"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 28(2018), No. 2, 5-18

# $m$-IRRESOLUTE MULTIFUNCTIONS IN FUZZY $m$-SPACES

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**Abstract.** In this paper a new type of fuzzy multifunction is introduced between a topological space and a fuzzy $m$-space [1]. In Section 4, several characterizations of this newly defined multifunction are done and in the last section some applications of it are shown.

## 1. Introduction

Fuzzy irresolute multifunction is introduced and studied in [7] between a topological space and a fuzzy topological space (fts, for short) in the sense of Chang [8]. The concept of minimal structure has been introduced in the paper [14]. In this paper instead of fuzzy topological space we use fuzzy $m$-space introduced by Alimohammady and Roohi [1] as follows : A family $\mathcal{M}$ of fuzzy sets in a non-empty set $Y$ is said to be fuzzy minimal structure on $Y$ if $\alpha 1_X \in \mathcal{M}$ for every $\alpha \in [0, 1]$. A more general version of fuzzy minimal structure (in the sense of Chang) are introduced in [6, 17] as follows : A family $m$ of fuzzy sets in a non-empty set $Y$ is a fuzzy minimal structure on $Y$ if $0_X \in m$ and $1_X \in m$. Then $(Y, m)$ is called fuzzy minimal space (fuzzy $m$-space, for short).

**Keywords and phrases:** Fuzzy $m$-space, fuzzy upper (lower) $m$-irresolute multifunction, fuzzy $m$-semiopen set, fuzzy $m$-semicompact set, fuzzy $m$-$S$-closed set, $s$-frontier of a set.
**(2010) Mathematics Subject Classification:** 54A40, 54C99

The members of $m$ are called fuzzy $m$-open sets and the complement of a fuzzy $m$-open set in $Y$ is called fuzzy $m$-closed set. Various forms of continuity between spaces possessing minimal structures instead of topologies have been studied, starting with the paper [21]. Throughout this paper $(X, \tau)$ stands for a topological space and $(Y, m)$ stands for a fuzzy $m$-space. Throughout this paper $(X, \tau)$ or simply by $X$ we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [8].

## 2. **Preliminaries**

Let $Y$ be a non-empty set and $I = [0, 1]$. Then a fuzzy set [24] $A$ in $Y$ is a mapping from $Y$ into $I$. The set of all fuzzy sets in $Y$ is denoted by $I^Y$. For a fuzzy set $A$ in $Y$, the support of $A$, denoted by $suppA$ [24] and is defined by $suppA = \{y \in Y : A(y) \neq 0\}$. A fuzzy point [22] with the singleton support $y \in Y$ and the value $t$ $(0 < t \leq 1)$ at $y$ will be denoted by $y_t$. $0_Y$ and $1_Y$ are the constant fuzzy sets taking respectively the constant values 0 and 1 on $Y$. The complement of a fuzzy set $A$ in $Y$ will be denoted by $1_Y \setminus A$ [24] and is defined by $(1_Y \setminus A)(y) = 1 - A(y)$, for all $y \in Y$. For two fuzzy sets $A$ and $B$ in $Y$, we write $A \leq B$ iff $A(y) \leq B(y)$, for each $y \in Y$, while we write $AqB$ to mean $A$ is quasi-coincident (q-coincident, for short) with $B$ [22] if there exists $y \in Y$ such that $A(y) + B(y) > 1$; the negation of $AqB$ is written as $A \not{q} B$. $clA$ and $intA$ of a set $A$ in $X$ (a fuzzy set $A$ in $Y$) respectively stand for the closure and interior of $A$ in $X$ (respectively, in $Y$). A (fuzzy) set $A$ in $X$ $(Y)$ is called (fuzzy) semiopen if $A \subseteq cl(intA)$ [12] $(A \leq cl(intA)$ [2]). The complement of a semiopen set is called semiclosed [12]. The intersection (union) of all semiclosed ( semiopen) sets containing (contained in) a subset $A$ in $X$ is called semiclosure [9] (semiinterior) of $A$ and is denoted by $sclA$ $(sintA)$ [10]. A subset $A$ of $X$ is semiopen (semiclosed) iff $A = sintA$ $(A = sclA)$. A subset $A$ of a topological space $(X, \tau)$ is called a semi neighbourhood (semi-nbd, for short) [10] of a point $x \in X$ if there is a semiopen set $U$ in $X$ containing $x$ such that $U \subseteq A$.

## 3. **Some Well-Known Definitions**

In this section we first recall some definitions, lemmas, proposition and theorems from [3, 4, 5] for ready references.

**Definition 3.1** [3]. Let $(Y, m)$ be a fuzzy $m$-space. For $A \in I^Y$, the fuzzy $m$-closure and fuzzy $m$-interior of $A$, denoted by $mclA$ and

$mintA$ respectively are defined as follows :

$$mclA = \bigwedge\{B : A \leq B, 1_Y \setminus B \in m\}$$

$$mintA = \bigvee\{B : B \leq A, B \in m\}$$

It is clear from definition that for a fuzzy set $A$ in $Y$, $mintA$ may not be a member of $m$. But if $m$ satisfies $M$-condition [3], i.e., $m$ is closed under arbitrary union, then for $A \in I^Y$, $mintA$ is $m$-open and $mclA$ is $m$-closed.

