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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 28(2018), No. 2, 49-62

SOME PROPERTIES OF $\mathcal{J}$ -HAUSDORFF, $\mathcal{J}$ -REGULAR AND $\mathcal{J}$ -NORMAL SPACES

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Abstract. In this paper we present new properties about the $\mathcal{J}$-Hausdorff, $\mathcal{J}$-regular and $\mathcal{J}$-normal spaces, introduced recently by Suriyakala-Vembu. Additionally we introduce the $\mathcal{J}$-Urysohn spaces, an intermediate concept between the Urysohn spaces and the $\mathcal{J}$-Hausdorff spaces. We present some of its properties.

1. Preliminaries

In 2016 Suriyakala and Vembu introduce the $\mathcal{J}$-Hausdorff, $\mathcal{J}$-regular and $\mathcal{J}$-normal spaces, as a simple and natural generalization of Hausdorff, T_3 and T_4 and spaces, respectively, through ideals. Many aspects of interest were not considered by the authors, such as the behavior of these new spaces with respect to products, sums and continuous functions. In this article we study these and other important aspects. We also introduce the $\mathcal{J}$-Urysohn spaces, an intermediate concept between the Urysohn spaces and the $\mathcal{J}$ -Hausdorff spaces. A relation between the $\mathcal{J}$-regular spaces and the $\mathcal{J}$-Urysohn spaces will be presented.

Keywords and phrases: $\mathcal{J}$ -Hausdorff, $\mathcal{J}$ -regular, $\mathcal{J}$ -normal.
(2010)Mathematics Subject Classification: 54A05, 54D10, 54D15

An ideal $\mathcal{I}$ in a set X is a subset of $\mathcal{P}(X)$, the power set of X, such that: (i) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some simple and useful ideals in X are: (i) $\mathcal{P}(A)$, where $A \subseteq X$, (ii) $\mathcal{I}_f$, the ideal of all finite subsets of X, (iii) $\mathcal{I}_c$, the ideal of all countable subsets of X, (iv) $\mathcal{I}_n$, the ideal of all nowhere dense subsets in a topological space (X,τ). If $f : X \to Y$ is a function and $\mathcal{I}$ is an ideal in X then the set $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal in Y [4]. Now, if f is injective and $\mathcal{J}$ is an ideal in Y then the set $f^{-1}(\mathcal{J}) = \{f^{-1}(J) : J \in \mathcal{J}\}$ is an ideal in X [4].

If (X,τ) is a topological space and $\mathcal{I}$ is an ideal in X, then $(X,\tau,\mathcal{I})$ is called an *ideal space.*

If (X,τ) is a topological space and $A \subseteq X$ then $\overline{A}$ (or $adh\,(A)$, or $adh_\tau\,(A)$) and $\overset{0}{A}$ (or $int(A)$, or $int_\tau(A)$) will, respectively, denote the closure and interior of A in (X,τ). The frontier of A is denoted by $Fr(A)$.

2. About $\mathcal{J}$-Hausdorff spaces

A previous Hausdorff version for ideal topological spaces was introduced by Dontchev [1] in 1995. This version was also studied by Nasef [3] and Hatir-Noiri [2]. The Suriyakala and Vembu version that we present here may sound more natural.

Recall that an ideal space $(X,\tau,\mathcal{I})$ is said to be *$\mathcal{J}$-Hausdorff* [6] if for each $\{a,b\} \subseteq X$, with $a \neq b$, there exists $\{U,V\} \subseteq \tau$ such that $a \in U$, $b \in V$ and $U \cap V \in \mathcal{I}$. If $A \subseteq X$ then A is defined to be *$\mathcal{J}$-Hausdorff* if the space $(A,\tau_A,\mathcal{I}_A)$ is $\mathcal{J}$ -Hausdorff, where $\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$.

It is noted that Hausdorff $\to$ $\mathcal{J}$-Hausdorff, and that each subspace of a $\mathcal{J}$-Hausdorff space is $\mathcal{J}$-Hausdorff.

Our first result is related to compact subsets of $\mathcal{J}$-Hausdorff spaces.

