk}) is a triple sequence of rough variables of real number β. The triple sequence (x_{mnk}) converges in trust to the rough variable x if

$$lim_{mnk\to\infty} Tr\{|x_{mnk} - x| \geq \beta + \epsilon\} = 0 \tag{2.2}$$

for every $\epsilon > 0$.

2.3. **Definition.** Let (x_{mnk}) is a triple sequence of rough variables with finite expected values. The triple sequence (x_{mnk}) converges in mean to the rough variable x if

$$lim_{mnk\to\infty} E\left[|x_{mnk} - x|\right] = 0 \tag{2.3}$$

2.4. **Definition.** Let $\Phi, \Phi_1, \Phi_2, \cdots,$ are the trust distributions of rough variables $x, x_{111}, x_{222}, \cdots$ respectively. If the triple sequence $\{\Phi_{mnk}\}$ converges weakly to Φ then (x_{mnk}) converges in distribution to x.

3. Main Results

3.1. **Theorem.** Let (x_{mnk}) is a triple sequence of rough variables of a real number β and Λ be a nonempty set. Then (x_{mnk}) converges almost surely to the rough variables x if and only if, for every $\epsilon > 0$ we have

$$lim_{rst\to\infty} Tr\left\{ \bigcup_{(m,n,k)=(r,s,t)}^{\infty} \{|x_{mnk} - x| \geq \beta + \epsilon\} \right\} = 0 \tag{3.1}$$

Proof: For every $m, n, k \geq 1$ and $\epsilon > 0$, we define
$X = \{\lambda \in \Lambda : lim_{mnk\to\infty} x_{mnk}(\lambda) \neq x(\lambda)\}, X_{mnk}(\beta+\epsilon) = \{\lambda \in \Lambda : |x_{mnk}(\lambda) - x(\lambda)| \geq \beta + \epsilon\}$.
It is clear that $X = \bigcup_{\beta+\epsilon>0} \left(\bigcap_{(r,s,t)=1}^{\infty} \bigcup_{(m,n,k)=(r,s,t)}^{\infty} X_{mnk}(\beta+\epsilon) \right)$.
Note that $x_{mnk} \to x$, almost surely if and only if $Tr(X) = 0$. That is $x_{mnk} \to x$, almost surely if and only if $Tr = \left\{ \bigcap_{(r,s,t)=1}^{\infty} \bigcup_{(m,n,k)=(r,s,t)}^{\infty} X_{mnk}(\beta+\epsilon) \right\} = 0$ for every $\epsilon > 0$. Since $\bigcup_{(m,n,k)=(r,s,t)}^{\infty} X_{mnk}(\beta+\epsilon) \downarrow \bigcap_{(r,s,t)=1}^{\infty} \bigcup_{(m,n,k)=(r,s,t)}^{\infty} X_{mnk}(\beta+\epsilon)$,
it follows from the continuity of trust measure that
$lim_{rst\to\infty} Tr = \left\{ \bigcup_{(m,n,k)=(r,s,t)}^{\infty} X_{mnk}(\beta+\epsilon) \right\} = Tr = \left\{ \bigcap_{(r,s,t)=1}^{\infty} \bigcup_{(m,n,k)=(r,s,t)}^{\infty} X_{mnk}(\beta+\epsilon) \right\} = 0.$

3.2. **Theorem.** Let (x_{mnk}) be a triple sequence of rough variables of a real number β. If (x_{mnk}) converges almost surely to the rough variables x, then (x_{mnk}) converges in trust to x.
Proof: It follows from the converges almost surely and Theorem (3.1) that

$$lim_{rst\to\infty} Tr\left\{ \bigcup_{(m,n,k)=(r,s,t)}^{\infty} \{|x_{mnk} - x| \geq \beta + \epsilon\} \right\} = 0$$

for each $\epsilon > 0$. For every $r, s, t \geq 1$, since we have

$$\{|x_{rst} - x| \geq \beta + \epsilon\} \subset \bigcup_{(m,n,k)=(r,s,t)}^{\infty} \{|x_{mnk} - x| \geq \beta + \epsilon\},$$

the theorem holds.

