"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 28(2018), No. 1, 115-130

FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE AND EXPANSIVE TYPE MAPPINGS OVER A C^*-ALGEBRA VALUED METRIC SPACE

KUSHAL ROY AND MANTU SAHA

Abstract. In this paper, fixed points of generalized contractive mappings and $n-$times reasonable expansive mappings over a C^*-algebra valued metric space have been investigated. The results obtained so far are the existence of fixed points of generalized contractive mappings via the notion of $d-$point of a lower semi-continuous function on the underlying space. Also a result on coincidence point of two mappings has been established. Some examples are given in support of fixed points of expansive mappings.

1. Introduction

The theory of fixed points is an emerging area for various branches of Mathematics, especially for non-linear functional analysis, due to its wide applicability. The field of the fixed point theory is vast and open for a great variety of techniques and ideas. Applications of metric fixed point theory play a fundamental role in numerical analysis and approximation theory, in solving matrix equations, functional equations, integral equations, etc.

Keywords and phrases: C^*-algebra, C^*-algebra valued metric space, $d-$point, C^*-algebra valued Hardy-Rogers type contractive mapping, C^*-algebra valued $n-$times reasonable expansive mapping, fixed point.
(2010)Mathematics Subject Classification: 47H10, 54H25.

functional differential equations, functional partial differential equations, functional integral equations etc. In 1977 J.D. Weston (See [12]) gave a necessary and sufficient condition for the completness of a metric space, introducing the notion of $d-$point for a lower-semi continuous function which is bounded from below. In 1973 Hardy and Rogers [5] proved a fixed point theorem for a mapping T satisfying the contractive condition given by

$$d(Tx,Ty) \leq a[d(x,Tx)+d(y,Ty)]+b[d(x,Ty)+d(y,Tx)]+cd(x,y) \tag{1}$$

for all $x, y \in X$, where a, b and c are non-negative numbers such that $2a+2b+c < 1$. In [3], J. Caristi and W.A. Kirk introduced an extension of Banach contraction principle [1], which states that: if (X, d) is a complete metric space and $\phi : X \to [0, \infty)$ is a lower semi continuous map, then a mapping $T : X \to X$ satisfying

$$d(x,Tx) \leq \phi(x) - \phi(Tx) \tag{2}$$

for each $x \in X$, has a fixed point in X. Subsequently, in 1983 a coincidence point theorem for two mappings has been proved by S. Park [9]. Shehwar et.al. [10] extended the Caristi's fixed point theorem for mappings defined on a C^*-algebra valued metric space. Motivated by the surveys on C^*-algebras (see [8]) and C^*-algebra valued metric spaces (see [7]), some fixed point theorems for mappings satisfying contractive conditions on C^*-algebra valued metric spaces have been established. It is worthwhile to note that fixed point theorems for mappings on these underlying spaces are obtained via the notion of $d-$point for a lower semi continuous function.

In light of the work of Chen and Zhu [4] we prove some fixed point theorems for various types of expansive mappings over a C^*-algebra valued metric space. Based on the concept of expansive mapping, the existence of fixed points for $n-$times reasonable expansive mappings over a C^*-algebra valued metric space has been proved.

2. Some important definitions and lemmas

Before going into discussion on a C^*-algebra valued metric space, we recall some notions from the theory of C^*-algebras (see [8]).

Let $\mathbb{A}$ be a unital algebra. An involution on $\mathbb{A}$ is a conjugate-linear mapping $x \to x^*$ on $\mathbb{A}$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$ for any $x, y \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*-$algebra.

A $*-$algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $||x^*|| = ||x||$ is called a Banach$^*-$algebra. Furthermore, a C^*-algebra $\mathbb{A}$ is a Banach$^*-$algebra with $||x^*x|| = ||x||^2$, for all $x \in \mathbb{A}$.

An element $a \in \mathbb{A}$ is said to be a positive element if $a^* = a$ and $\sigma(a) \subset \mathbb{R}^+$, where $\sigma(a) = \{\lambda \in \mathbb{C}$: $\lambda I - a$ is not invertible, I is the unity in $\mathbb{A}$ $\}$. If a is positive we write it as $a \succeq \theta$, where θ is the zero element of $\mathbb{A}$.

A partial ordering on $\mathbb{A}$ is defined by $a \succeq b$ if and only if $(a - b) \succeq \theta$ whenever $a, b \in \mathbb{A}$.

We denote the set of all positive elements of $\mathbb{A}$ by $\mathbb{A}_+$ that is $\mathbb{A}_+$=$\{ a \in \mathbb{A}$: $a \succeq \theta$ $\}$. Also by $\acute{\mathbb{A}}_+$ we denote the set of all positive elements in $\mathbb{A}$ which commutes with each element of $\mathbb{A}$. Each positive element of a C^*-algebra $\mathbb{A}$ has a unique positive square root.

Definition 2.1. [7] Let X be a non-empty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfying the following conditions.

1. $d(x, y) \succeq \theta$ $\forall x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ $\forall x, y \in X$,
3. $d(x, y) \preceq d(x, z) + d(z, y)$ $\forall x, y, z \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Example 2.2. [7] Let $X = \mathbb{R}$ and $\mathbb{A} = M_{2\times2}(\mathbb{R})$ with $||A|| = max\{|a_1|, |a_2|, |a_3|, |a_4|\}$, where a_i's are the entries of the matrix $A \in M_{2\times2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space, where

$$d(x, y) = \begin{pmatrix} |x - y| & 0 \\ 0 & \alpha|x - y| \end{pmatrix}, \alpha \geq 0$$

and the partial ordering on $\mathbb{A}$ is given by

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

iff $a_i \geq b_i$ for $i = 1, 2, 3, 4$.

