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THE m-TH ROOT FINSLER GEOMETRY OF THE BOGOSLOVSKY-GOENNER METRIC

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Abstract. In this paper we present the m-th root Finsler geometries of the three and four dimensional Bogoslovsky-Goenner metrics (good Finslerian anisotropic models in Special Relativity), in the sense of their Cartan torsion and curvature distinguished tensors or vertical Einstein-like equations.

1. Introduction

The physical studies of Asanov [1], Bogoslovsky [2] or Minguzzi [4] emphasize the importance of the Finsler geometry in the theory of space-time structure, gravitation and electromagnetism. For this reason, one emphasizes the important role played by the Finsler-Asanov metric (see Miron et al., [5], pp. 54)

$$F : TM \to \mathbb{R}, \quad F(x,y) = \left(a^1 a^2 ... a^n\right)^{\frac{1}{n}},$$

where a^α $(\alpha = 1, 2, ..., n)$ are linearly independent 1-forms. The above Finsler metric was initially considered, in diverse particular forms, by Riemann and Berwald-Moór (see Minguzzi [4] or Bogoslovsky and Goenner [3] and references therein). Moreover, considering that we have $a^\alpha = a^\alpha_\beta y^\beta$, where $a^\alpha_\beta \in \mathbb{R}$, the above Finsler-Asanov metric becomes a Minkowski metric of n-th root type. These kind of metrics were intensively studied by Shimada [6].

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In such an anisotropic physical context, Bogoslovski and Goenner introduced the locally Minkowski metric of a flat space-time with entirely broken isotropy, which is given by

(1) for $n = 3$:

$$L(y) = \sqrt[3]{(y^1 - y^2 - y^3)(y^1 - y^2 + y^3)(y^1 + y^2 - y^3)}; \tag{1}$$

(2) for $n = 4$:

$$\begin{aligned} L(y) \;=\; & \sqrt[4]{(y^1 - y^2 - y^3 - y^4)(y^1 - y^2 + y^3 + y^4)\cdot} \\ & \overline{\cdot(y^1 + y^2 - y^3 + y^4)(y^1 + y^2 + y^3 - y^4)}. \end{aligned} \tag{2}$$

2. The Finsler geometry of a locally Minkovski space

Let us consider that we have a locally Minkowski space $(M^n, L = L(y))$, where $L : TM \to \mathbb{R}$ is a Finsler metric depending only on directional variables y^i, where $i = 1, 2, ..., n$. Note that, in this Section, the Latin letters $i, j, k, ...$ run from 1 to n, and the Einstein convention of summation is adopted all over this work. It follows that the fundamental metrical d-tensor of the Minkowski space is

$$g_{ij}(y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j},$$

whose inverse d-tensor is given by $g^{jk}(y)$. Taking into account that the Finsler function L is a locally Minkowski metric, we deduce that the Euler-Lagrange equations of L produce the canonical nonlinear connection $N^i_j = 0$, where

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \frac{\partial}{\partial y^j}\left[g^{iu}\left(\frac{\partial^2 L^2}{\partial x^v \partial y^u}y^v - \frac{\partial L^2}{\partial x^u} + \frac{\partial^2 L^2}{\partial t \partial y^u}\right)\right].$$

As a consequence, in the Finsler geometrical study of the Minkowski metric are important only the vertical geometrical objects like (see [5]):

(1) the Cartan d-torsion:

$$C_{jkm} = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^m} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^j \partial y^k \partial y^m} \quad \text{(covariant form)},$$

$$C^i_{jk} = g^{im} C_{jkm} = \frac{g^{im}}{2}\frac{\partial g_{jk}}{\partial y^m} \quad \text{(contravariant form)};$$

(2) the Cartan d-curvature:

$$S^l_{ijk} = \frac{\partial C^l_{ij}}{\partial y^k} - \frac{\partial C^l_{ik}}{\partial y^j} + C^u_{ij}C^l_{uk} - C^u_{ik}C^l_{uj} \quad \text{(contravariant form)},$$

$$S_{imjk} = g_{ml}S^l_{ijk} = g^{uv}\left(C_{ujm}C_{vik} - C_{ukm}C_{vij}\right) \quad \text{(covariant form)}.$$

(3) the vertical Einstein-like equations for $n > 2$:

$$S_{ij} - \frac{S}{2}g_{ij} = k\widetilde{T}_{ij},$$

where

- $S_{ij} = S^m_{ijm}$ is the vertical Ricci d-tensor;
- $S = g^{uv}S_{uv}$ is the scalar curvature;
- $\widetilde{T}_{ij}$ are the new vertical components of the non-isotropic stress-energy d-tensor of matter $\mathbb{T}$;
- k is the Einstein constant.