**Proposition 3.2** [3]. Let $(Y, m)$ be a fuzzy $m$-space. Then for any $A \in I^Y$, a fuzzy point $y_\alpha \in mclA$ if and only if for any $U \in m$ with $y_\alpha qU$, $UqA$.

**Lemma 3.3** [3]. Let $(Y, m)$ be a fuzzy $m$-space and $A, B \in I^Y$. Then the following statements are true :
(i) $A \leq B \Rightarrow mintA \leq mintB$, $mclA \leq mclB$
(ii)$mint0_Y = 0_Y$, $mint1_Y = 1_Y$, $mcl0_Y = 0_Y$, $mcl1_Y = 1_Y$
(iii) $mintA \leq A \leq mclA$
(iv) $mclA = A$ if $1_Y \setminus A \in m$, $mintA = A$ if $A \in m$
(v) $mcl(1_Y \setminus A) = 1_Y \setminus mintA$, $mint(1_Y \setminus A) = 1_Y \setminus mclA$
(vi) $mcl(mcl(A) = A$, $mint(mintA) = A$.
**Theorem 3.4** [4]. Let $(Y, m)$ be a fuzzy $m$-space and $A, B \in I^Y$. then
(i) $mclA \bigvee mclB \leq mcl(A \bigvee B)$
(ii) $mint(A \wedge B) \leq mintA \bigwedge mintB$.

**Definition 3.5** [4]. Let $(Y, m)$ be a fuzzy $m$-space. Then $A(\in I^Y)$ is called fuzzy $m$-semiopen in $Y$ is $A \leq mcl(mintA)$.

The complement of a fuzzy $m$-semiopen set is called fuzzy $m$-semiclosed.

**Definition 3.6** [4]. The intersection (union) of all fuzzy $m$-semiclosed (fuzzy $m$-semiopen) sets in a fuzzy $m$-space $(Y, m)$ containing (contained in) a fuzzy set $A$ in $Y$ is called fuzzy $m$-semiclosure (fuzzy $m$-semiinterior) of $A$, denoted by $msclA$ ($msintA$).

A fuzzy set $A$ in a fuzzy $m$-space $(Y, m)$ is fuzzy $m$-semiclosed (fuzzy $m$-semiopen) if $A = msclA$ ($A = msintA$).

**Definition 3.7** [5]. A fuzzy set $B$ is called fuzzy $m$-semi neighbourhood (fuzzy $m$-seminbd, for short) of a fuzzy set $A$ in a fuzzy $m$-space

$(Y, m)$ if there is a fuzzy $m$-semiopen set $U$ in $Y$ such that $A \leq U \leq B$.

**Definition 3.8** [5]. A fuzzy set $A$ in a fuzzy $m$-space $(Y, m)$ is said to be a fuzzy $m$-semi $q$-neighbourhood (fuzzy $m$-semi $q$-nbd, for short) of a fuzzy point $y_\alpha$ in $Y$ if there is a fuzzy $m$-semiopen set $V$ in $Y$ such that $y_\alpha q V \leq A$.

**Result 3.9** [5]. Let $A \in I^Y$. Then a fuzzy point $y_\alpha \in msclA$ if and only if $UqA$ where $U$ is fuzzy $m$-semi $q$-nbd of $y_\alpha$.

**Result 3.10** [4]. If a fuzzy set $A$ in a fuzzy $m$-space $(Y, m)$ is fuzzy $m$-semiopen, then $mclA$ is also fuzzy $m$-semiopen in $Y$.

**Note 3.11** [4]. Union of arbitrary collection of fuzzy $m$-semiopen sets in a fuzzy $m$-space $(Y, m)$ is fuzzy $m$-semiopen in $Y$.

## 4. Fuzzy Upper (Lower) $m$-Irresolute Multifunction

In this section we first recall the following definitions and theorem from [20, 16] for ready references.

**Definition 4.1** [20]. A fuzzy multifunction $F : X \to Y$ assigns to each $x \in X$, a unique fuzzy subset $F(x)$ in $Y$.

Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

**Definition 4.2** [20, 16]. For a fuzzy multifunction $F : X \to Y$, the upper inverse $F^+$ and lower inverse $F^-$ are defined as follows :

For any fuzzy set $A$ in $Y$, $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x)qA\}$.

There is a following relationship between the upper and the lower inverses of a fuzzy multifunction.

**Theorem 4.3** [16]. For a fuzzy multifunction $F : X \to Y$, we have $F^-(1_Y \setminus A) = X \setminus F^+(A)$, for any fuzzy set $A$ in $Y$.

**Definition 4.4** [7]. A fuzzy multifunction $F : X \to Y$ is said to be fuzzy

(a) upper irresolute at a point $x_0 \in X$ if for every fuzzy semiopen set $V$ in $Y$ with $x_0 \in F^+(V)$, there exists a semiopen set $U$ in $X$ with $x_0 \in U$ such that $U \subseteq F^+(V)$,

(b) lower irresolute at a point $x_0 \in X$ if for every fuzzy semiopen set $V$ in $Y$ with $x_0 \in F^-(V)$, there exists a semiopen set $U$ in $X$ with $x_0 \in U$ such that $U \subseteq F^-(V)$,

(c) upper (lower) irresolute on $X$ if it is upper (lower) irresolute at each point of $X$.

