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Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 27(2017), No. 2, 73-82

DETERMINING THE LAPLACIAN SPECTRUM IN PARTICULAR CLASSES OF GRAPHS

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Abstract. During the last three decades, different types of decompositions have been processed in the field of graph theory. Among these we mention: decompositions based on the additivity of some characteristics of the graph, decompositions where the adjacency law between the subsets of the partition is known, decompositions where the subgraph induced by every subset of the partition must have predeterminate properties, as well as combinations of such decompositions. In this paper we characterize threshold graphs using the weakly decomposition, determine the Laplacian spectrum in threshold graphs.

Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. Introduction

Threshold graphs play an important role in graph theory as well as in several applied areas such as set-packing problem (Chvátal and Hammer [4]), parallel processing (Henderson and Zalcstein [12]), allocation problems (Ordman [16]).

Keywords and phrases: Threshold graph, weakly decomposition, Laplacian spectrum.
(2010) Mathematics Subject Classification: 05C99, 68R10.

When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weakly decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally non-adjacent. The structure of the paper is the following. In Section 2 we present the notations to be used, in Section 3 we give the notion of weakly decomposition and in Section 4 we characterize threshold graphs and determine the Laplacian spectrum in threshold graphs.

2. General Notations

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges ([3]), having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G. If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. A vertex $z \in V$ distinguishes the non-adjacent vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \not\sim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say that A and B are *non-adjacent*.

The *neighbourhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we simply write $N(v)$ and $N[v]$, when G appears clearly from the context. The neighbourhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighbourhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A *clique* is a subset Q of V with the property that $G(Q)$ is complete. The *clique number* or *density* of G, denoted by $\omega(G)$, is the size of the maximum clique. A clique cover is a partition of the vertices set such that each part is a clique. $\theta(G)$ is the size of a smallest possible clique cover of G; it is called the *clique cover number* of G. A stable set is a subset X of vertices where every two vertices are not adjacent. $\alpha(G)$ is the number of vertices is a stable set o maximum cardinality; it is called the *stability number* of G. $\chi(G) = \omega(\overline{G})$ and it is called *chromatic number*.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A graph is called *cograph* if it does not contain P_4 as an induced subgraph.

A *split* graph is a graph in which the vertices can by partitioned a clique and an independent set.

Let F denote a family of graphs. A graph G is called F-*free* if none of its subgraphs is in F. The *Zykov sum* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having:

$$V(G) = V(G_1) \cup V(G_2),$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

3. Preliminary results

3.1. Weakly decomposition

At first, we recall the notions of weakly component and weakly decomposition.

Definition 1. ([7], [19], [20]) *A set $A \subset V(G)$ is called a weakly set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weakly set, maximal with respect to set inclusion, then $G(A)$ is called a weakly component. For simplicity, the weakly component $G(A)$ will be denoted with A.*

Definition 2. ([7], [19], [20]) *Let $G = (V, E)$ be a connected and non-complete graph. If A is a weakly set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weakly decomposition of G with respect to A.*

Below we remind a characterization of the weakly decomposition of a graph.

The name of "*weakly component*" is justified by the following result.

Theorem 1. ([8], [19], [20]) *Every connected and non-complete graph $G = (V, E)$ admits a weakly component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 2. ([19], [20]) *Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weakly component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weakly decomposition (A, B, C), such that $G(A)$ is a weakly component and $G(V - A) = G(B) + G(C)$.*

Theorem 2 provides an $O(n+m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

Algorithm for the weakly decomposition of a graph ([22])
Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.

begin
  $A :=$ any set of vertices such that
  $A \cup N(A) \neq V$
  $N := N(A)$
  $R := V - A \cup N(A)$
  while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*
    begin
      $A := A \cup \{n\}$
      $N := (N - \{n\}) \cup (N(n) \cap R)$
      $R := R - (N(n) \cap R)$
    end
end

3.2. Threshold graphs

In this subsection we remind some results on threshold graphs.

A graph G is called *threshold* graph if $N_G(x) \subseteq N_G[y]$ or $N_G(y) \subseteq N_G[x]$ for any pair of vertices x and y in G.

Threshold graphs were first introduced by Chvátal and Hammer ([5]).

Theorem 3. ([4]) A graph G is a threshold graph if and only if G does not contain a C_4, $\overline{C}_4$, P_4 as an induced subgraph.

Chvátal and Hammer also showed that threshold graphs can be recognizing in $O(n^2)$ time.