3. The 3-rd root Finsler geometry of the three dimensional Bogoslovsky-Goenner metric

In this Section we have $M = \mathbb{R}^3$, that is $n = 3$, and the Latin indices $i, j, k, ...$ run from 1 to 3. Let us consider the following notations $S_\alpha = (y^1)^\alpha + (y^2)^\alpha + (y^3)^\alpha$, where $\alpha \in \mathbb{Z}$, $P_3 = y^1y^2y^3$ and

$$A = (y^1 - y^2 - y^3)(y^1 - y^2 + y^3)(y^1 + y^2 - y^3).$$

In such a context, by direct computations, the three dimensional Bogoslovsky-Goenner metric (1) takes the 3-rd root metric form

$$L = \sqrt[3]{A} = \sqrt[3]{(y^1)^3 + (y^2)^3 + (y^3)^3 - y^1(y^2)^2 - y^1(y^3)^2 - y^2(y^1)^2 - y^2(y^3)^2 - y^3(y^1)^2 - y^3(y^2)^2 + 2y^1y^2y^3} = \sqrt[3]{2S_3 - S_1S_2 + 2P_3}.$$

Working on the domain in which $2S_3 - S_1S_2 + 2P_3 \neq 0$, the fundamental metrical d-tensor produced by the Bogoslovsky-Goenner metric of order three (1) is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial^2 A^{2/3}}{\partial y^i \partial y^j} = \frac{A^{-1/3}}{3}A_{ij} - \frac{A^{-4/3}}{9}A_iA_j, \tag{3}$$

where

$$A_i = \frac{\partial A}{\partial y^i} = 6(y^i)^2 - S_2 - 2y^iS_1 + 2\frac{P_3}{y^i},$$

$$A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j} = -2y^i - 2y^j + 2\frac{P_3}{y^iy^j} + \left[12y^i - 2S_1 - 2\frac{P_3}{(y^i)^2}\right]\delta_{ij}.$$

Putting the coefficients A_{ij} into a matrix, $i,j=\overline{1,3}$, we get the matrix

$$(A_{ij})=\left(\begin{array}{ccc} 6y^1-2y^2-2y^3 & -2y^1-2y^2+2y^3 & -2y^1+2y^2-2y^3 \\ -2y^1-2y^2+2y^3 & -2y^1+6y^2-2y^3 & 2y^1-2y^2-2y^3 \\ -2y^1+2y^2-2y^3 & 2y^1-2y^2-2y^3 & -2y^1-2y^2+6y^3 \end{array}\right),$$

whose determinant is given by $\det(A_{ij})=-8D$, where $D=-4(y^1)^3-4(y^2)^3-4(y^3)^3+4(y^1)^2y^2+4(y^1)^2y^3+4y^1(y^2)^2+4y^1(y^3)^2+4(y^2)^2y^3+4y^2(y^3)^2-8y^1y^2y^3=-8S_3+4S_1S_2-8P_3$. If we have $D\neq 0$, then the inverse matrix of (A_{ij}) is the matrix $(A^{jk})_{j,k=\overline{1,3}}=D^{-1}\cdot A^*$, where

$$A^*=\left(\begin{array}{ccc} 2(y^2)^2-4y^2y^3+2(y^3)^2 & S_2-2y^1y^3-2y^2y^3 & S_2-2y^1y^2-2y^2y^3 \\ S_2-2y^1y^3-2y^2y^3 & 2(y^1)^2-4y^1y^3+2(y^3)^2 & S_2-2y^1y^2-2y^1y^3 \\ S_2-2y^1y^2-2y^2y^3 & S_2-2y^1y^2-2y^1y^3 & 2(y^1)^2-4y^1y^2+2(y^2)^2 \end{array}\right).$$

As a general formula, we have

$$A^{jk}=\frac{1}{D}\left[S_2-2(y^j+y^k)\frac{P_3}{y^jy^k}-(y^j)^2\delta_{jk}\right].$$