Let us now introduce fuzzy upper and lower irresolute multifunction in fuzzy $m$-space.

**Definition 4.5**. A fuzzy multifunction $F:(X,\tau)\to(Y,m)$ is said to be fuzzy

(a) upper $m$-irresolute at a point $x_0\in X$ if for every fuzzy $m$-semiopen set $V$ in $Y$ with $F(x)\le V$, there exists a semiopen set $U$ in $X$ containing $x_0$ such that $F(U)\le V$, i.e, $U\subseteq F^+(V)$,

(b) lower $m$-irresolute at a point $x_0\in X$ if for every fuzzy $m$-semiopen set $V$ in $Y$ with $F(x)qV$, there exists a semiopen set $U$ in $X$ containing $x_0$ such that $F(u)qV$, for all $u\in U$, i.e., $U\subseteq F^-(V)$,

(c) upper (lower) $m$-irresolute on $X$ if it is upper (lower) $m$-irresolute on $X$ at each point of $X$.

When $m$ is a fuzzy topology on $Y$ we get fuzzy upper (lower) irresolute multifunction as defined in Definition 4.4.

**Theorem 4.6**. A fuzzy multifunction $F:(X,\tau)\to(Y,m)$ is fuzzy upper $m$-irresolute if and only if $F^+(U)$ is semiopen in $X$ for every fuzzy $m$-semiopen set $U$ in $Y$.

**Proof**. Let $F:(X,\tau)\to(Y,m)$ be fuzzy upper $m$-irresolute and $U$ be any fuzzy $m$-semiopen set in $Y$ with $x\in F^+(U)$. Then $F(x)\le U$. By hypothesis, there exists a semiopen set $V$ in $X$ containing $x$ such that $V\subseteq F^+(U)$. But $sint(F^+(U))$ is the union of all semiopen sets contained in $F^+(U)$ and $V$ is a semiopen set contained in $F^+(U)$ and so $V\subseteq sint(F^+(U))$. As $x\in V$, $x\in sint(F^+(U))$ and so $F^+(U)\subseteq sint(F^+(U))$ and hence $F^+(U)$ is semiopen in $X$.

Conversely, let $F^+(U)$ be semiopen in $X$ for every fuzzy $m$-semiopen set $U$ in $Y$. Let $x\in X$ and $V$ be a fuzzy $m$-semiopen set in $Y$ with $x\in F^+(V)$. Let $U=F^+(V)$. By hypothesis, $U$ is semiopen in $X$ containing $x$ such that $F(U)=F(F^+(V))\le V$.

In a similar manner we can state the following theorem.

**Theorem 4.7**. A fuzzy multifunction $F:(X,\tau)\to(Y,m)$ is fuzzy lower $m$-irresolute if and only if $F^-(U)$ is semiopen in $X$ for every fuzzy $m$-semiopen set $U$ in $Y$.

From last two theorems and by Theorem 4.3, we can easily state the following theorem.

**Theorem 4.8**. A fuzzy multifunction $F:(X,\tau)\to(Y,m)$ is fuzzy upper (lower) $m$-irresolute if and only if $F^-(U)$ (resp., $F^+(U)$) is semiclosed in $X$ for every fuzzy $m$-semiclosed set $U$ in $Y$.

**Theorem 4.9**. For a fuzzy multifunction $F:(X,\tau)\to(Y,m)$ where $m$ satisfies $M$-condition, the following statements are equivalent :

(a) $F$ is fuzzy upper $m$-irresolute,
(b) for each $x \in X$ and each fuzzy $m$-semi nbd $V$ of $F(x)$ in $Y$, $F^+(V)$ is a semi-nbd of $x$ in $X$,
(iii) for each $x \in X$ and each fuzzy $m$-semi nbd $V$ of $F(x)$ in $Y$, there exists a semi-nbd $U$ of $x$ in $X$ such that $F(U) \leq V$,
(d) $scl(F^-(B)) \subseteq F^-(msclB)$.