Theorem 2.1. If $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-Hausdorff then:

(1) If $K \subseteq X$ is compact and $a \in X\backslash K$, then there exists $\{U,V\}$ $\subseteq \tau$ such that $K \subseteq U$, $a \in V$ and $U \cap V \in \mathcal{I}$.

(2) If K_1 and K_2 are disjoint compact subsets of X, then there is a $\{U,V\} \subseteq \tau$ such that $K_1 \subseteq U$, $K_2 \subseteq V$ and $U \cap V \in \mathcal{I}$.

Proof. (1) If $x \in K$ then there exists $\{U_x,V_x\} \subseteq \tau$ such that $x \in U_x$, $a \in V_x$ and $U_x \cap V_x \in \mathcal{I}$. Since K is compact, there exists a finite $K_0 \subseteq K$ such that $K \subseteq \bigcup\limits_{x\in K_0} U_x$. If $U = \bigcup\limits_{x\in K_0} U_x$ and V

$= \bigcap_{x\in K_0} V_x$ then $K \subseteq U$, $a \in V$ and $U \cap V = \bigcup_{x\in K_0} (U_x \cap V) \subseteq \bigcup_{x\in K_0} (U_x \cap V_x) \in \mathcal{I}$, and so $U \cap V \in \mathcal{I}$.

(2) If $x \in K_2$ then there is a $\{U_x, V_x\} \subseteq \tau$ such that $K_1 \subseteq U_x$, $x \in V_x$ and $U_x \cap V_x \in \mathcal{I}$, by part (1). There exists a finite $K \subseteq K_2$ such that $K_2 \subseteq \bigcup_{x\in K} V_x$. If $V = \bigcup_{x\in K} V_x$ and $U = \bigcap_{x\in K} U_x$ then $K_1 \subseteq U$, $K_2 \subseteq V$ and $U \cap V \in \mathcal{I}$. □

The following result is related to continuous functions whose codomain is a $\mathcal{J}$-Hausdorff space.

Theorem 2.2. (1) Suppose that $f : (X, \tau) \to (Y, \beta)$ is a continuous function, $\mathcal{J}$ is an ideal in $Y \backslash f(X)$ and $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-Hausdorff. Then the set $A = \{(u, v) \in X \times X : f(u) = f(v)\}$ is closed in $(X \times X, \tau \times \tau)$.

(2) Suppose that $f, g : (X, \tau) \to (Y, \beta)$ are continuous, $\mathcal{J}$ is an ideal in $Y \backslash f(X)$ and $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-Hausdorff.
Then the set $A = \{u \in X : f(u) = g(u)\}$ is closed.

Proof. (1) Suppose that $(u, v) \in (X \times X) \backslash A$. There exists $\{U, V\} \subseteq \beta$ such that $f(u) \in U$, $f(v) \in V$ and $U \cap V \in \mathcal{J}$. Thus $(u, v) \in f^{-1}(U) \times f^{-1}(V)$ and $[f^{-1}(U) \times f^{-1}(V)] \cap A = \varnothing$, because if $(a, b) \in [f^{-1}(U) \times f^{-1}(V)] \cap A$ then $f(a) = f(b) \in U \cap V$, and this implies that $\{f(a)\} \in \mathcal{J}$. However this is not possible since $\mathcal{J}$ is an ideal in $Y \backslash f(X)$.

(2) Suppose that $u \in X \backslash A$. There is a $\{U, V\} \subseteq \beta$ such that $f(u) \in U$, $g(u) \in V$ and $U \cap V \in \mathcal{J}$. Hence $u \in f^{-1}(U) \cap g^{-1}(V)$ and $[f^{-1}(U) \cap g^{-1}(V)] \cap A = \varnothing$, because if $a \in [f^{-1}(U) \cap g^{-1}(V)] \cap A$ then $f(a) = g(a) \in U \cap V$, and this implies that $\{f(a)\} \in \mathcal{J}$. But this is not possible since $\mathcal{J}$ is an ideal in $Y \backslash f(X)$. □

Theorem 2.3. Let $(X, \tau, \mathcal{I})$ be an ideal space. If F and G are disjoint closed sets and if there exists $\{U, V\} \subseteq \tau$ such that $FrF \subseteq U$, $FrG \subseteq V$ and $U \cap V \in \mathcal{I}$, then there is a $\{W, T\} \subseteq \tau$ such that $F \subseteq W$, $G \subseteq T$ and $W \cap T \in \mathcal{I}$.