Using the above geometrical entities, we deduce that the inverse d-tensor of the fundamental metrical d-tensor (3) has the form

$$g^{jk}=3A^{1/3}A^{jk}+\frac{A^{-2/3}}{1-\frac{A^{-1}}{3}A^{uv}A_uA_v}A^jA^k, \tag{4}$$

where $A^j=A^{jw}A_w$. Moreover, by direct computations, the covariant Cartan d-torsion produced by the three dimensional Bogoslovsky-Goenner metric (1) is given by

$$C_{jkm}=\frac{A^{-1/3}}{6}A_{jkm}-\frac{A^{-4/3}}{18}(A_{jk}A_m+A_{km}A_j+A_{mj}A_k)-\frac{A^{-7/3}}{18}A_jA_kA_m,$$

where

$$A_{jkm}=\frac{\partial A_{jk}}{\partial y^m}=\frac{\partial^3 A}{\partial y^j\partial y^k\partial y^m}=\left\{\begin{array}{ll} 6, & \text{if } j=k=m \\ 2, & \text{if } j\neq k\neq m\neq j \\ -2, & \text{otherwise.}\end{array}\right.$$

In other words, if we denote by $A_{(m)} = (A_{jkm})_{j,k=\overline{1,3}}$, where $m \in \{1,2,3\}$, we get

$$A_{(1)} = \begin{pmatrix} 6 & -2 & -2 \\ -2 & -2 & 2 \\ -2 & 2 & -2 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} -2 & -2 & 2 \\ -2 & 6 & -2 \\ 2 & -2 & -2 \end{pmatrix},$$

$$A_{(3)} = \begin{pmatrix} -2 & 2 & -2 \\ 2 & -2 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

4. The 4-th root Finsler geometry of the four dimensional Bogoslovsky-Goenner metric

In this Section we have $M = \mathbb{R}^4$, that is $n = 4$, and the Latin indices $i, j, k, ...$ run from 1 to 4. Let us consider the following notations $S_\alpha = (y^1)^\alpha + (y^2)^\alpha + (y^3)^\alpha + (y^4)^\alpha$, where $\alpha \in \mathbb{Z}$, $P_4 = y^1 y^2 y^3 y^4$ and

$$A = (y^1 - y^2 - y^3 - y^4)(y^1 - y^2 + y^3 + y^4)(y^1 + y^2 - y^3 + y^4)(y^1 + y^2 + y^3 - y^4).$$

In such a context, by direct computations, the four dimensional Bogoslovsky-Goenner metric (2) takes the 4-th root metric form

$$L = \sqrt[4]{A} = \sqrt[4]{(y^1)^4 + (y^2)^4 + (y^3)^4 + (y^4)^4 - 2\left(y^1\right)^2 (y^2)^2 - \ldots}$$

$$\overline{-2\left(y^1\right)^2 (y^3)^2 - 2\left(y^1\right)^2 (y^4)^2 - 2\left(y^2\right)^2 (y^3)^2 - 2\left(y^2\right)^2 (y^4)^2 - \ldots}$$

$$\overline{-2\left(y^3\right)^2 (y^4)^2 - 8y^1y^2y^3y^4} = \sqrt[4]{2S_4 - S_2^2 - 8P_4}.$$

Working on the domain in which $2S_4 - S_2^2 - 8P_4 > 0$, the fundamental metrical d-tensor produced by the Bogoslovsky-Goenner metric of order four (2) is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial^2 A^{1/2}}{\partial y^i \partial y^j} = \frac{A^{-1/2}}{4} A_{ij} - \frac{A^{-3/2}}{8} A_i A_j, \tag{5}$$

where

$$A_i = \frac{\partial A}{\partial y^i} = 4(y^i)^3 - 4y^i \left[S_2 - \left(y^i\right)^2\right] - 8\frac{P_4}{y^i},$$

$$A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j} = -8y^i y^j - 8\frac{P_4}{y^i y^j} + \left[24\left(y^i\right)^2 - 4S_2 + 8\frac{P_4}{(y^i)^2}\right]\delta_{ij}.$$

Putting the coefficients A_{ij} into a matrix, we get the matrix

$$(A_{ij})_{i,j=\overline{1,4}} = \begin{pmatrix} p & a & b & c \\ a & q & c & b \\ b & c & r & a \\ c & b & a & s \end{pmatrix},$$

where

$$\begin{aligned} p &= 12(y^1)^2 - 4(y^2)^2 - 4(y^3)^2 - 4(y^4)^2, \\ q &= -4(y^1)^2 + 12(y^2)^2 - 4(y^3)^2 - 4(y^4)^2, \\ r &= -4(y^1)^2 - 4(y^2)^2 + 12(y^3)^2 - 4(y^4)^2, \\ s &= -4(y^1)^2 - 4(y^2)^2 - 4(y^3)^2 + 12(y^4)^2, \\ a &= -8y^1y^2 - 8y^3y^4, \\ b &= -8y^1y^3 - 8y^2y^4, \\ c &= -8y^1y^4 - 8y^2y^3, \end{aligned}$$