Definition 2.3. [7] (i) A sequence $\{ x_n \}$ in a C^*-algebra valued metric space $(X, \mathbb{A}, d)$ is called convergent and converges to some $x \in X$ with respect to $\mathbb{A}$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $||d(x_n, x)||< \varepsilon$ whenever $n \geq N$ i.e. $d(x_n, x) \to \theta$ as $n \to \infty$ and we write $x_n \to x$ as as $n \to \infty$ in X.

(ii) A sequence $\{ x_n \}$ in a C^*-algebra valued metric space $(X, \mathbb{A}, d)$ is said to be cauchy with respect to $\mathbb{A}$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $||d(x_n, x_m)||< \varepsilon$ whenever $n, m \geq N$ i.e. $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.

(iii) A C^*-algebra valued metric space $(X, \mathbb{A}, d)$ is called complete if any cauchy sequence in X with respect to $\mathbb{A}$ is convergent.

Definition 2.4. [2] Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $T : X \to X$ is said to be continuous at $a \in X$ with respect to $\mathbb{A}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $||d(Tx, Ta)|| < \varepsilon$ whenever $||d(x, a)|| < \delta$. T is called continuous on X with respect to $\mathbb{A}$ if it is continuous at any point $a \in X$. Evidently if $\{x_n\}$ be a sequence in X such that $\{x_n\} \to x$ as n$\to \infty \Rightarrow Tx_n \to Tx$ as $n \to \infty$ in X that is $d(x_n, x) \to \theta \Rightarrow d(Tx_n, Tx) \to \theta$ as $n \to \infty$ then T is continuous at $x \in X$.

Definition 2.5. [10] Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $\phi : X \to \mathbb{A}$ is called lower semi-continuous at t_0 with respect to $\mathbb{A}$ if

$$||\phi(t_0)|| \leq liminf_{t \to t_0} ||\phi(t)|| \tag{3}$$

that is for any $\varepsilon > 0$ $||\phi(t_0)|| < ||\phi(t)|| + \varepsilon$ $\forall t \in B(t_0, \delta_\varepsilon)$ for some $\delta_\varepsilon > 0$, where $B(t_0, \delta_\varepsilon) = \{x \in X : ||d(x, t_0)|| < \delta_\varepsilon\}$.

Lemma 2.6. [6] *We state some results that hold in a C^*-algebra with the partial ordering generated by the positive elements.*

1. $\mathbb{A}_+ = \{a^* a : a \in \mathbb{A}\}$,

2. If $a^* = a$, $b^* = b$, $a \preceq b$ and $c \in \mathbb{A}$ then $c^* ac \preceq c^* bc$,

3. For all $a, b \in \mathbb{A}$ such that $a^* = a$, $b^* = b$ if $\theta \preceq a \preceq b$ then $||a|| \leq ||b||$.

Lemma 2.7. [7] *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unity I.*

1. If $a \in \mathbb{A}_+$ with $||a|| < \frac{1}{2}$, then $I - a$ is invertible and $||a(I - a)^{-1}|| < 1$,

2. Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$ then $ab \succeq \theta$,

3. By $\acute{\mathbb{A}}$ we denote the set $\{a \in \mathbb{A} : ab = ba \ \forall\ b \in \mathbb{A}\}$. Let $a \in \acute{\mathbb{A}}$. If $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I - a \in \acute{\mathbb{A}}_+$ is invertible, then $(I - a)^{-1} b \succeq (I - a)^{-1} c$.

Lemma 2.8. *Let $\mathbb{A}$ be an unital C^*-algebra with unity I.*

1.[8] *If a, b are positive elements of $\mathbb{A}$, then the inequality $a \preceq b$ implies $a^{\frac{1}{2}} \preceq b^{\frac{1}{2}}$.*

2.[11] *If $\mathbb{A}$ be commutative then the ordered structure on $\mathbb{A}$ forms a lattice.*

3. Main results on fixed points of some C^*-algebra valued generalized contractive type mappings

Definition 3.1. Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and T be a self map on X. Then T is said to be orbitally continuous at $u \in X$ if for any $x \in X$ $||d(T^{n_i}x, u)|| \to 0$ as $i \to \infty$ implies $||d(T^{n_i+1}x, Tu)|| \to 0$ as $i \to \infty$.

Lemma 3.2. $\mathbb{A}$ *is a commutative* C^**-algebra then*

(a) For some positive element h *in* $\mathbb{A}$, $(h - I)$ *is invertible implying that* $(\sqrt{h} - I)$ *is invertible in* $\mathbb{A}$.

(b) $h \succeq I$, $h \neq I$ *and* $(h - I)$ *is invertible then* $(h - I)^{-1} \succeq \theta$.

Proof. (a) If $(h - I)^{-1} = u$ then we get $(h - I)u = I$ implies $(\sqrt{h} - I)(\sqrt{h}+I)u = I$, if we take $(\sqrt{h}+I)u = v$ then we have $(\sqrt{h}-I)v = I$ and thus $(\sqrt{h} - I)$ is invertible in $\mathbb{A}$.

