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APPROXIMATE DIFFERENTIABILITY IN NEWTONIAN SPACES BASED ON BANACH FUNCTION SPACES

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Abstract. In this note we investigate the approximate differentiability of Newtonian functions on a doubling metric measure space. The Newtonian space under consideration consists of functions belonging to a rearrangement invariant Banach function space $\mathbf{E}$ and possesing an upper gradient which also belongs to $\mathbf{E}$. Our main tools are a Poincaré inequality and a noncentered maximal operator, both defined via the Banach function space $\mathbf{E}$. Under our assumptions, considering for a Newtonian function u an upper gradient g belonging to the given Banach function space $\mathbf{E}$, it turns out that a Hajłasz gradient of u is a constant multiple of the maximal function $\mathcal{M}_{\mathbf{E}}g$, which is Borel measurable and finite almost everywhere.

1. Introduction

Approximate differentiability is a generalization of the concept of differentiability, obtained by replacing the limit by an approximate limit at a point of Lebesgue density 1. The almost everywhere approximate differentiability of a real function on a measurable subset A of $\mathbb{R}^n$ can be characterized using the existence almost everywhere of the partial derivatives [13, Theorem 3.1.4]. Important examples of classes of maps between Euclidean spaces, that are almost everywhere differentiable, are Sobolev classes $W^{1,p}$ and the BV class [12].

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The notion of approximate differentiability is very useful in the theory of optimal transport in noncompact Riemannian manifolds, playing an important role in establishing the existence and differentiability of optimal transport maps [2], [28].

The extension of first order differential calculus to metric spaces progressed from the introduction of the notion of upper gradient [16] to the study of measurable differentiable structures [7]. The following theorem of Cheeger, a deep extension of Rademacher's theorem, shows that a metric space equipped with a doubling measure and supporting a weak $(1,p)-$Poincaré inequality admits a differentiable structure for which Lipschitz functions are a.e. differentiable.

Theorem 1.1. [7] *Let* (X,d,μ) *be a doubling metric measure space supporting a weak* $(1,p)-$*Poincaré inequality for some* $1\leq p<\infty$. *Then there exists a countable collection* $\{(X_\alpha,\varphi_\alpha)\}_{\alpha\in\Lambda}$ *of measurable sets* $X_\alpha\subset X$ *with positive measure and Lipschitz coordinates* $\varphi_\alpha=(\varphi_\alpha^1,...,\varphi_\alpha^{N(\alpha)}):X\to\mathbb{R}^{N(\alpha)}$, *with the following properties:*

(i) $\mu(X\setminus\bigcup_{\alpha\in\Lambda}X_\alpha)=0$;

(ii) There exists a non-negative integer N *such that* $N(\alpha)\leq N$ *for each* $\alpha\in\Lambda$;

(iii) If $f:X\to\mathbb{R}$ *is Lipschitz, then for each* $(X_\alpha,\varphi_\alpha)$, *there exists a unique (up to a set of zero measure) measurable bounded vector valued function* $d^\alpha f:X_\alpha\to\mathbb{R}^{N(\alpha)}$ *such that*

$$\lim_{\substack{y\to x\\ y\neq x}}\frac{|f(y)-f(x)-d^\alpha f(x)\cdot(\varphi_\alpha(y)-\varphi_\alpha(x))|}{d(y,x)}=0, \tag{1}$$

for $\mu-$*almost every* $x\in X_\alpha$.

In (1), " $\cdot$ " denotes the usual inner product on $\mathbb{R}^{N(\alpha)}$.

The collection $\{(X_\alpha,\varphi_\alpha)\}_{\alpha\in\Lambda}$ of Theorem 1.1. is said to be a *strong measurable differentiable structure* for (X,d,μ). A function $f:X\to\mathbb{R}$ is said to be *(Cheeger) differentiable* at a point $x\in X_\alpha$ with respect to the strong measurable differentiable structure $\{(X_\alpha,\varphi_\alpha)\}_{\alpha\in\Lambda}$ if there exists a unique vector $d^\alpha f(x)\in\mathbb{R}^{N(\alpha)}$ (called *Cheeger differential* of f at x) such that (1) holds.