**Proof**. (a) $\Rightarrow$ (b) Let $x \in X$ and $V$ be a fuzzy $m$-semi nbd of $F(x)$ in $Y$. Then by definition there is a fuzzy $m$-semiopen set $U$ in $Y$ such that $F(x) \leq U \leq V$. Since $F$ is fuzzy upper $m$-irresolute, by Theorem 4.6, $F^+(U)$ is semiopen in $X$ and $x \in F^+(U) \subseteq F^+(V)$ implies that $F^+(V)$ is a semi-nbd of $x$ in $X$.
(b)$\Rightarrow$ (c) Obvious.
(c)$\Rightarrow$ (a) Let $x \in X$ and $V$ be a fuzzy $m$-semiopen set in $Y$ such that $x \in F^+(V)$, i.e., $F(x) \leq V$. Then $V$ is a fuzzy $m$-semi nbd of $F(x)$ in $Y$. By (c), there exists a semi-nbd $U$ of $x$ in $X$ such that $F(U) \leq V$. Then by definition, there exists a semiopen set $W$ in $X$ such that $x \in W \subseteq U$. Then $F(x) \leq F(W) \leq F(U) \leq V$ implies that $F(W) \leq V$. Hence $F$ is fuzzy upper $m$-irresolute.
(a)$\Rightarrow$(d) Let $B \in I^Y$. As $m$ satisfies $M$-condition, $msclB$ is fuzzy $m$-semiclosed in $Y$ and so by (a), $F^-(msclB)$ is semi-closed in $X$ (by Theorem 4.8) and $F^-(B) \subseteq F^-(msclB)$. So $scl(F^-(B)) \subseteq F^-(msclB)$.
(d)$\Rightarrow$(a) Let $B(\in I^Y)$ be fuzzy $m$-semiclosed in $Y$. By (d), $scl(F^-(B)) \subseteq F^-(msclB) = F^-(B)$ (as $m$ satisfies $M$-condition) implies that $F^-(B)$ is semiclosed in $X$. Hence by Theorem 4.8, $F$ is fuzzy upper $m$-irresolute.

**Theorem 4.10**. For a fuzzy multifunction $F : (X, \tau) \to (Y, m)$ where $m$ satisfies $M$-condition, the following statements are equivalent :
(a) $F$ is fuzzy lower $m$-irresolute,
(b) for each $x \in X$ and each fuzzy $m$-semi $q$-nbd $V$ of $F(x)$ in $Y$, $F^-(V)$ is a semi-nbd of $x$ in $X$,
(c) for each $x \in X$ and each fuzzy $m$-semi $q$-nbd $V$ of $F(x)$ in $Y$, there is a semiopen set $U$ containing $x$ in $X$ such that $U \subseteq F^-(V)$,
(d) $F(sclA) \leq mscl(F(A))$, for any subset $A$ of $X$,
(e) $scl(F^+(B)) \subseteq F^+(msclB)$, for any $B \in I^Y$,
(f) $F(int(clA)) \leq mscl(F(A))$, for any subset $A$ of $X$.

**Proof**. (a)$\Rightarrow$ (b) Let $x \in X$ and $V$ be a fuzzy $m$-semi $q$-nbd of $F(x)$ in $Y$. Then by definition, there exists a fuzzy $m$-semiopen set $U$ in $Y$ such that $F(x)qU \leq V$. So $x \in F^-(U) \subseteq F^-(V)$. By (a), $F^-(U)$ is semiopen in $X$ (by Theorem 4.7) and so $F^-(V)$ is a semi-nbd of $x$ in

$X$.

(b)⇒(c) Let $x \in X$ and $V$ be a fuzzy $m$-semi $q$-nbd of $F(x)$ in $Y$. Then by (b), $F^-(V)$ is a semi-nbd of $x$ in $X$. So by definition there exists a semiopen set $U$ in $X$ containing $x$ such that $U \subseteq F^-(V)$.

(c)⇒(a) Let $x \in X$ and $V$ be a fuzzy $m$-semiopen set in $Y$ such that $x \in F^-(V)$, i.e., $F(x)qV$. Then $V$ being a fuzzy $m$-semi $q$-nbd of $F(x)$, by (c), there is a semiopen set $U$ containing $x$ in $X$ such that $U \subseteq F^-(V)$. Hence $F$ is fuzzy lower $m$-irresolute.

(a)⇒(d) Let $A$ be any subset of $X$. Then $mscl(F(A))$ is fuzzy $m$-semiclosed in $Y$ (as $m$ satisfies $M$-condition). Then by (a), $F^+(mscl(F(A)))$ is semiclosed in $X$ (by Theorem 4.8) and $F(A) \leq mscl(F(A))$ implies that $A \subseteq F^+(mscl(F(A)))$ and so $sclA \subseteq scl(F^+(mscl(F(A))))$. Then $sclA \subseteq F^+(mscl(F(A)))$ and hence $F(sclA) \leq mscl(F(A))$.

(d)⇒(a) Let $V$ be a fuzzy $m$-semiclosed set in $Y$. Then $F^+(V)$ is a subset of $X$. By (d), $F(scl(F^+(V))) \leq mscl(F(F^+(V))) \leq msclV = V$ which shows that $scl(F^+(V)) \subseteq F^+(V)$ and so $F^+(V)$ is semiclosed in $X$. By Theorem 4.8, $F$ is fuzzy lower $m$-irresolute.

(d)⇒(e) Let $B \in I^Y$. Then $F^+(B) \subseteq X$. By (d), $F(scl(F^+(B))) \leq mscl(F(F^+(B))) \leq msclB$ and so $scl(F^+(B)) \subseteq F^+(msclB)$.

(e)⇒(d) Let $A \subseteq X$. Then $F(A) \in I^Y$. By (e), $scl(F^+(F(A))) \subseteq F^+(mscl(F(A)))$ which implies that $sclA \subseteq scl(F^+(F(A))) \subseteq F^+(mscl(F(A)))$ and hence $F(sclA) \leq mscl(F(A))$.