whose determinant is given by $D = \det(A_{ij}) = a^4 - 2a^2c^2 - 2b^2c^2 - 2a^2b^2 + b^4 + c^4 - a^2pq - b^2pr - a^2rs - b^2qs - c^2ps - c^2qr + 2abcp + 2abcq + 2abcr + 2abcs + pqrs$. If we have $D \neq 0$, then the inverse matrix of (A_{ij}) is the matrix $(A^{jk})_{j,k=\overline{1,4}} = D^{-1} \cdot A^*$, where $A^* =$

$$\left(\begin{matrix} -qa^2 + 2abc - rb^2 - sc^2 + qrs & a^3 - ac^2 - ab^2 + bcr + bcs - ars \\ a^3 - ac^2 - ab^2 + bcr + bcs - ars & -pa^2 + 2abc - sb^2 - rc^2 + prs \\ b^3 - bc^2 - a^2b + acq + acs - bqs & c^3 - b^2c - a^2c + abp + abs - cps \\ c^3 - b^2c - a^2c + abq + abr - cqr & b^3 - bc^2 - a^2b + acp + acr - bpr \end{matrix}\right.$$

$$\left.\begin{matrix} b^3 - bc^2 - a^2b + acq + acs - bqs & c^3 - b^2c - a^2c + abq + abr - cqr \\ c^3 - b^2c - a^2c + abp + abs - cps & b^3 - bc^2 - a^2b + acp + acr - bpr \\ -sa^2 + 2abc - pb^2 - qc^2 + pqs & a^3 - ac^2 - ab^2 + bcp + bcq - apq \\ a^3 - ac^2 - ab^2 + bcp + bcq - apq & -ra^2 + 2abc - qb^2 - pc^2 + pqr \end{matrix}\right).$$

Using the above geometrical entities, we deduce that the inverse d-tensor of the fundamental metrical d-tensor (5) has the form

$$g^{jk} = 4A^{1/2}A^{jk} + \frac{A^{-1/2}}{2 - A^{-1}A^{uv}A_uA_v}A^jA^k, \tag{6}$$

where $A^j = A^{jw}A_w$. Moreover, by direct computations, the covariant Cartan d-torsion produced by the four dimensional Bogoslovsky-Goenner metric (2) is given by

$$C_{jkm} = \frac{A^{-1/2}}{8}A_{jkm} - \frac{A^{-3/2}}{16}(A_{jk}A_m + A_{km}A_j + A_{mj}A_k) + \frac{3A^{-5/2}}{32}A_jA_kA_m,$$

where

$$A_{jkm} = \frac{\partial A_{jk}}{\partial y^m} = \frac{\partial^3 A}{\partial y^j \partial y^k \partial y^m}.$$

If we denote by $A_{(m)} = (A_{jkm})_{j,k=\overline{1,4}}$, where $m \in \{1,2,3,4\}$, we get

$$A_{(1)} = \begin{pmatrix} 24y^1 & -8y^2 & -8y^3 & -8y^4 \\ -8y^2 & -8y^1 & -8y^4 & -8y^3 \\ -8y^3 & -8y^4 & -8y^1 & -8y^2 \\ -8y^4 & -8y^3 & -8y^2 & -8y^1 \end{pmatrix},$$

$$A_{(2)} = \begin{pmatrix} -8y^2 & -8y^1 & -8y^4 & -8y^3 \\ -8y^1 & 24y^2 & -8y^3 & -8y^4 \\ -8y^4 & -8y^3 & -8y^2 & -8y^1 \\ -8y^3 & -8y^4 & -8y^1 & -8y^2 \end{pmatrix},$$

$$A_{(3)} = \begin{pmatrix} -8y^3 & -8y^4 & -8y^1 & -8y^2 \\ -8y^4 & -8y^3 & -8y^2 & -8y^1 \\ -8y^1 & -8y^2 & 24y^3 & -8y^4 \\ -8y^2 & -8y^1 & -8y^4 & -8y^3 \end{pmatrix},$$

$$A_{(4)} = \begin{pmatrix} -8y^4 & -8y^3 & -8y^2 & -8y^1 \\ -8y^3 & -8y^4 & -8y^1 & -8y^2 \\ -8y^2 & -8y^1 & -8y^4 & -8y^3 \\ -8y^1 & -8y^2 & -8y^3 & 24y^4 \end{pmatrix}.$$

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