Keith [17] obtained a refinement of Cheeger's theorem, showing that the existence of a strong measurable differentiable structure is still obtained if we replace the conditions that the metric measure space supports a doubling measure and a $(1,p)-$Poincaré inequality by some weaker ones: the property of chunky measure, respectively the comparability of the pointwise upper and lower Lipschitz constants of Lipschitz functions.

In [11] $\{(X_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is said to be an *approximate measurable differentiable structure* for (X, d, μ) if the metric measure space satisfies the conclusion of Theorem 1.1, but with the limit in (1) replaced by an approximate limit. Furthermore, a function $f : X \to \mathbb{R}$ is said to be *approximately differentiable* at $x \in X_\alpha$ with respect to the approximate measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ if there exists a vector $d^\alpha f(x) \in \mathbb{R}^{N(\alpha)}$ such that the analogue of (1) with the limit replaced by approximate limit holds [11]. Note that the notion of approximate differentiability in the setting of metric measure spaces has been considered earlier by Keith [18].

Using Theorem 1.1, Balogh, Rogovin and Zürcher [6] obtained the following generalization of Stepanov's theorem: if (X, d, μ) is a doubling metric measure space, with a strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, then every function $f : X \to \mathbb{R}$ is $\mu-$a.e. differentiable in the set $S_f := \{x \in X : \mathit{Lip}\, f(x) < \infty\}$ with respect to the given strong measurable differentiable structure. The pointwise upper Lipschitz constant of f at x is given by

$$\mathit{Lip}\, f(x) = \limsup_{y\to x} \frac{|f(y) - f(x)|}{d(y, x)}.$$

This Stepanov theorem is very useful in proving the a.e. Cheeger differentiability of some classes of functions on a metric measure space endowed with a strong measurable differentiable structure, e.g. in extending to metric measure spaces Cesari-Calderón theorem [6, Theorem 4.1] and the theorem on the differentiability of monotone continuous functions $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and $p > n - 1$ [23, Theorem 4.2].

Durand-Cartagena, Ihnatsyeva, Korte and Szumańska [11] extended to the metric setting a theorem of Whitney, that characterizes approximately differentiable functions on sets in $\mathbb{R}^n$ as functions having Lipschitz Luzin approximations. Then they proved the following Stepanov type characterization of approximate differentiability [11, Corollary 2.4]: given a complete doubling metric measure space (X, d, μ), endowed with an approximate differentiable structure, a function $f : X \to \mathbb{R}$ is $\mu-$a.e. approximately differentiable in a bounded measurable set $E \subset X$ if and only if

$$\text{(2)} \qquad \text{ap} \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y,x)} < \infty, \ \mu - \text{a.e. in } E.$$

It was shown in [11, Corollary 3.1] that the above theorem yields the approximate differentiability of functions from the Newtonian space $N^{1,p}(X)$ if μ is doubling and (X, d, μ) supports a $(1, p)-$Poincaré inequality, where $1 \leq p < \infty$.

This result was extended to Orlicz-Sobolev spaces in [22, Theorem 3.1], proving that every $f \in N^{1,\Phi}(X)$ satisfies the condition (5), provided that the Young function Φ is strictly increasing and satisfies the Δ_2-condition, while the metric measure space (X, d, μ) is complete, doubling and supports a $(1, \Phi) -$ Poincaré inequality.

The purpose of this note is to prove a further extension of the result regarding the approximate differentiability of Newtonian functions from [11, Corollary 3.1], where the role of the Lebesgue space L^p is played by a Banach function space $\mathbf{E}$, defined as in [4].