In [1], Babel showed that if G is a threshold graph then the algorithms that determine $\omega(G)$, $\chi(G)$, $\alpha(G)$ and $\theta(G)$ are $O(n+m)$ time.

Theorem 4. ([4]) A graph G is a threshold graph if and only if G is a cograph and G is a split graph.

In [6] (as well as in [10] and [14]) linear algorithms for recognizing a cograph can be found. Hammer and Simeone [11]) give an $O(n+m)$ algorithm for recognizing a split graph. Therefore, an algorithm that recognizes a threshold graph is $O(n(n+m))$.

In [15] a linear algorithm for recognizing a threshold graph can be found.

3.3. Quasi-threshold graphs

In this subsection we remind some results on quasi-threshold graphs. A graph G which is P_4-free is called a *cograph*. A cograph which is C_4-free is called a *quasi-threshold graph*. Graphs obtained from a vertex by recursively applying the following operations: (i) adding a new vertex, (ii) adding a new vertex that is adjacent to all old vertices, and (iii) disjoint union of two graphs are precisely the quasi-threshold graphs (see Yan et al. [13]).

3.4. Weakly quasi-threshold graphs

In this subsection we remind some results on weakly quasi-threshold graphs. In ([2]) we study the class of weakly quasi-threshold graphs that are obtained from a vertex by recursively applying the operations (i) adding a new isolated vertex, (ii) adding a new vertex and making it adjacent to all old vertices, (iii) disjoint union of two old graphs, and (iv) adding a new vertex and making it adjacent to all neighbours of an old vertex.

Fie $G = (V, E)$ un graf conex. Atunci urmatoarele sunt echivalente ([2]):
(a) G este un graf weakly quasi-threshold
(b) G este un cograf si nu exista $C_4 = [v_1, v_2, v_3, v_4]$ cu $N(v_1) \neq N(v_3)$ si $N(v_2) \neq N(v_4)$.

A graph G is weakly quasi-threshold ([18]) if and only if G does not contain any P_4 or *co*-($2P_3$) as induced subgraphs.

3.5. Mock threshold graphs

In this subsection we remind some results on mock threshold graphs.

Theorem 4. *A graph G is threshold if and only if G has a vertex ordering* $v_1, \ldots, v_n$ *such that for every i (*$1 \leq i \leq n$*) the degree of* v_i *in G(*$v_1, \ldots, v_i$*) is 0 or i-1.*

By relaxing this characterization slightly we get a new, bigger class of graphs.

Definition 3. *A graph G is said to be mock threshold1 if there is a vertex ordering* $v_1, \ldots, v_n$ *such that for every i (*$1 \leq i \leq n$*) the degree of* v_i *in G(*$v_1, \ldots, v_i$*) is 0, 1, i-2, or i-1. We write* G_{MT} *for the class of mock threshold graphs.*

We call such an ordering an MT-ordering. Note that a graph can have several MT-orderings. There are several easy but important consequences of the definition.

4. New results on threshold graphs

4.1. Characterization of a threshold graph using the weakly decomposition

In this paragraph we give a new characterization of threshold graphs using the weakly decomposition. Some of the following result is found in [21], but we give a demonstration for a fully presentation.

Theorem 5. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a threshold graph if and only if:*
i) $A \sim N \sim R$;
ii) N is clique and R is stable set;
iii) G(A) is threshold graph.

Proof. Let $G = (V, E)$ be a connected, uncomplete graph and (A, N, R) a weakly decomposition of G, with $G(A)$ as the weakly component.
At first, we assume that G is threshold. Then $N \sim R$ and $A \sim N$ also, as otherwise $a \in A$, $n \in N$ would exists such that $an \notin E$. Because $N = N(A)$ it follows that there exists $a_1 \in A$ such that $na_1 \in E$. As $G(A)$ is connected, a path P_{aa_1} exists. On the path from a to a_1 in P_{aa_1}, let $a_2 \in A$ the last vertex with $a_2n \notin E$ and $a_3 \in A$ the first vertex with $a_3n \in E$. Then $G(\{a_2, a_3, n, r\}) \simeq P_4$, for every $r \in R$, so i) holds.