3.3. **Theorem.** Let (x_{mnk}) be a triple sequence of rough variables and f be a nonnegative Borel measurable function. If f is even increasing on $[0, \infty)$, then for any number $t > 0$, we have

$$Tr\{|x| \geq t\} \leq \frac{E[f(x)]}{f(t)} \tag{3.2}$$

Proof: It is clear that $Tr\{|x| \geq f^{-1}(\eta)\}$ is a monotone decreasing function from η on $[0, \infty)$. It follows from the nonnegativity of $f(x)$ that

$$\begin{aligned} E[f(x)] &= \int_0^\infty Tr\{f(x) \geq \eta\}\, d\eta \\ &= \int_0^\infty Tr\{|x| \geq f^{-1}\eta\}\, d\eta \\ &\geq \int_0^{f(t)} Tr\{|x| \geq f^{-1}(\eta)\}\, d\eta \\ &\geq \int_0^{f(t)} d\eta \cdot Tr\{|x| \geq f^{-1}(f(t))\} \\ &= f(t) \cdot Tr\{|x| \geq t\}. \end{aligned}$$

3.4. **Theorem.** Let (x_{mnk}) be a triple sequence of rough variables. Then for any given numbers $t > 0$ and $p > 0$, we have

$$Tr\{|x| \geq t\} \leq \frac{E[|x^p|]}{t^p} \tag{3.3}$$

Proof: It is following Theorem(3.3) when $f(x) = |x|^p$.

3.5. **Theorem.** Let (x_{mnk}) be a triple sequence of rough variables such that the variance $V[x]$ exists. Then for any given number $t > 0$, we have

$$Tr\{|x - E[x]| \geq t\} \leq \frac{V[|x|]}{t^2} \tag{3.4}$$

Proof: It is following Theorem(3.3) when the triple sequence of rough variable x is replaced with $x - E[x]$ and $f(x) = x^2$.

3.6. **Theorem.** Let (x_{mnk}) be a triple sequence of rough variables of a real number β and Λ be a nonempty set. If the triple sequence (x_{mnk}) converges in mean to a rough variable x, then (x_{mnk}) converges in trust to x.

Proof: It follows from Theorem (3.4) that, for any given number $\epsilon > 0$,
(3.5)
$$Tr\left\{|x_{mnk} - x| \geq \beta + \epsilon\right\} \leq \frac{E\left[|x_{mnk} - x|\right]}{\beta + \epsilon} \to 0 \, as \, m, n, k \to \infty.$$
Thus (x_{mnk}) converges in trust to x. The trust distribution $\Phi : \Re \to [0, 1]$ of a rough variable x is defined as

$$\Phi\left(z\right) = Tr\left\{\lambda \in \Lambda : x\left(\lambda\right) \leq z\right\} \tag{3.6}$$

for all $z \in \Re$. That is $\Phi\left(z\right)$ is the trust that the rough variable $x \leq z$.

3.7. **Theorem.** Let $x, x_{111}.x_{222}, \cdots ,$ are triple sequence of rough variables. If the triple sequence (x_{mnk}) converges in trust to x, then the triple sequence of (x_{mnk}) converges in distribution to x.
Proof: Let u be any given continuity point of the distribution Φ. On the other hand, for any $v > u$, we have
$\{x_{mnk} \leq u\} = \{x_{mnk} \leq u, x \leq v\} \bigcup \{x_{mnk} \leq u, x > v\} \subset \{x \leq v\} \bigcup \{|x_{mnk} - x| \geq v - u\}$
$\Rightarrow \Phi_{mnk}\left(u\right) \leq \Phi\left(v\right) + Tr\left\{|x_{mnk} - x| \geq v - u\right\}.$ Since (x_{mnk}) converges in trust to x, we have $Tr\left\{|x_{mnk} - x| \geq v - u\right\} \to 0$. Thus we obtain
$lim_{mnk\to\infty} sup \, \Phi_{mnk}\left(u\right) \leq \Phi\left(v\right)$ for any $v \geq u$. Letting $v \to u$, we get

$$lim_{mnk\to\infty} sup \, \Phi_{mnk}\left(u\right) \leq \Phi\left(u\right). \tag{3.7}$$

On the other hand, for any $w < u$, we have
$\{x \leq w\} = \{x \leq w, x \leq u\} \bigcup \{x \leq w, x_{mnk} > u\} \subset \{x_{mnk} \leq u\} \bigcup \{|x_{mnk} - x| \geq u - w\}$ which implies that $\Phi\left(w\right) \leq \Phi_{mnk}\left(u\right) + Tr\left\{|x_{mnk} - x| \geq u - w\right\}.$
Since $Tr\left\{|x_{mnk} - x| \geq u - w\right\} \to 0$, we obtain
$\Phi\left(w\right) \leq lim_{mnk\to\infty} inf \, \Phi_{mnk}\left(u\right)$ for any $w < u$. Letting $w \to u$, we get

$$\Phi\left(u\right) \leq lim_{mnk\to\infty} inf \, \Phi_{mnk}\left(u\right) \tag{3.8}$$

It follows from (3.7) and (3.8) that $\Phi_{mnk}\left(u\right) = \Phi\left(u\right).$

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