Given a doubling metric measure space (X, d, μ) and a rearrangement invariant Banach function space $\mathbf{E}$ on X and assuming that (X, d, μ) supports a weak $(1, \mathbf{E})-$Poincaré inequality, it follows that every function u belonging to the Newtonian space $N^{1,\mathbf{E}}(X)$ satisfies the condition (2) for $\mu-$a.e. $x \in X$. In particular, when (X, d, μ) is complete and endowed with an approximate differentiable structure, we see that every $u \in N^{1,\mathbf{E}}(X)$ is approximately differentiable $\mu-$a.e. in X with respect to the given approximate differentiable structure.

We use as a basic tool a noncentered maximal operator associated to the Banach function space $\mathbf{E}$ [24], [21], an analogue of the maximal operator studied in Euclidean spaces by [3], defined for any $\mu-$measurable real function f on X, by

$$\mathcal{M}_{\mathbf{E}} f(x) = \sup_{B \ni x} \frac{\|f\chi_B\|_{\mathbf{E}}}{\|\chi_B\|_{\mathbf{E}}},$$

where the supremum is taken over all balls $B \subset X$ containing the point $x \in X$.

We use the $(1, \mathbf{E})-$Poincaré inequality introduced in [24] as an extension of several types of Poincaré inequalities on metric measure spaces: the weak $(1, p)-$Poincaré inequality with $1 \leq p < \infty$ [16], a version of the Orlicz-Poincaré inequality introduced by Aïssaoui [1], the Poincaré inequality based on Lorentz spaces [8] and the $\infty-$Poincaré inequality [9]. For $u \in N^{1,\mathbf{E}}(X)$ and $g \in \mathbf{E}$ an upper gradient of u, using $(1,\mathbf{E})-$Poincaré inequality it turns out that $\mathcal{M}_{\mathbf{E}}g$ is a Hajłasz gradient of u, that is finite a.e. under our assumptions.

2. Preliminaries

A metric measure space (X,d,μ) is a metric space (X,d) with an outer Borel regular measure $\mu:\mathcal{P}(X)\to[0,\infty]$.

A measure μ in a metric space (X,d) is said to be doubling if there is a constant $C_d>1$ such that

$$\mu(B(x,2r))\le C_d\,\mu(B(x,r))$$

for every ball $B(x,r)\subset X$. The metric measure space (X,d,μ) is called doubling if the measure μ is doubling.

Lebesgue's differentiation theorem, having as a special case Lebesgue's density theorem, holds in doubling metric measure spaces [15, Theorem 1.8].

In the setting of metric spaces, a substitute for the length of the gradient of a smooth function is the notion of *upper gradient* [16]. Newtonian spaces are analogues of Sobolev spaces on metric measure spaces, based on the notion of upper gradient [26]. A Borel function $g:X\to[0,\infty]$ is said to be an *upper gradient* of a function $u:X\to\mathbb{R}$ if for all compact rectifiable curves $\gamma:[a,b]\to X$ we have

$$|u(\gamma(b))-u(\gamma(a))|\le\int_\gamma g\,ds. \tag{3}$$

We use the definition given by Bennet and Sharpley [4] for a Banach function norm $\boldsymbol{\rho}$ defined on the set of all *non-negative* measurable functions on a $\sigma-$ finite measure space (X,μ). The collection $\mathbf{E}$ of all $\mu-$measurable functions $f:X\to\overline{\mathbb{R}}$ for which $\boldsymbol{\rho}(|f|)<\infty$ is called a *Banach function space* on X. For $f\in\mathbf{E}$ define $\|f\|_{\mathbf{E}}=\boldsymbol{\rho}(|f|)$.

Then $(\mathbf{E},\|\cdot\|_{\mathbf{E}})$ is a seminormed space, that induces a normed space via the equivalence of functions that coincide $\mu-$a.e. The corresponding normed space, that will be still denoted by $(\mathbf{E},\|\cdot\|_{\mathbf{E}})$, is complete [4, Theorem I.1.6]. Recall that $\mathbf{E}\subset L^1_{loc}(X)$.

Notable examples of Banach function spaces are Orlicz spaces and Lorentz spaces, both generalizing Lebesgue spaces.