(a)⇒(f) Let $A \subseteq X$. By (a), $F^+(mscl(F(A)))$ is semiclosed in $X$ (by Theorem 4.8). Then $int(cl(F^+(mscl(F(A))))) \subseteq F^+(mscl(F(A)))$. Then $int(clA) \subseteq int(cl(F^+(mscl(F(A))))) \subseteq F^+(mscl(F(A)))$, i.e., $F(int(clA)) \leq mscl(F(A))$.

(f)⇒(d) Let $A \subseteq X$. Since $sclA = A \bigcup int(clA)$, $F(sclA) = F(A \bigcup int(clA)) = F(A) \bigvee F(int(clA)) \leq F(A) \bigvee mscl(F(A))$ (by (f)) $= mscl(F(A))$ and so $F(sclA) \leq mscl(F(A))$.

**Lemma 4.11** [18]. Let $A$ be an open subset of a topological space $(X, \tau)$. Then the following two statements are true :

(a) $U(\subset X)$ is semiopen in $X$ implies that $U \bigcap A$ is semiopen in $A$.

(b) $U(\subset A)$ is semiopen in $A$ implies that $U$ is semiopen in $X$.

**Theorem 4.12**. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a topological space $(X, \tau)$. A fuzzy multifunction $F : (X, \tau) \to (Y, m)$ is fuzzy upper $m$-irresolute if and only if $F/U_\alpha : U_\alpha \to (Y, m)$ is fuzzy upper $m$-irresolute for each $\alpha \in \Lambda$.

**Proof**. Let $F$ be fuzzy upper $m$-irresolute and $x \in U_\alpha$, for each $\alpha \in \Lambda$. Let $V$ be a fuzzy $m$-semiopen set in $Y$ such that $x \in F^+(V)$.

Then by definition, there is a semiopen set $U$ in $X$ containing $x$ such that $U \subseteq F^+(V)$. By Lemma 4.11(a), $U \bigcap U_\alpha$ is semiopen in $U_\alpha$ containing $x$ and $U \bigcap U_\alpha \subseteq (F/U_\alpha)^+(V)$ implies that $F/U_\alpha$ is fuzzy upper $m$-irresolute.

Conversely, let $x \in X$ and $V$ be a fuzzy $m$-semiopen set in $Y$ with $x \in F^+(V)$. By hypothesis, there is some $\alpha \in \Lambda$ such that $x \in U_\alpha$. Again $F/U_\alpha : U_\alpha \to (Y, m)$ is fuzzy upper $m$-irresolute. So there is a semiopen set $U$ in $U_\alpha$ such that $(F/U_\alpha)(U) \leq V$. By Lemma 4.11(b), $U$ is semiopen in $X$ and also $U \subseteq U_\alpha$ and so $F(U) = (F/U_\alpha)(U) \leq V$ implies that $U \subseteq F^+(V)$. Hence $F$ is fuzzy upper $m$-irresolute.

**Theorem 4.13**. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a topological space $(X, \tau)$. A fuzzy multifunction $F : (X, \tau) \to (Y, m)$ is fuzzy lower $m$-irresolute if and only if $F/U_\alpha : U_\alpha \to (Y, m)$ is fuzzy lower $m$-irresolute for each $\alpha \in \Lambda$.

**Proof**. The proof is quite similar to that of Theorem 4.12 and hence omitted.

**Definition 4.14**. Let $F : (X, \tau) \to (Y, m)$ be a fuzzy multifunction. Then $msclF : (X, \tau) \to (Y, m)$ is defined by $(msclF)(x) = mscl(F(x))$, for each $x \in X$.

**Lemma 4.15**. For a fuzzy multifunction $F : (X, \tau) \to (Y, m)$, $(msclF)^-(V) = F^-(V)$, for every fuzzy $m$-semiopen set $V$ in $Y$.

**Proof**. Let $x \in F^-(V)$. Then $F(x)qV$ implies that $(mclF)(x)qV$ and so $x \in (msclF)^-(V)$ which implies that $F^-(V) \subseteq (msclF)^-(V)$.

Conversely, let $x \in (msclF)^-(V)$. Then $(msclF)(x)qV$. Then there exists $y \in Y$ such that $[(msclF)(x)](y) + V(y) > 1$. Let $[(msclF)(x)](y) = \alpha$. Then $V(y) + \alpha > 1$ and so $y_\alpha qV$ and $y_\alpha \in (mscl(F(x))$. By Result 3.9, $VqF(x)$ and so $x \in F^-(V)$ and so $(msclF)^-(V) \subseteq F^-(V)$.

**Lemma 4.16**. For a fuzzy multifunction $F : (X, \tau) \to (Y, m)$, $(msclF)^+(V) = F^+(V)$, for every fuzzy $m$-semiclosed set $V$ in $Y$.

**Proof**. Let $x \in (msclF)^+(V)$. Then $(msclF)(x) \leq V$ which implies that $msclF(x) \leq V$ and so $F(x) \leq V$. Therefore, $x \in F^+(V)$ and hence $(msclF)^+(V) \subseteq F^+(V)$.