If N would not be a clique then (as $A \sim N \sim R$) an induced C_4 would exists. This would be a contradiction, as G is threshold. So N is a clique and $A \sim N \sim R$.
Suppose that R is not stable. Then an edge r_1r_2 $(r_1, r_2 \in R)$ exists such that $G(\{r_1, r_2, a_1, a_2\}) \simeq 2K_2$, for every $a_1 \in A$ and every $a_2 \in A$, as $|A| \geq 2$. Indeed, if $|A| = 1$ then because R is not stable there exists $R' \subseteq R$ such that $G(R')$ is connected. Suppose that R' is maximal with respect to inclusion. Then $G(R')$ is a weakly component as R' is a weakly set ($N_G(R') = N \neq A \cup N \cup (R - R') = V - R'$, $G(R')$ is connected) and R' is maximal with respect to inclusion. We have $|R'| > |A|$, contradicting the maximality of A. As $A \neq \emptyset$, it follows that $|A| \geq 2$. So R is stable. So ii) also holds.

As G is threshold we have that $G(A)$ is threshold, so iii) holds, too.

Conversely, we suppose that i), ii) and iii) hold. If we suppose that $X \subset V$ exists such that $G(X) \simeq 2K_2$ then, as $A \sim N \sim R$, N clique and R stable, it follows that $X \subseteq A$, contradicting that $G(A)$ is threshold. If we suppose that $G(X) \simeq P_4$ then $X \subseteq A$, contradicting iii). In a similar manner we can prove that G is C_4-free. So G is threshold.

The above results lead to a recognition algorithm with the total execution time $O(n(n+m))$.

4.2. Characterization of a quasi-threshold graph using the weakly decomposition

In this paragraph we give a characterization of quasi-threshold graphs using the weakly decomposition.

Theorem 6. [21] *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a quasi-threshold graph if and only if:*
i) $A \sim N \sim R$;
ii) N is clique;
iii) G(A),G(R) are quasi-threshold graph.

4.3. Characterization of a weakly quasi-threshold graph using the weakly decomposition

In this paragraph we give a new characterization of weakly quasi-threshold graphs using the weakly decomposition.

Theorem 7. [22] *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a weakly quasi-threshold graph if and only if:*
i. $A \sim N \sim R$;
ii. G(A),G(N),G(R) are $\overline{P}_3$-free graph;
iii. G(A),G(N),G(R) are weakly quasi-threshold graph.

4.4. Determine the Laplacian spectrum in threshold graphs

The survey spectrum graph is an important subject in the algebraic theory of graphs [23]. Laplacian is also related to the eigenvalues graph of the Wiener index [24]

Theorem 8. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. If G is a weakly quasi-threshold graph then* :
i) $d_G(r) = |N|, \forall r \in R$;
ii) $d_G(a) = |A| + d_{G(A)}(a), \forall a \in A$;
iii) $d_G(x) = |V| - 1, \forall x \in N$.
Proof. It folows of Theorem 7.

Definition 4. *Let P be a hereditary property a vertices of a graph (ie, if v have the property P in G and G' is subgaph of G so that v is in V(G'), then it has property P in G'). The graph G is called locally*

$P-perfect$ if every induced subgraph H of G it is a vertex that has the property P in H.

Proposition 1. *The graph G is locally P-perfect if and only if there is an order $x_1, ...x_n$ of the vertices set of G such that in $G_i = [x_1, ..., x_i]_G$ the vertex x_i has the property P, $\forall i = 1, ..., n$.*

Demonstration. Suppose G is locally P-perfect. Consider G_n = G . Then there is x_n with the property P in G_n. Let $G_{n-1} = G_n - x_n$. Then there is x_{n-1} with the property P in G_{n-1}. We continue the process. Obtain an order $x_1, ..., x_n$ such that x_i has the property P in G_i, $\forall i = 1, ..., n$.

Suppose that there is an order $x_1, ..., x_n$ in $G_i = [x_1, ..., x_i]_G$ such that x_i has the property P in G_i, $\forall i = 1, ..., n$. Let G ' be an induced subgraph of G. The order of the vertices of G induce an order $x_{k_1}...x_{k_n}$ of the set of vertices of G'. If $k_p = j$ then x_{k_p} it is the property P in G_j. According to the election of $x_{k_p}(= x_j)$ we have V(G')⊆V (G_j). As the property P is hereditary the result that x_{k_p} has the property P in G'. So G is locally P-perfect.

In [1a], R. Behr, V. Sivaraman and T. Zaslavsky specifies:

Proposition 2. *Every induced subgraph of a mock threshold graph is mack threshold.*

So, the mock threshold graphs are hereditary in the sense of induced subgraphs.

We consider the property P:

$(*) : The degree of v i in G(v_1, \ldots, v_i) is 0, 1, i - 2 or i - 1.$

Corollary 2. *A graph G is mock threshold if and only if G is locally P-perfect, where P is the property specified in (*).*

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