The $\mathbf{E}-$modulus of a family Γ of curves in X is defined by $Mod_{\mathbf{E}}(\Gamma)=\inf\|\rho\|_{\mathbf{E}}$, where the infimum is taken over all Borel functions $\rho : X \to [0,\infty]$ with $\int\limits_{\gamma} \rho \, ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$ [25]. Note that, in the case $\mathbf{E} = L^p(X)$ we have $Mod_{\mathbf{E}}(\Gamma) = (Mod_p(\Gamma))^{1/p}$ for $1 \leq p < \infty$ and $Mod_{\mathbf{E}}(\Gamma) = Mod_{\infty}(\Gamma)$ for $p = \infty$. Here $Mod_p(\Gamma)$ is the $p-$modulus of Γ [15], [9, Theorem 4.7].

A non-negative Borel function g on X is said to be a $\mathbf{E}-$*weak upper gradient* of $u : X \to \mathbb{R}$ if inequality (3) holds for all rectifiable curves γ except those that belong to a family Γ with $Mod_{\mathbf{E}}(\Gamma) = 0$. Note that for every $\mathbf{E}-$weak upper gradient g of a function u on X there is a decreasing sequence $(g_i)_{i\geq 1}$ of upper gradients of u, such that $\lim\limits_{i\to\infty} \|g_i - g\|_{\mathbf{E}} = 0$ [25, Proposition 2].

The Newtonian space $N^{1,\mathbf{E}}(X)$ based on a Banach function space $\mathbf{E}$ was introduced in [25]. Let $\widetilde{N}^{1,\mathbf{E}}(X)$ be the class of all functions $u \in \mathbf{E}$ such that u has a $\mathbf{E}-$weak upper gradient in $\mathbf{E}$. Then that each $u \in \mathbf{E}$ has an upper gradient in $\mathbf{E}$. For $u \in \widetilde{N}^{1,\mathbf{E}}(X)$ we define $\|u\|_{\widetilde{N}^{1,\mathbf{E}}(X)} = \|u\|_{\mathbf{E}} + \inf \|g\|_{\mathbf{E}}$, where the infimum is taken over all $\mathbf{E}-$weak upper gradients $g \in \mathbf{E}$ of u. The quotient space $N^{1,\mathbf{E}}(X) = \widetilde{N}^{1,\mathbf{E}}(X) / \sim$, where $u \sim v$ if and only if $\|u - v\|_{\widetilde{N}^{1,\mathbf{E}}(X)} = 0$, is a vector space, equipped with the norm $\|u\|_{N^{1,\mathbf{E}}(X)} := \|u\|_{\widetilde{N}^{1,\mathbf{E}}(X)}$. Note that for $\mathbf{E} = L^p(X)$ the space $N^{1,\mathbf{E}}(X)$ is the Newtonian space $N^{1,p}(X)$ introduced in [26].

Let $(X, \mathcal{A}, \mu)$ be a measure space. Given any all real-valued measurable function u in X, we denote by $u^* : [0,\infty) \to [0,\infty]$ its *decreasing rearrangement*, defined for $s \geq 0$ as

$$u^*(s) = \inf \{t > 0 : \mu(\{x \in X : |u(x)| > t\}) > s\}.$$

Let $f : X \to \overline{\mathbb{R}}$ be a $\mu-$measurable function. *The distribution function* of f is defined by $d_f(t) = \mu(\{x \in X : |f(x)| > t\})$, where $t \geq 0$. The *nonincreasing rearrangement* of f is defined by $f^*(t) = \inf \{s \geq 0 : d_f(s) \leq t\}$, where $t \geq 0$.

A Banach function space $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is said to be *rearrangement invariant* if $f^* = g^*$ implies $\|f\|_{\mathbf{E}} = \|g\|_{\mathbf{E}}$, whenever $f, g \in \mathbf{E}$. The most well-known examples of rearrangement invariant Banach function spaces are the Orlicz spaces $L^{\Psi}(X)$ and the Lorentz spaces $L^{p,q}(X)$ with $1 \leq q \leq p < \infty$ [4, Theorem IV.4.3].