(b) It is sufficient to prove that if $a \succeq \theta$ and a^{-1} exists then $a^{-1} \succeq \theta$. If $a \succeq \theta$ and a^{-1} exists then $\sigma(a) = \{\lambda \in \mathbb{C} : (\lambda I - a)$ is not invertible$\} \subset (0, \infty)$. Now $\sigma(a^{-1}) = \{\mu \in \mathbb{C} : (\mu I - a^{-1})$ is not invertible$\} \subset \mathbb{R}$ since $(a^{-1})^* = a^{-1}$. If $\lambda \in \sigma(a)$ then $\lambda \neq 0$ and $\lambda I - a$ is not invertible implying that $(\frac{1}{\lambda}a - I)$ is not invertible that is $a^{-1}(\frac{1}{\lambda}a - I)$ is not invertible. So $(\frac{1}{\lambda}I - a^{-1})$ is not invertible and we get $\frac{1}{\lambda} \in \sigma(a^{-1})$. Similarly if $\mu \in \sigma(a^{-1})$ then clearly $\frac{1}{\mu} \in \sigma(a)$. Therefore it follows that $\sigma(a^{-1}) \subset (0, \infty)$. Hence $a^{-1} \succeq \theta$. ∎

Lemma 3.3. *Let* $(X, \mathbb{A}, d)$ *be a* C^*-*algebra valued metric space. If* $\phi : X \to \mathbb{A}_+$ *is a lower semi-continuous function then the set* $K\eta = \{x \in X : ||\phi(x)|| \leq \eta\}, \eta > 0$ *is closed in* X.

Proof. Let $z \in \overline{K_\eta}$, for some $\eta > 0$. So for all $\delta > 0$, $B(z, \delta) \cap K_\eta \neq \phi$. Let $y_\delta \in B(z, \delta) \cap K_\eta$ for each $\delta > 0$. This implies that $\forall\ \delta > 0$, $||d(y_\delta, z)|| < \delta$ and $||\phi(y_\delta)|| \leq \eta$ for each $\delta > 0$. Since ϕ is lower semi-continuous in X, then for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $||\phi(z)|| < ||\phi(x)||+\varepsilon\ \forall x \in B(z, \delta_\varepsilon)$. Now $y_{\delta_\varepsilon} \in K_\eta$ and $||d(y_{\delta_\varepsilon}, z)|| < \delta_\varepsilon$. Therefore $||\phi(y_{\delta_\varepsilon})|| \leq \eta$. Therefore $||\phi(z)|| < ||\phi(y_{\delta_\varepsilon})||+\varepsilon \leq \eta+\varepsilon$ Since $\varepsilon > 0$ is arbitrary, $||\phi(z)|| \leq \eta$. So $z \in K_\eta$ and hence K_η is closed. ∎

Lemma 3.4. *Every lower semi-continuous function* ϕ *from a complete* C^**-algebra valued metric space* X *into* $\mathbb{A}_+$ *has a* $d-$*point* p *in* X, *that is we get*

$$\phi(p) - \phi(x) \preceq d(p, x) \text{ and } \phi(p) - \phi(x) \neq d(p, x) \tag{4}$$

for each point $x(\neq p) \in X$.

Proof. If possible, let ϕ has no $d-$point in X. Then for each $p \in X$ there exists some $x_p \in X$ such that $p \neq x_p$ and $d(p, x_p) \preceq \phi(p) - \phi(x_p)$. For each $p \in X$ we choose one and only one $x_p \in X$ and then we get a map $h : X \to X$ defined by $h(p) = x_p, \forall p \in X$.
Then $d(p, h(p)) \preceq \phi(p) - \phi(h(p))$. So by applying Caristi-Kirk fixed point theorem on a C^*-algebra valued metric space we see that h has a fixed point in X, a contradiction. Therefore ϕ has a $d-$point p in X. ∎

Lemma 3.5. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. Also let $\phi : X \to \mathbb{A}_+$ be a lower semi-continuous function. We define a relation $\ll \phi$ on X by $x \ll \phi y$ iff $\phi(y) - \phi(x) \succeq d(x, y)(\neq \theta)$. If f is a function on X and $\ll \phi$ has the property that if $f(x) \neq x$ then $f(x) \ll \phi x$, then any $d-$point of ϕ is a fixed point of f.*

Proof. Let p be a $d-$point of ϕ. Then $\phi(p) - \phi(x) \preceq d(p, x)$ and $\phi(p) - \phi(x) \neq d(p, x)$ for all $x \neq p$. Now let $f(p) \neq p$ then by the given property $f(p) \ll \phi p$.
Therefore $\phi(p) - \phi(f(p)) \succeq d(p, f(p))$ but since p is a $d-$point of ϕ then $\phi(p) - \phi(f(p)) \preceq d(p, f(p))$ and $\phi(p) - \phi(f(p)) \neq d(p, f(p))$, a contradiction.
Thus $f(p) = p$ and so p is a fixed point of f. ∎

Lemma 3.6. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $Y \subset X$. Suppose $f, g : Y \to X$ satisfy the following conditions:*
(1) f is surjective mapping,
(2)There exists a function $\phi : X \to \mathbb{A}_+$ which is lower semi-continuous such that $d(f(x), g(x)) \preceq \phi(f(x)) - \phi(g(x))\ \forall x \in Y$. Then f and g have a coincidence point in X.