Conversely, let $x \in F^+(V)$ where $V$ is fuzzy $m$-semiclosed in $Y$. Then $F(x) \leq V$ implies that $mscl(F(x)) \leq msclV = V$ and so $mscl(F(x)) \leq V$. Then $(msclF)(x) \leq V$. Therefore, $x \in (msclF)^+(V)$ and hence $F^+(V) \subseteq (msclF)^+(V)$.

**Theorem 4.17**. A fuzzy multifunction $F : (X, \tau) \to (Y, m)$ is fuzzy lower $m$-irresolute if and only if $msclF : (X, \tau) \to (Y, m)$ is fuzzy lower $m$-irresolute.

**Proof**. Follows from Lemma 4.15.

**Theorem 4.18**. A fuzzy multifunction $F : (X, \tau) \to (Y, m)$ is fuzzy upper $m$-irresolute if and only if $msclF : (X, \tau) \to (Y, m)$ is fuzzy upper $m$-irresolute.

**Proof**. Follows from Lemma 4.16.

## 5. Applications

In this section we first recall some definitions from [5, 8, 10, 11] for ready references.

**Definition 5.1** [10]. A topological space $(X, \tau)$ is called semicompact if every semiopen cover of $X$ has a finite subcover.

**Definition 5.2** [8, 11]. Let $A$ be a fuzzy set in an fts $Y$. A collection $\mathcal{U}$ of fuzzy sets of $Y$ is called a fuzzy cover of $A$ if $sup\{U(x) : U \in \mathcal{U}\} = 1$ for each $x \in suppA$. If, in particular, $A = 1_Y$, we get the definition of fuzzy cover of $Y$.

**Definition 5.3** [5]. A fuzzy set $A$ in a fuzzy $m$-sapce $(Y, m)$ is said to be fuzzy $m$-semicompact set if every cover $\mathcal{U}$ of $A$ by fuzzy $m$-semiopen sets in $Y$ has a finite subfamily $\mathcal{U}_0$ such that $\bigcup \mathcal{U}_0 \geq A$. If, in particular, $A = 1_Y$, we get the definition of fuzzy $m$-semicompact space $Y$.

**Theorem 5.4**. Let $F : (X, \tau) \to (Y, m)$ be a surjective fuzzy multifunction and $F(x)$ be a fuzzy $m$-semicompact set in $Y$ for each $x \in X$. If $F$ is fuzzy upper $m$-irresolute and $X$ is semicompact, then $Y$ is fuzzy $m$-semicompact.

**Proof**. Let $\mathcal{U} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of $Y$ by fuzzy $m$-semiopen sets in $Y$. As for each $x \in X$, $F(x)$ is fuzzy $m$-semicompact set in $Y$, there is a finite subset $\Lambda_x$ of $\Lambda$ such that $F(x) \leq \bigcup\limits_{\alpha \in \Lambda_x} A_\alpha$. Let $A_x = \bigcup\limits_{\alpha \in \Lambda_x} A_\alpha$. Then $F(x) \leq A_x$ and $A_x$ is fuzzy $m$-semiopen in $Y$ (by Note 3.11). Since $F$ if fuzzy upper $m$-irresolute, there exists a semiopen set $U_x$ in $X$ containing $x$ such that $U_x \subseteq F^+(A_x)$. Then the family $\mathcal{V} = \{U_x : x \in X\}$ is a semiopen cover of $X$. $X$ being semicompact, there exist finitely many members $x_1, x_2, ..., x_n$ in $X$ such that $X = \bigcup\limits_{i=1}^{n} U_{x_i}$. As $F$ is surjective, $1_Y = F(X) = F(\bigcup\limits_{i=1}^{n} U_{x_i}) = \bigcup\limits_{i=1}^{n} F(U_{x_i}) \leq \bigcup\limits_{i=1}^{n} A_{x_i} \leq \bigcup\limits_{i=1}^{n} \bigcup\limits_{\alpha \in \Lambda_{x_i}} A_\alpha$. Hence $Y$ is fuzzy $m$-semicompact space.

**Definition 5.5**. A subset $A$ of a topological space $X$ is called an

$S$-closed set [19] if every cover $\mathcal{U}$ of $A$ by semiopen sets of $X$ has a finite proximate subcover, i.e., there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\bigcup U_0 \geq A$. If, in particular, $A = 1_Y$, we get the definition of $S$-closed space [23].

**Definition 5.6** [5]. A fuzzy set $A$ in a fuzzy $m$-space $(Y, m)$ is said to be fuzzy $m$-$S$-closed, if every cover $\mathcal{V}$ of $A$ by fuzzy $m$-semiopen sets of $Y$ has a finite proximate subcover, i.e., there is a finite subfamily $\mathcal{V}_0$ of $\mathcal{V}$ such that $\bigcup\limits_{U \in \mathcal{V}_0} mclU \geq A$. If, in particular $A = 1_Y$, we get the definition of fuzzy $m$-$S$-closed space.

**Lemma 5.7** [19]. A topological space $(X, \tau)$ is $S$-closed if and only if for every cover $\mathcal{V}$ of $X$ by semiopen sets of $X$ has a finite subcollection $\mathcal{V}_0$ such that $X = scl(\bigcup\limits_{V \in \mathcal{V}_0} V)$.