Let $\mathbf{E}$ be a rearrangement invariant space over (X, μ). *The fundamental function* of $\mathbf{E}$ is defined by $\Phi_{\mathbf{E}}(t) = \|\chi_A\|_{\mathbf{E}}$, for $t \geq 0$, where $A \subset X$ is any $\mu-$measurable set with $\mu(A) = t$. The definition is unambiguous, since the characteristic functions of two sets with equal measures have the same distribution function.

We recall that (X, μ) is resonant if μ is $\sigma-$finite and nonatomic. If the measure μ is doubling, then μ has as atoms only singletons consisting of isolated points [19]. Thus, a doubling metric measure space without isolated point is resonant.

3. Approximate differentiability of measurable functions with upper gradients in a Banach function spaces

In order to discuss about approximate differentiability, we need to define the notion of approximate limit. We say that $l \in \mathbb{R}$ is the approximate limit of a function $f : X \to \mathbb{R}$ at a point $x \in X$ and write $\operatorname{ap}\lim_{y \to x} f(y) = l$ if x is a density point for all the sets $\{y \in X : |f(y) - l| < \varepsilon\}$.

Definition 3.1. [11, Definition 1.8]A function $f : X \to \mathbb{R}$ is said to be *approximately differentiable* at a point $x \in X_\alpha$ with respect to the approximate differentiable structure $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ if there exists a vector $L^\alpha f(x) \in \mathbb{R}^{N(\alpha)}$ (called approximate differential of f at x) such that

$$\operatorname{ap}\lim_{y \to x} \frac{|f(y) - f(x) - L^\alpha f(x) \cdot (\varphi_\alpha(y) - \varphi_\alpha(x))|}{d(y,x)} = 0, \tag{4}$$

It is proved in [11, Lemma 1.10] that for every $\alpha \in \Lambda$ the approximate differential $L^\alpha f(x)$ is unique for almost every point $x \in X_\alpha$. Moreover, if $f : X \to \mathbb{R}$ is a measurable function that is approximately differentiable at almost every $x \in X_\alpha$, then the approximate differential $L^\alpha f : X_\alpha \to \mathbb{R}^{N(\alpha)}$ is $\mu-$measurable on X_α.

Durand-Cartagena, Ihnatsyeva, Korte and Szumańska proved, in the setting of metric measure spaces, a Whitney-type characterization of approximate differentiability and a Stepanov-type characterization of functions that can be approximated by Lipschitz functions in Luzin sense. These results yield the following

Theorem 3.2. [11, Corollary 2.4] *Let (X, d, μ) be a complete and doubling metric measure space, endowed with an approximate differentiable structure $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$. A function $f : X \to \mathbb{R}$ is $\mu-$a.e. approximately differentiable in a bounded measurable set $E \subset X$ if* *and only if*

$$\text{ap } \limsup_{y\to x} \frac{|f(y)-f(x)|}{d(y,x)} < \infty \tag{5}$$

for $\mu-$a.e. $x \in E$.

For a function $v : X \to \mathbb{R}$, the approximate upper limit ap $\limsup_{y\to x} v(y)$ is defined as the infimum of all numbers $a \in \mathbb{R}$ for which the set $\{y \in X : v(y) > a\}$ has density zero at the point $x \in X$.

Below we will call the left hand side in (5) the approximate pointwise upper Lipschitz constant of the function f at x.

The existence of a Hajłasz gradient that is finite a.e., for a real-extended measurable function f that is finite a.e., implies the finiteness a.e. of the approximate pointwise upper Lipschitz constant of f appearing in (5).

Lemma 3.3. *Let (X, d, μ) be a doubling metric measure space. Assume that $f : X \to \overline{\mathbb{R}}$ is a $\mu-$measurable function, which is finite almost everywhere for which there exists a $\mu-$measurable function $h : X \to [0, \infty]$, which is also finite almost everywhere, such that*

$$|f(x) - f(y)| \leq C\ d(x, y)(h(x) + h(y)) \text{ for } \mu - \text{a.e. } x, y \in X.$$

Then (5) holds.