Proof. Since ϕ is lower semi-continuous on X, by Theorem 3.4 we have ϕ has a $d-$point p in X. So $\phi(p) - \phi(x) \preceq d(p, x)$ with $\phi(p) - \phi(x) \neq d(p, x)$ for every $x(\neq p) \in X$.
Since f is surjective, there exists $t_0 \in Y$ such that $f(t_0) = p$ and we get $\phi(f(t_0)) - \phi(x) \preceq d(f(t_0), x)$ with $\phi(f(t_0)) - \phi(x) \neq d(f(t_0), x)$ for every $x(\neq f(t_0)) \in X$. Now if $g(t_0) \neq f(t_0)$ then $\phi(f(t_0)) - \phi(g(t_0)) \preceq d(f(t_0), g(t_0))$ with $\phi(f(t_0)) - \phi(g(t_0)) \neq d(f(t_0), g(t_0))$.
But it is given that $d(f(t_0), g(t_0)) \preceq \phi(f(t_0)) - \phi(g(t_0))$, a contradiction. So $f(t_0) = g(t_0)$. Thus f and g have a coincidence point. ∎

Lemma 3.7. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $f : X \to X$ be a C^*-algebra valued Hardy-Rogers type* *mapping that is for any* $x, y \in X$

$$d(fx, fy) \preceq a[d(x, fx) + d(y, fy)] + b[d(x, fy) + d(y, fx)] + cd(x, y)$$

where $a, b, c \in \acute{\mathbb{A}}_+$ *and* $2||a|| + 2||b|| + ||c|| < 1$. *Let* $X = \overline{o(x_0)}$ *for some* $x_0 \in X$, *where* $o(x_0) = \{x_0, f(x_0), f^2(x_0), ...\}$. *Then the function* $\phi : X \to \mathbb{A}_+$ *defined by* $\phi(x) = \beta d(x, fx)$ *for all* $x \in X$*;* $\beta \in \acute{\mathbb{A}}_+$, *is lower semi-continuous function on* X.

Proof. Let $K\eta = \{x \in X : ||\phi(x)|| \leq \eta\}, \eta > 0$ and let $y \in \overline{K\eta}$. Then $y \in \overline{o(x_0)}$. Therefore clearly there exists a sequence $\{y_n\}$ in $o(x_0) \cap K\eta$ such that $y_n \to y$ as $n \to \infty$. Since f is orbitally continuous on X so $f(y_n) \to f(y)$ as $n \to \infty$.
Now,

$$\begin{aligned} \phi(y) = \beta d(y, fy) &\preceq \beta\left[d(y, y_n) + d(y_n, fy_n) + d(fy_n, fy)\right] \\ &= \beta\left[d(y, y_n) + d(fy_n, fy)\right] + \phi(y_n). \end{aligned}$$

Which implies that,

$$\begin{aligned} ||\phi(y)|| &\leq ||\beta||\left[||d(y, y_n)|| + ||d(fy_n, fy)||\right] + ||\phi(y_n)|| \\ &\leq ||\beta||\left[||d(y, y_n)|| + ||d(fy_n, fy)||\right] + \eta, \ \forall n \in \mathbb{N}. \end{aligned}$$

Taking $n \to \infty$ we get, $||\phi(y)|| \leq \eta$. Therefore $y \in K\eta$ and so $K\eta$ is closed. Since $\eta > 0$ is arbitrary therefore ϕ is a lower semi-continuous function on X. ∎

Theorem 3.8. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^*-*algebra valued metric space and* $f : X \to X$ *be a* C^*-*algebra valued Hardy-Rogers type mapping that is for any* $x, y \in X$

$$d(fx, fy) \preceq a[d(x, fx) + d(y, fy)] + b[d(x, fy) + d(y, fx)] + cd(x, y)$$

where $a, b, c \in \acute{\mathbb{A}}_+$ *and* $2||a|| + 2||b|| + ||c|| < 1$. *If* $X = \overline{o(x_0)}$ *for some* $x_0 \in X$, *where* $o(x_0) = \{x_0, f(x_0), f^2(x_0), ...\}$ *and* $\phi : X \to \mathbb{A}_+$ *is defined by* $\phi(x) = [I - (2a + 2b + c)]^{-1}[I - (a + b)]d(x, fx), \forall x \in X$ *then by defining the relation* $\ll \phi$ *in* X *we get* f *has a fixed point in* X *which is also unique.*

Proof. For any $x \in X$ we get

$$\begin{aligned} d(fx, f^2x) &\preceq a[d(x, fx) + d(fx, f^2x)] + b[d(x, f^2x) + d(fx, fx)] \\ &\quad + cd(x, fx) \\ &\preceq a[d(x, fx) + d(fx, f^2x)] + b[d(x, fx) + d(fx, f^2x)] \\ &\quad + cd(x, fx), \end{aligned}$$

which implies that $(I-a-b)\,d(fx,f^2x) \preceq (a+b+c)\,d(x,fx)$ that is $d(fx,f^2x) \preceq (I-a-b)^{-1}(a+b+c)\,d(x,fx)$. Now,

$$\begin{aligned}
\phi(x)-\phi(fx) &= [I-(2a+2b+c)]^{-1}[I-(a+b)] \\
&\quad .[d(x,fx)-d(fx,f^2x)] \\
&\succeq [I-(2a+2b+c)]^{-1}[I-(a+b)][I-(a+b)]^{-1} \\
&\quad .[I-(2a+2b+c)]d(x,fx),
\end{aligned}$$

implying that $\phi(x)-\phi(fx) \succeq d(x,fx)$.
Thus $\ll \phi$ has the property that whenever $fx \neq x$ then $fx \ll \phi x$. Now by Lemma 3.4 ϕ has atleast one $d-$point in X, so f has atleast one fixed point in X by Lemma 3.5.
Now let u and v be two fixed points of f then $d(u,v)=d(fu,fv) \preceq (2a+2b+c)d(u,v)$ implying that

$$\begin{aligned}
||d(u,v)|| &\leq ||2a+2b+c||||d(u,v)|| \\
&\leq (2||a||+2||b||+||c||)||d(u,v)||.
\end{aligned}$$