**Theorem 5.8**. Let $F : (X, \tau) \to (Y, m)$ be a fuzzy multifunction and $F(x)$ be a fuzzy $m$-$S$-closed set in $Y$, for each $x \in X$. If $F$ is fuzzy upper $m$-irresolute and lower $m$-irresolute and $X$ is $S$-closed, then $Y$ is fuzzy $m$-$S$-closed.

**Proof**. Let $\mathcal{U} = \{A_\alpha : \alpha \in \Lambda\}$ be any fuzzy $m$-semiopen cover of $Y$. For each $x \in X$, $F(x)$ is a fuzzy $m$-$S$-closed set in $Y$. So there is a finite subset $\Lambda_x$ of $\Lambda$ such that $F(x) \leq \bigcup\limits_{\alpha \in \Lambda_x} mclA_\alpha$ (=$U_x$, say). Then $U_x$ is a fuzzy $m$-semiopen set in $Y$ containing $F(x)$ (by Result 3.10). Since $F$ is fuzzy upper $m$-irresolute, $F^+(U_x)$ is semiopen set in $X$, by Theorem 4.6. Then $\mathcal{V} = \{F^+(U_x) : x \in X\}$ is a semiopen cover of $X$. $X$ being $S$-closed, there are finitely many points $x_1, x_2, ..., x_n$ of $X$ such that $X = scl(\bigcup\limits_{i=1}^{n} F^+(U_{x_i}))$ (by Lemma 5.7). Then $X = scl(F^+(\bigcup\limits_{i=1}^{n} U_{x_i}))$. Since $F$ is fuzzy lower $m$-irresolute, by Theorem 4.10, $1_Y = F(X) = F(scl(F^+(\bigcup\limits_{i=1}^{n} U_{x_i}))) \leq F(F^+(mscl(\bigcup\limits_{i=1}^{n} U_{x_i}))) \leq mscl(\bigcup\limits_{i=1}^{n} U_{x_i}) = mscl(\bigcup\limits_{i=1}^{n}(\bigcup\limits_{\alpha \in \Lambda_{x_i}} mclA_\alpha)) = \bigcup\limits_{i=1}^{n}(\bigcup\limits_{\alpha \in \Lambda_{x_i}} mclA_\alpha)$.

**Definition 5.9** [13]. A subset $A$ of a topological space $(X, \tau)$ is said to be $s$ -closed if every cover $\mathcal{U}$ of $A$ by semiopen sets of $X$ has a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \bigcup\limits_{U \in \mathcal{U}_0} sclU$.

**Theorem 5.10**. Let $F : (X, \tau) \to (Y, m)$ be a fuzzy multifunction such that $F(x)$ is fuzzy $m$-$S$-closed set in $Y$ for each $x \in X$. If $F$ is fuzzy upper as well as fuzzy lower $m$-irresolute and $A$ is $s$-closed set in $X$, then $F(A)$ is fuzzy $m$-$S$-closed set in $Y$.

**Proof**. Let $\mathcal{U} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of $F(A)$ by fuzzy $m$-semiopen sets in $Y$. For each $x \in A$, $F(x)$ is fuzzy $m$-$S$-closed in $Y$. So there is a finite subset $\Lambda_x$ of $\Lambda$ such that $F(x) \leq \bigcup_{\alpha \in \Lambda_x} mclA_\alpha$. Let $B_x = \bigcup_{\alpha \in \Lambda_x} mclA_\alpha$. Then $B_x$ is fuzzy $m$-semiopen set in $Y$ and $F(x) \leq B_x$. As $F$ is fuzzy upper $m$-irresolute, $F^+(B_x)$ is semiopen in $X$. Then $\mathcal{V} = \{F^+(B_x) : x \in A\}$ is a cover of the $s$-closed set $A$ by semiopen sets of $X$. Since $A$ is $s$-closed, there exist finitely many points $x_1, x_2, ..., x_n$ in $A$ such that $A \subseteq \bigcup_{i=1}^{n} scl(F^+(B_{x_i}))$. As $F$ is fuzzy lower $m$-irresolute, by Theorem 4.10, $F(A) \leq F(\bigcup_{i=1}^{n} scl(F^+(B_{x_i}))) \leq F(F^+(mscl(\bigcup_{i=1}^{n} B_{x_i}))) \leq mscl(\bigcup_{i=1}^{n} B_{x_i}) = mscl(\bigcup_{i=1}^{n}(\bigcup_{\alpha \in \Lambda_{x_i}} mclA_\alpha)) = \bigcup_{i=1}^{n}(\bigcup_{\alpha \in \Lambda_{x_i}} mclA_\alpha)$. Hence $F(A)$ is fuzzy $m$-$S$-closed set in $Y$.

**Definition 5.11**. The $s$-frontier of a subset $A$ of a topological space $X$, denoted by $sfr(A)$, is defined by $sfr(A) = sclA \bigcap scl(X \setminus A) = sclA \setminus sintA$.