Proof. There exist some sets $E_1, E_2 \subset X$ with $\mu(E_1) = \mu(E_2) = 0$ such that

$$\frac{|f(x) - f(y)|}{d(x, y)} \leq C\ \ (h(x) + h(y))$$

for every distinct $x, y \in X \setminus E_1$ and $h(x) < \infty$ for every $x \in E_2$.

Since μ is doubling, a Luzin type theorem, Theorem 1.4 from [11], shows that the $\mu-$measurable function h is approximately continuous outside some set F with $\mu(F) = 0$.

Let $x \in X \setminus (E_1 \cup E_2 \cup F)$. Taking approximate supremum limits for $y \to x$ in both members of the above inequality we get ap $\limsup_{y\to x} \frac{|f(y)-f(x)|}{d(y,x)} \leq 2Ch(x) < \infty$. ∎

We introduced in [24] a new type of first order Poincaré inequality for functions defined on a metric measure space, that extends the Orlicz-Poincaré inequality introduced by Aïssaoui [1] and the Poincaré inequality based on Lorentz spaces, introduced by Costea and Miranda

[8], that in turn generalize the well-known weak $(1,p)-$Poincaré inequality [16]. Note that the Orlicz-Poincaré inequalities introduced by Aïssaoui [1] and by Tuominen [27] are different, but each implies the other under some assumptions on the underlying Young function [24, Remark 2.2].

Let $u \in L^1_{loc}(\Omega)$, where $\Omega \subset X$ is an open set. Let $g : \Omega \to [0,\infty]$ be Borel measurable and $1 \leq p < \infty$. We denote the average value of u on A by $u_A = \frac{1}{\mu(A)} \int_A u \, d\mu$ for every measurable set $A \subset \Omega$ with $0 < \mu(A) < \infty$.

Definition 3.4. Let (X, d, μ) be a metric measure space and let $\mathbf{E}$ be a Banach function space over (X, μ). Consider a locally integrable function $u : X \to \overline{\mathbb{R}}$ and a Borel measurable function $g : X \to [0,\infty]$. We say that the pair (u, g) satisfies a *weak* $(1,\mathbf{E})-$*Poincaré inequality* if there exist some constants $C > 0$ and $\tau \geq 1$ such that for all balls $B = B(x,r) \subset X$ we have

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq Cr \frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}. \tag{6}$$

The space (X, d, μ) is said to support a *weak* $(1,\mathbf{E})-$*Poincaré inequality* if (6) holds for every pair (u, g) with $u : X \to \overline{\mathbb{R}}$ locally integrable and g an upper gradient of u, with fixed constants $C > 0$ and $\tau \geq 1$.

Here $\|g\chi_{\tau B}\|_{\mathbf{E}}$ stands for $\rho(g\chi_{\tau B})$, even in the case $\boldsymbol{\rho}(g\chi_{\tau B}) = \infty$, where $\boldsymbol{\rho}$ is the Banach function norm from the definition of $\mathbf{E}$. See also [24, Definition 2.3].

Note that, if (X, d, μ) supports a weak $(1,\mathbf{E})-$Poincaré inequality, then (6) holds whenever g is a $\mathbf{E}-$weak upper gradient of u, as we see using the approximation of weak upper gradients by upper gradients in the norm of $\mathbf{E}$ [25, Proposition 2].

If a metric measure space supports a weak $(1,\mathbf{E})-$Poincaré inequality, for some Banach function space $\mathbf{E}$, then it supports a weak $\infty-$Poincaré inequality [24, Lemma 2.5] and also it supports a first order Poincaré inequality for $\mathcal{F} = N^{1,\infty}(X)$, versions of Poincaré inequality that have been introduced and studied in [9], respectively in [10].

Let $\mathbf{E}$ be a Banach function space over (X, μ). We will consider an analogue of the maximal operator from [3], which was defined in the case $X = \mathbb{R}^n$ using cubes. In our case, balls are replacing cubes.