This implies $||d(u,v)||=0$ and so $u=v$. Therefore f has a unique fixed point in X. ∎

Corollary 3.9. *(a) If in Theorem 3.8 we put $a=b=\theta$ then f will reduce to a Banach contraction mapping and for this case ϕ is given by $\phi(x)=(I-c)^{-1}d(x,fx)$ $\forall x \in X$ consequently f will have a unique fixed point in X.*

(b) If in Theorem 3.8 we put $b=c=\theta$ then f will reduce to a Kannan type mapping and for this case ϕ is given by $\phi(x)=(I-2a)^{-1}(I-a)d(x,fx)$ $\forall x \in X$ and hence f has a unique fixed point in X.

(c) If in Theorem 3.8 we put $a=c=\theta$ then f will reduce to a Chatterjee type mapping and for this case ϕ is given by $\phi(x)=(I-2b)^{-1}(I-b)d(x,fx)$ $\forall x \in X$. So f must have a unique fixed point in X.

4. Main results on fixed point of C^*-algebra valued $n-$times reasonable expansive mapping

Now let $(X,\mathbb{A},d)$ be a commutative C^*-algebra valued metric space. Here the following definitions are analouge of the work of Chen and Zhu [4].

Definition 4.1. A mapping $T : X \to X$ is called C^*-algebra valued expansive mapping if there exists a fixed element(constant) $h \succeq I$ such that

$$d(Tx, Ty) \succeq hd(x, y) \ \forall \ x, y \in X. \tag{5}$$

We now give the following definitions.

Definition 4.2. A mapping $T : X \to X$ is called C^*-algebra valued generalized expansive mapping of metric-1 type if there exists a fixed element(constant) $h \succeq I, (h \neq I)$ such that

$$d(Tx, Ty) \succeq h \ \inf\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{6}$$

for all $x, y \in X$.

Definition 4.3. A mapping $T : X \to X$ is called C^*-algebra valued generalized expansive mapping of metric-2 type if there exists a fixed element(constant) $h \succeq I, (h \neq I)$ such that

$$\begin{aligned} d^2(Tx, Ty) \succeq h \ \inf\{&d^2(x, y), d(x, y).d(x, Tx), d(x, Tx).d(y, Ty), \\ &d^2(x, Tx), d(y, Ty).d(x, Ty), d(y, Ty).d(y, Tx)\} \end{aligned} \tag{7}$$

for all $x, y \in X$.

Definition 4.4. A mapping $T : X \to X$ is called C^*-algebra valued $n-$times reasonable expansive mapping if there exists a fixed element(constant) $h \succeq I(h \neq I)$ such that

$$d(x, T^n x) \succeq h \, d(x, Tx) \, \forall x \in X(n \geq 2, n \in \mathbb{N}). \tag{8}$$

Definition 4.5. A mapping $T : X \to X$ is called C^*-algebra valued $n-$times reasonable expansive mapping of metric-1 type if there exists a fixed element(constant) $h \succeq I, h \neq I$ such that

$$\begin{aligned} d(T^{n-1}x, T^{n-1}y) \succeq h \ \inf\{&d(x, y), d(x, Tx), d(T^{n-2}y, T^{n-1}y), \\ &d(x, T^{n-1}y), d(T^{n-2}y, T^{n-1}x)\} \end{aligned} \tag{9}$$

for all $x, y \in X(n \geq 2, n \in \mathbb{N})$.

Definition 4.6. A mapping $T : X \to X$ is called C^*-algebra valued $n-$times reasonable expansive mapping of metric-2 type if there exists a fixed element(constant) $h \succeq I, h \neq I$ such that

$$\begin{aligned} d^2(T^{n-1}x, T^{n-1}y) \succeq\ & h\ \inf\{d^2(x,y), d(x,y).d(x,Tx), \\ & d(x,Tx).d(T^{n-2}y, T^{n-1}y), \\ & d^2(x,Tx), d(T^{n-2}y, T^{n-1}y).d(x, T^{n-1}y), \\ & d(T^{n-2}y, T^{n-1}y).d(T^{n-2}y, T^{n-1}x)\} \end{aligned} \tag{10}$$

for all $x, y \in X (n \geq 2, n \in \mathbb{N})$.

Theorem 4.7. *Let $(X, \mathbb{A}, d)$ be a complete commutative C^*-algebra valued metric space and let $T : X \to X$ be a continuous and surjective mapping. If there exists a fixed element(constant) $h \succeq I$ such that $(h - I)$ is invertible $(h \neq I)$ and*

$$d(T^{n-1}x, T^n x) \succeq h\, d(x, Tx)\ \forall x \in X (n \geq 2, n \in \mathbb{N}). \tag{11}$$

Then T has a fixed point in X.