Proof. If we do $W = \overset{0}{F} \cup (U \backslash G)$ and $T = \overset{0}{G} \cup (V \backslash F)$ then $\{W, T\} \subseteq \tau$, $F \subseteq W$, $G \subseteq T$ and $W \cap T = (U \backslash G) \cap (V \backslash F) \in \mathcal{I}$. □

Corollary 2.4. Let $(X, \tau, \mathcal{I})$be a $\mathcal{J}$-Hausdorff space such that FrU is compact, for each $U \in \tau$. If F and G are disjoint closed sets then there exists $\{U, V\} \subseteq \tau$ such that $F \subseteq U$, $G \subseteq V$ and $U \cap V \in \mathcal{I}$.

Proof. It is a consequence of Theorems 2.1 and 2.3. □

The result that follows is related to the convergence of sequences in $\mathcal{J}$-Hausdorff spaces.

Theorem 2.5. If $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-Hausdorff and $\{x_n\}$ is a succession in X such that there is no a positive integer M such that $\{x_n : n \geq M\} \in \mathcal{I}$, then if $\{x_n\}$ converge to a and b, we have that $a = b$.

Proof. If $a \neq b$ then there exists $\{U,V\} \subseteq \tau$ such that $a \in U$, $b \in V$ and $U \cap V \in \mathcal{I}$. Now, since $\{x_n\}$ converge to a and b, there is a $M \in \mathbb{Z}^+$ such that $\{x_n : n \geq M\} \in U \cap V$. This implies that $\{x_n : n \geq M\} \in \mathcal{I}$, absurd. □

Next we obtain a property of the product of $\mathcal{J}$-Hausdorff spaces.

If $\{X_i : i \in \Lambda\}$ is a collection of sets and if $\mathcal{I}_i$ is an ideal in X_i, for each $i \in \Lambda$, we will denote by $\bigotimes\limits_{i\in\Lambda} \mathcal{I}_i$ the set of all $A \subseteq \prod\limits_{i\in\Lambda} X_i$ such that there exists a finite $\Lambda_0 \subseteq \Lambda$ with $A \subseteq \bigcup\limits_{i\in\Lambda_0} p_i^{-1}(I_i)$, for some $I_i \in \mathcal{I}_i$, for each $i \in \Lambda_0$. Here p_i represents the i-th projection. It is clear that $\bigotimes\limits_{i\in\Lambda} \mathcal{I}_i$ is an ideal in $\prod\limits_{i\in\Lambda} X_i$.

Theorem 2.6. If $\{(X_i,\tau_i,\mathcal{I}_i) : i \in \Lambda\}$ is a collection of non-empty $\mathcal{J}$-Hausdorff spaces, with $|\Lambda| > 1$, then $\left(\prod\limits_{i\in\Lambda} X_i, \prod\limits_{i\in\Lambda} \tau_i, \bigotimes\limits_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{J}$-Hausdorff.

Proof. Suppose that $\{x,y\} \subseteq \prod\limits_{i\in\Lambda} X_i$, with $x \neq y$, that $x = (x_i)_{i\in\Lambda}$ and that $y = (y_i)_{i\in\Lambda}$. There exists $\theta \in \Lambda$ such that $x_\theta \neq y_\theta$. Given that $(X_\theta,\tau_\theta,\mathcal{I}_\theta)$ is $\mathcal{J}$-Hausdorff, there is a $\{U_\theta, V_\theta\} \subseteq \tau_\theta$ such that $x_\theta \in U_\theta$, $y_\theta \in V_\theta$ and $U_\theta \cap V_\theta = I_\theta \in \mathcal{I}_\theta$. If p_θ represents the θ-th projection we have that $\left\{p_\theta^{-1}(U_\theta), p_\theta^{-1}(V_\theta)\right\} \subseteq \prod\limits_{i\in\Lambda} \tau_i$, $x \in p_\theta^{-1}(U_\theta)$, $y \in p_\theta^{-1}(V_\theta)$ and $p_\theta^{-1}(U_\theta) \cap p_\theta^{-1}(V_\theta) = p_\theta^{-1}(U_\theta \cap V_\theta) \in \bigotimes\limits_{i\in\Lambda} \mathcal{I}_i$. □