Assume that $f : X \to \mathbb{R}$ is a $\mu-$measurable function. If $f\chi_B \notin \mathbf{E}$ for some ball B, then $\boldsymbol{\rho}(f\chi_B) = \infty$ and we write $\|f\chi_B\|_{\mathbf{E}} = \infty$.

Definition 3.5. [21] The *noncentered maximal operator* associated with the Banach function space $\mathbf{E}$ is defined by

$$\mathcal{M}_{\mathbf{E}}f(x) = \sup_{B \ni x} \frac{\|f\chi_B\|_{\mathbf{E}}}{\|\chi_B\|_{\mathbf{E}}},$$

where the supremum is taken over all balls $B \subset X$ containing the point x. Here $f : X \to \overline{\mathbb{R}}$ is any $\mu-$measurable function.

Note that in the case when $\mathbf{E} = L^p(X)$, $1 \leq p < \infty$, we have $\mathcal{M}_{\mathbf{E}}f = (\mathcal{M}^*(|f|^p))^{1/p}$.

As in the classical case of the noncentered version of the Hardy-Littlewood maximal operator, it turns out that, for every measurable function f, the function $\mathcal{M}_{\mathbf{E}}f : X \to [0, \infty]$ is Borel measurable, since the preimage of any interval $(\alpha, \infty]$ under $\mathcal{M}_{\mathbf{E}}f$ is an open subset of X.

We proved in [24] that the validity of the Poincaré inequality based on a Banach function space, on a doubling metric measure space, implies a pointwise estimate involving an appropriate maximal operator. The following result partially extends Theorem 3.2 from [14].

Lemma 3.6. [24, Proposition 3.2] *Let (X, d, μ) be a doubling metric measure space and let $\mathbf{E}$ be a Banach function space over (X, μ). Let $u : X \to \overline{\mathbb{R}}$ be a locally integrable function and $g : X \to [0, \infty]$ be a Borel measurable function. Assume that the pair (u, g) satisfies a weak $(1, \mathbf{E})$ $-$Poincaré inequality with constants C and τ. Then*

$$|u(x) - u(y)| \leq C' d(x, y) \left(\mathcal{M}_{\mathbf{E}}g(x) + \mathcal{M}_{\mathbf{E}}g(y)\right) \tag{7}$$

for almost every $x, y \in X$. Here C' is some constant depending only on C and the doubling constant C_d of the measure μ.

Remark 3.7. The above lemma shows that $\mathcal{M}_{\mathbf{E}}g$ is a Hajłasz gradient of u, provided that (u, g) satisfies a weak $(1, \mathbf{E})$ $-$Poincaré inequality.

We need some estimates for the distribution function of this operator, that generalize a result proved by Bastero, Milman and Ruiz [3, Theorem 1] in the Euclidean case and a similar result proved by Costea and Miranda [8, Lemma 6.3] for Newtonian Lorentz spaces on doubling metric spaces. These results are generalizations of the classical weak-type estimate, the Hardy-Littlewood maximal inequality for L^1, which has been extended to doubling metric measure spaces in [15, Theorem 2.2].

Lemma 3.8. [21, Theorem 2] *Let* $\mathbf{E}$ *be a rearrangement invariant Banach function space on a doubling metric measure space* (X, d, μ) *without isolated points. Let* $\Phi_{\mathbf{E}}$ *and* $\mathcal{M}_{\mathbf{E}}$ *be the corresponding fundamental function and maximal operator, respectively. If* $\mathbf{E}$ *satisfies a lower* $\Phi_{\mathbf{E}}-$*estimate, then there exists a positive constant* $C < \infty$ *such that for every* $f \in \mathbf{E}$ *we have*

$$\Phi_{\mathbf{E}}\left(\mu\left(\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\}\right)\right) \leq \frac{C}{\lambda}\|f\|_{\mathbf{E}}. \tag{8}$$