Proof. Given that, $d(T^{n-1}x, T^n x) \succeq h\, d(x, Tx)\ \forall\, x \in X$. Then we get, $d(T^{n-1}x, T^n x) - d(x, Tx) \succeq hd(x, Tx) - d(x, Tx)$ that is $(h-I)d(x, Tx) \preceq d(T^{n-1}x, T^n x) - d(x, Tx)$, which implies that $d(x, Tx) \preceq (h - I)^{-1}[d(T^{n-1}x, T^n x) - d(x, Tx)]\ \forall x \in X$.
Let us take $\phi(x) = (h - I)^{-1}[d(x, Tx) + d(Tx, T^2x) + \ldots + d(T^{n-3}x, T^{n-2}x)$
$+ d(T^{n-2}x, T^{n-1}x)]$ for all $x \in X$. So $d(x, Tx) \preceq \phi(Tx) - \phi(x)\ \forall x \in X$. From the continuity of T, we know that ϕ is continuous.

Now in Lemma 3.6 if we take $Y = X$ and $g = I_X$ then $T = f$ satisfies $d(f(x), x) \preceq \phi(f(x)) - \phi(x)\ \forall x \in X$ (i.e Caristi fixed point theorem in C^*-algebra valued metric space). Hence f has a fixed point in X. Therefore T has a fixed point in X. ∎

Theorem 4.8. *Let $(X, \mathbb{A}, d)$ be a complete commutative C^*-algebra valued metric space and let $T : X \to X$ be a continuous, surjective C^*-algebra valued $n-$times reasonable expansive mapping which satisfies $d(T^{n-1}x, T^n x) \succeq d(x, Tx)\ \forall x \in X$. Assume that either (i) or (ii) holds.*

(i) T is a C^*-algebra valued $n-$times reasonable expansive mapping of metric-1 type.

(ii) T is a C^*-algebra valued $n-$times reasonable expansive mapping of metric-2 type.

For the same constant $h \succeq I (h \neq I)$ such that $(h - I)$ is invertible, then T has a fixed point in $X (n \geq 2, n \in \mathbb{N})$.

Proof. **case (i):** Let $x \in X$. If $Tx = x$ then T has a fixed point in X. So let $Tx \neq x$.
Taking $y = Tx$ in (9) we get,

$$\begin{aligned} d(T^{n-1}x, T^n x) &\succeq h \, \inf\{d(x,Tx), d(x,Tx), d(T^{n-1}x, T^n x), d(x, T^n x), \\ &\quad d(T^{n-1}x, T^{n-1}x)\} \\ &= h \, \inf\{d(x,Tx), d(T^{n-1}x, T^n x), d(x, T^n x)\}. \end{aligned}$$

As T is a C^*-algebra valued n-times resonable expansive mapping, we have $d(x, T^n x) \succeq h \, d(x,Tx) \succeq d(x,Tx)$ and $h \, d(x,Tx) \neq d(x,Tx)$. Thus we obtain, $d(T^{n-1}x, T^n x) \succeq h \, \inf\{d(x,Tx), d(T^{n-1}x, T^n x)\}$.
Since $d(T^{n-1}x, T^n x) \succeq d(x,Tx) \; \forall x \in X$ then $d(T^{n-1}x, T^n x) \succeq h \, d(x,Tx) \; \forall x \in X$. So from Theorem 4.7 we obtain that T has a fixed point in X.
case (ii): Let us take $x \in X$. If $Tx = x$ then x is a fixed point of T in X.

So let $Tx \neq x$. Taking $y = Tx$ in (10) we get,

$$\begin{aligned} d^2(T^{n-1}x, T^n x) &\succeq h \, \inf\{d^2(x,Tx), d(x,Tx).d(x,Tx), \\ &\quad d(x,Tx).d(T^{n-1}x, T^n x), d^2(x,Tx), \\ &\quad d(T^{n-1}x, T^n x).d(x, T^n x), \\ &\quad d(T^{n-1}x, T^n x).d(T^{n-1}x, T^{n-1}x)\} \\ &= h \, \inf\{d^2(x,Tx), d(x,Tx).d(T^{n-1}x, T^n x), \\ &\quad d(T^{n-1}x, T^n x).d(x, T^n x)\}. \end{aligned}$$

We have, $d(x, T^n x) \succeq h \, d(x,Tx) \succeq d(x,Tx)$ and $h \, d(x,Tx) \neq d(x,Tx)$, since T is a C^*-algebra valued n-times resonable expansive mapping. Hence $d(x, T^n x).d(T^{n-1}x, T^n x) \succeq d(x,Tx).d(T^{n-1}x, T^n x)$ and $d(x, T^n x).d(T^{n-1}x, T^n x) \neq d(x,Tx).d(T^{n-1}x, T^n x)$. Therefore, we have $d^2(T^{n-1}x, T^n x) \succeq h \, \inf\{d^2(x,Tx), d(x,Tx).d(T^{n-1}x, T^n x)\}$.