**Theorem 5.12**. Let $F : (X.\tau) \to (Y, m)$ be a fuzzy multifunction where $m$ satisfies $M$-condition. Let $A = \{x \in X : F$ is not fuzzy upper $m$-irresolute at $x\}$, $B = \bigcup\{sfr(F^+(V)) : F(x) \leq V$ and $V$ is fuzzy $m$-semiopen set in $Y\}$. Then $A = B$.

**Proof**. Let $x \in X$ and $F$ be not fuzzy upper $m$-irresolute at $x$. Then there exists a fuzzy $m$-semiopen set $V$ in $Y$ with $F(x) \leq V$, but $U \not\subseteq F^+(V)$, for all semiopen sets $U$ in $X$ containing $x$. Then $U \bigcap (X \setminus F^+(V)) \neq \phi$ which shows that $x \in scl(X \setminus F^+(V)) = X \setminus sint(F^+(V))$ and so $x \notin sint(F^+(V))$. Again, $x \in F^+(V) \subseteq scl(F^+(V))$ and hence $x \in sfr(F^+(V))$.

Conversely, let $x \in X$ and $V$ be a fuzzy $m$-semiopen set in $Y$ with $F(x) \leq V$ such that $x \in sfr(F^+(V))$. If possible, let $F$ be fuzzy upper $m$-irresolute at $x$. Then there exists a semiopen set $U$ in $X$ containing $x$ such that $U \subseteq F^+(V)$. Then $x \in U = sintU \subseteq sint(F^+(V))$ which implies that $x \in sint(F^+(V))$ and so $x \notin sfr(F^+(V))$, a contradiction and hence $F$ is not fuzzy upper $m$-irresolute.

**Theorem 5.13**. Let $F : (X.\tau) \to (Y, m)$ be a fuzzy multifunction where $m$ satisfies $M$-condition. Let $A = \{x \in X : F$ is not fuzzy lower $m$-irresolute at $x\}$, $B = \bigcup\{sfr(F^-(V)) : F(x)qV$ and $V$ is fuzzy $m$-semiopen set in $Y\}$. Then $A = B$.

**Proof**. Let $x \in X$ and $F$ be not fuzzy lower $m$-irresolute at $x$. Then there exists a fuzzy $m$-semiopen set $V$ in $Y$ with $F(x)qV$, but $U \not\subseteq F^-(V)$, for all semiopen sets $U$ in $X$ containing $x$. Then $U \bigcap (X \setminus F^-(V)) \neq \phi$ which implies that $x \in scl(X \setminus F^-(V)) = X \setminus sint(F^-(V))$ and so $x \notin sint(F^-(V))$. Again, $x \in F^-(V) \subseteq scl(F^-(V))$ . Hence $x \in sfr(F^-(V))$.

Conversely, let $x \in X$ and $V$ be a fuzzy $m$-semiopen set in $Y$ with $F(x)qV$ such that $x \in sfr(F^-(V))$. If possible, let $F$ be fuzzy lower $m$-irresolute at $x$. Then there exists a semiopen set $U$ in $X$ containing $x$ such that $U \subseteq F^-(V)$. Then $x \in U = sintU \subseteq sint(F^-(V))$ which implies that $x \in sint(F^-(V))$ and so $x \notin sfr(F^-(V))$, a contradiction and hence $F$ is not fuzzy lower $m$-irresolute.

**Definition 5.14**. A fuzzy set $A$ in a fuzzy $m$-space $(Y, m)$ is said to be a fuzzy lower (upper) $m$-semi nbd of a fuzzy set $B$ in $Y$ if there exists a fuzzy $m$-semipen set $V$ of $Y$ such that $BqV$ (resp., $B \leq V$) and $V \not{q} (1_Y \setminus A)$.

**Theorem 5.15**. A fuzzy multifunction $F : (X, \tau) \to (Y, m)$ is fuzzy upper (lower) $m$-irresolute on $X$ if and only if for each point $x_0 \in X$ and each fuzzy upper (lower) $m$-semi nbd $M$ of $F(x_0)$, $F^+(M)$ is a semi-nbd of $x_0$.

**Proof**. Let $F$ be fuzzy upper (lower) $m$-irresolute multifunction. Then for any $x_0 \in X$ and for any fuzzy upper (lower) $m$-semi nbd $M$ of $F(x_0)$, there exists a fuzzy $m$-semiopen set $V$ of $Y$ such that $F(x_0) \leq V$ $(F(x_0)qV)$ and $V \not{q} (1_Y \setminus M)$ and so $V \leq M$. Since $F$ is fuzzy upper (lower) $m$-irresolute, there exists a semiopen set $U$ in $X$ containing $x_0$ such that $U \subseteq F^+(V)$ $(U \subseteq F^-(V))$ which implies that $F(U) \leq V$ $(F(u)qV$, for all $u \in U)$ and so $F(U) \leq V \leq M$ $(x_0 \in U \subseteq F^-(V) \subseteq F^-(M))$. Hence $U \subseteq F^+(M)$ $(U \subseteq F^-(M))$ where $U$ is semiopen in $X$ containing $x_0 \Rightarrow F^+(M)(F^-(M))$ is a semi-nbd of $x_0$.

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