Moreover, if the norm of $\mathbf{E}$ *is absolutely continuous, then*

$$\lim_{\lambda\to\infty} \lambda \Phi_{\mathbf{E}}\left(\mu\left(\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\}\right)\right) = 0. \tag{9}$$

Corollary 3.9. *Assume that* $\mathbf{E}$ *is a rearrangement invariant Banach function space on a doubling metric measure space* (X, d, μ) *without isolated points, such that* $\mathbf{E}$ *satisfies a lower* $\Phi_{\mathbf{E}}-$*estimate. Then* $\mathcal{M}_{\mathbf{E}} f$ *is finite almost everywhere for every* $f \in \mathbf{E}$.

Proof. Since $\Phi_{\mathbf{E}}$ is strictly increasing [4, Corollary II.5.3.], inequality (8) implies

$$\begin{aligned} \Phi_{\mathbf{E}}\left(\mu\left(\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) = \infty\}\right)\right) &\leq \Phi_{\mathbf{E}}\left(\mu\left(\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\}\right)\right) \\ &\leq \frac{C}{\lambda}\|f\|_{\mathbf{E}}, \end{aligned}$$

for all $\lambda > 0$.

The claim follows letting λ tend to infinity.

Theorem 3.10. *Let* (X, d, μ) a *doubling metric measure space. Let* $\mathbf{E}$ *be a rearrangement invariant Banach function space over* (X, μ), *such that* $\mathbf{E}$ *satisfies a lower* $\Phi_{\mathbf{E}}-$*estimate. Assume that* (X, d, μ) *supports a weak* $(1, \mathbf{E})-$*Poincaré inequality. Then every function* $u \in N^{1,\mathbf{E}}(X)$ *satisfies the condition*

$$\text{ap} \limsup_{y\to x} \frac{|u(y) - u(x)|}{d(y, x)} < \infty \tag{10}$$

for $\mu-$*a.e.* $x \in X$.

If in addition (X, d, μ) *is complete and endowed with an approximate differentiable structure, then* u *is approximately differentiable* $\mu-$*a.e. in* X *with respect to the given approximate differentiable structure.*

Proof. As (X, d, μ) supports a weak $(1, \mathbf{E})-$Poincaré inequality, it supports a weak $\infty-$Poincaré inequality [24, Lemma 2.5]. A space supporting a Poincaré inequality is connected and thus has no isolated points.

Let $u \in N^{1,\mathbf{E}}(X)$. Then $u \in \mathbf{E} \subset L^1_{loc}(X)$ and u has an upper gradient $g \in \mathbf{E}$. Since (X, d, μ) supports a weak $(1, \mathbf{E})-$Poincaré inequality, the pair (u, g) satisfies a weak $(1, \mathbf{E})$ $-$Poincaré inequality. By Lemma 3.6, the pointwise estimate (7) holds for almost all points $x, y \in X$.

As $\mathcal{M}_{\mathbf{E}}g : X \to [0, \infty]$ is $\mu-$measurable and finite almost everywhere (by Corollary 3.9), applying Lemma 3.3 with $h = \mathcal{M}_{\mathbf{E}}g$ we obtain that (10) holds for $\mu-$a.e. $x \in X$.

In case (X, d, μ) is complete and endowed with an approximate differentiable structure, we appeal to Theorem 3.2 to finish the proof. ∎

Remark 3.11. Every Orlicz space $\mathbf{E} = L^{\Psi}(X)$ corresponding to an $N-$function Ψ satisfies a lower $\Phi_{\mathbf{E}}-$estimate [21, Corollary 1]. Every Lorentz space $\mathbf{E} = L^{p,q}(X)$ with $1 \leq q \leq p < \infty$ satisfies a a lower $\Phi_{\mathbf{E}}-$estimate, see [8, Proposition 2.4]. Since the above Banach function spaces are also rearrangement invariant, Theorem 3.11 can be used for the corresponding Newtonian spaces $N^{1,\mathbf{E}}(X)$, provided that the doubling metric measure space supports the $(1, \mathbf{E})-$Poincaré inequality.

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