Now, $d(T^{n-1}x, T^n x) \succeq d(x,Tx)$ implying that $d(x,Tx).d(T^{n-1}x, T^n x)$
$\succeq d^2(x,Tx)$. Therefore $d^2(T^{n-1}x, T^n x) \succeq h \, d^2(x,Tx)$ which implies that, $d(T^{n-1}x, T^n x) \succeq \sqrt{h} \, d(x,Tx) \; \forall x \in X (n \geq 2, n \in \mathbb{N})$.
It is given that $h \succeq I$ and $h \neq I$ so $\sqrt{h} \succeq I, \sqrt{h} \neq I$ and $(\sqrt{h} - I)^{-1}$ exists. So by Theorem 4.7 we get T has a fixed point in X. ∎

Corollary 4.9. *Let $(X, \mathbb{A}, d)$ be a complete commutative C^*-algebra valued metric space. If $T : X \to X$ is a continuous and surjective generalized expansive mapping of metric-1 type and a C^*-algebra valued two times reasonable expansive mapping which satisfies* $d(Tx, T^2 x) \succeq$

$d(x, Tx)$ $\forall x \in X$, where the fixed element(constant) h satisfies $h \succeq I, h \neq I$ and $(h-I)^{-1}$ exists, then T has a fixed point in X.

Proof. Taking $n = 2$ in (9) and denoting $y = T^0 y$ we get T is a generalized expansive mapping of metric-1 type. So using theorem 4.8 under condition 4.8(i) it follows that T has a fixed point in X. ∎

Corollary 4.10. *Let $(X, \mathbb{A}, d)$ be a complete commutative C^*-algebra valued metric space. If $T : X \to X$ is a continuous and surjective generalized expansive mapping of metric-2 type and a C^*-algebra valued two times reasonable expansive mapping which satisfies $d(Tx, T^2x) \succeq d(x, Tx)$ $\forall x \in X$, where the fixed element(constant) h satisfies $h \succeq I, h \neq I$ and $(h-I)^{-1}$ exists, then T has a fixed point in X.*

Proof. In (10) taking $n = 2$ and denoting $y = T^0 y$ we get T is a generalized expansive mapping of metric-2 type. So using theorem 4.8 under condition 4.8(ii) it follows that T has a fixed point in X. ∎

Theorem 4.11. *Let $(X, \mathbb{A}, d)$ be a complete commutative C^*-algebra valued metric space. If $T : X \to X$ is a continuous and surjective C^*-algebra valued n-times resonable expansive mapping which satisfies*
(12)
$$d(T^n x, T^n y) \succeq h \ \inf\{d(x, y), d(y, T^n y)\} \quad \forall\, x, y \in X (n \geq 2, n \in \mathbb{N})$$
and the fixed element(constant) $h \succeq I$, $h \neq I$ with $(h-I)$ is invertible, then T has a fixed point in X.

Proof. Letting $x = Ty$ in (12) we get $d(T^{n+1}y, T^n y) \succeq h \ \inf\{d(Ty, y), d(y, T^n y)\}$ $\forall\, y \in X$.
Since T is a C^*-algebra valued $n-$times resonable expansive mapping then $d(y, T^n y) \succeq h\, d(y, Ty) \succeq d(y, Ty)$ $\forall\, y \in X$. Therefore we get $d(T^n y, T^{n+1} y) \succeq h\, d(y, Ty)$ $\forall\, y \in X$.
So by Theorem 4.7 it follws that T has a fixed point in X. ∎

Here we cite the following examples which are on $n-$times reasonable expansive mapping with fixed points.

Example 4.12. Let $X = \mathbb{R}$ be the metric space with usual metric. Then it is a C^*-algebra valued metric space with the C^*-algebra $\mathbb{A} = \mathbb{R}$.

(a) The identity mapping on X is clearly a $n-$times reasonable expansive mapping for any constant $h > 1$ and for any $n(\geq 2) \in \mathbb{N}$.

(b) The mapping $T : X \to X$ is defined by $Tx = k\, x$ $\forall x \in X$, where $k > 1$, is clearly a $n-$times reasonable expansive mapping for the constant $h = k + 1$ and for any $n(\geq 2) \in \mathbb{N}$.

Example 4.13. Let $B_1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : e^{i\theta} \in \mathbb{C}, -\infty < \theta < \infty\}$ be a unit circular ring in the complex plane $\mathbb{C}$. Then B_1 is a C^*-algebra valued metric space with the C^*-algebra $\mathbb{A} = \mathbb{R}$. Suppose that for $n(\geq 2) \in \mathbb{N}$ $T : B_1 \to B_1$ is a mapping defined as follows:

$$Tz = T(e^{i\theta}) = \begin{cases} e^{i(\theta+\frac{2\pi}{3n})}, & if \;\; \theta \neq 2m\pi, m \in \mathbb{Z} \\ 1, & if \;\; \theta = 2m\pi, m \in \mathbb{Z} \end{cases} \tag{13}$$

For every $z = e^{i\theta} \in B_1$, $\theta \neq 2m\pi, m \in \mathbb{Z}$ we have
$Tz = Te^{i\theta} = e^{i(\theta+\frac{2\pi}{3n})}$,
$T^2z = T(Tz) = T(Te^{i\theta}) = Te^{i(\theta+\frac{2\pi}{3n})} = e^{i(\theta+2(\frac{2\pi}{3n}))}$,
...
$T^nz = e^{i(\theta+n(\frac{2\pi}{3n}))} = e^{i(\theta+\frac{2\pi}{3})}$
Thus we obtain,

$$\begin{aligned} d(z, T^nz) &= |T^nz - z| = |e^{i(\theta+\frac{2\pi}{3})} - e^{i\theta}| = |e^{i\theta}||e^{i\frac{2\pi}{3}} - 1| \\ &= |\cos(\frac{2\pi}{3}) + i\;\sin(\frac{2\pi}{3}) - 1| = |-\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1| = \sqrt{3} \\ d(z, Tz) &= |Tz - z| = |e^{i(\theta+\frac{2\pi}{3n})} - e^{i\theta}| = |e^{i\theta}||e^{i\frac{2\pi}{3n}} - 1| \\ &= |\cos(\frac{2\pi}{3n}) + i\;\sin(\frac{2\pi}{3n}) - 1| = 2\sin(\frac{\pi}{3n}). \end{aligned}$$

Since $n \geq 2$ then $\sin(\frac{\pi}{3n}) \leq \frac{1}{2}$ and so $d(T^nz, z) \geq \sqrt{3}d(z, Tz)$, $\forall z(\neq 1) \in B_1$. Now if $\theta = 2m\pi$ for some $m \in \mathbb{Z}$ then $z = 1$ and by the definition of mapping $Tz = 1$ and consequently $T^nz = 1$ so in this case also clearly $d(T^nz, z) \geq \sqrt{3}d(z, Tz)$. Therefore T is a $n-$times reasonable expansive mapping with the constant $h = \sqrt{3}$.
Here 1 is the unique fixed point of T. Clearly the mapping is surjective but not continuous on B_1.

But the following examples (See Chen and Zhu [4]) are given for $n-$times reasonable expansive mapping without having any fixed point.

Example 4.14. [4] Let B_1 be the unit circle in the complex plane having origin as its center and 1 as its radius, that is $B_1 = \{z \in \mathbb{C} : |z| = 1\}$. B_1 can also be written as $\{e^{i\theta} : e^{i\theta} \in \mathbb{C}, -\infty < \theta < \infty\}$. Then B_1 is a C^*-algebra valued metric space with the C^*-algebra $\mathbb{A} = \mathbb{R}$. Suppose that for $n(\geq 2) \in \mathbb{N}$, $T : B_1 \to B_1$ is a mapping defined as follows:

$$Tz = T(e^{i\theta}) = e^{i(\theta+\frac{2\pi}{3n})} \;\; \forall\, \theta \in \mathbb{R} \tag{14}$$

Then T is a $n-$times reasonable expansive mapping with the constant $h = \sqrt{3}$. Since $e^{i\theta} \neq e^{i(\theta + \frac{2\pi}{3n})}$, then $Tz \neq z$, $\forall z \in B_1$. This shows that T has no fixed points in X. Clearly the mapping is surjective and continuous in B_1.

Example 4.15. [4] Let $X = \mathbb{R}$ be the metric space with usual metric. Then it is a C^*-algebra valued metric space with the C^*-algebra $\mathbb{A} = \mathbb{R}$. Suppose that $T : X \to X$ is a mapping defined as $Tx = x + 1$ $\forall x \in X$. Then T is a $n-$times reasonable expansive mapping with the constant $h = 2$. Clearly T is continuous and Surjective in X but it has no fixed point in X.

Acknowledgments. The first author wishes to thank the Council of Scientific and Industrial Research, New Delhi, India, for providing the research fellowship to continue the research work.

References

[1] Banach, S., *Sur les operations dans les ensembles abstraits et leur application aux equations untegrales*, Fund. Math., **3** (1922), 133-181.

[2] Batul, S. and Kamran, T., *C^*-valued contractive type mappings*, Fixed Point Theory and Applications, **142** (2015), DOI 10.1186/s13663-015-0393-3.

[3] Caristi, J., *Ph.D. Thesis*, University of lowa, (1975).

[4] Chen, C. and Zhu, C., *Fixed Point Theorems for n Times Reasonable Expansive Mapping*, Fixed point Theory and Applications, Hindwai Publishing Corporation, (2008), doi:10.1155/2008/302617.

[5] Hardy, G. and Rogers, T., *A generalization of a fixed point theorem of Reich*, Bull. Cond. Math. Soc., **16** (1973).

[6] Kadelburg, Z. and Radenovic, S., *Fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts*, Fixed Point Theory and Applications, **53** (2016), DOI 10.1186/s13663-016-0544-1.

[7] Ma, Z., Jiang, L. and Sun, H., *C^*-algebra-valued metric spaces and related fixed point theorems*, Fixed Point Theory and Applications, **206** (2014), DOI 10.1186/1687-1812-2014-206.

[8] Murphy, G.J., *C^*-algebras and Operator Theory*, Academic Press, Boston, (1990).

[9] Park, S., *On Extension Of The Caristi-Kirk Fixed Point Theorem*, J. Korean Math. Soc., **19**(2) (1983).

[10] Shehwar, D., Batul, S., Kamran, T. and Ghiura, A., *Caristi fixed point theorem on C^*-algebra valued metric spaces*, Journal of Nonlinear Science and Applications, **9** (2016), 584-588.

[11] Sherman, S., *Order in Operator Algebras*, American Journal of Mathematics, **73**(1) (1951), 227-232.

[12] Weston, J.D., *A characterization of metric completeness*, Proceedings Of The American Mathematical Society, **64**(1) (1977), 186-188.

Department of Mathematics
The University of Burdwan
Burdwan-713104, West Bengal, India
e-mail: kushal.roy93@gmail.com

Department of Mathematics
The University of Burdwan
Burdwan-713104, West Bengal, India
e-mail: mantusaha.bu@gmail.com