In the next theorem we analyze the sum of $\mathcal{J}$-Hausdorff spaces. If $\{X_i : i \in \Lambda\}$ is a collection of sets such that $X_i \cap X_j = \varnothing$, for each $\{i,j\} \subseteq \Lambda$ with $i \neq j$, and if $\mathcal{I}_i$ is an ideal in X_i, for each $i \in \Lambda$, we will denote by $\sum\limits_{i\in\Lambda} \mathcal{I}_i$ the set $\left\{\sum\limits_{i\in\Lambda} I_i : I_i \in \mathcal{I}_i, \text{ for each } i \in \Lambda\right\}$. It is clear that $\sum\limits_{i\in\Lambda} \mathcal{I}_i$ is an ideal in $\sum\limits_{i\in\Lambda} X_i$. On the other hand, if τ_i is a topology on X_i, for each $i \in \Lambda$, then the topology $\sum\limits_{i\in\Lambda} \tau_i$ is the set $\left\{A \subseteq \sum\limits_{i\in\Lambda} X_i : A \cap X_i \in \tau_i, \text{ for each } i \in \Lambda\right\}$.

Theorem 2.7. If $\{(X_i, \tau_i, \mathcal{I}_i) : i \in \Lambda\}$ is a non-empty collection of non-empty $\mathcal{J}$-Hausdorff spaces, with $X_i \cap X_j = \varnothing$ for each $i \neq j$, then the space $\left(\sum\limits_{i\in\Lambda} X_i, \sum\limits_{i\in\Lambda} \tau_i, \sum\limits_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{J}$-Hausdorff.

Proof. Suppose that $\{a, b\} \subseteq \sum\limits_{i\in\Lambda} X_i$ and $a \neq b$. There exists $\{j, k\} \subseteq \Lambda$ with $a \in X_j$ and $b \in X_k$. If $j \neq k$ then $\{X_j, X_k\} \subseteq \sum\limits_{i\in\Lambda} \tau_i$ and $X_j \cap X_k = \varnothing \in \sum\limits_{i\in\Lambda} \mathcal{I}_i$. If $j = k$ then there is a $\{U, V\} \subseteq \tau_j \subseteq \sum\limits_{i\in\Lambda} \tau_i$ such that $a \in U$, $b \in V$ and $U \cap V \in \mathcal{I}_j \subseteq \sum\limits_{i\in\Lambda} \mathcal{I}_i$. □

We finished this section by presenting some functional properties of the $\mathcal{J}$-Hausdorff spaces. If $\mathcal{J}$ is an ideal in Y and if $f : X \to Y$ is a function, we will denote by $\mathcal{I}_{f,\mathcal{J}}$ the set $\{A \subseteq X : A \subseteq f^{-1}(J), \text{ for some } J \in \mathcal{J}\}$. It's very simple to show that $\mathcal{I}_{f,\mathcal{J}}$ is an ideal in X. Moreover, if β is a topology in Y then the set $f^{-1}(\beta) = \{f^{-1}(V) : V \in \beta\}$ is a topology in X.

Theorem 2.8. (1) If $f : X \to Y$ is inyective and $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-Hausdorff, then $(X, f^{-1}(\beta), f^{-1}(\mathcal{J}))$ is $\mathcal{J}$-Hausdorff.
(2) If $(Y, \beta, \mathcal{J})$ is an ideal space, $f : X \to Y$ is a surjective function and $(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{J}})$ is $\mathcal{J}$-Hausdorff, then $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-Hausdorff.

Proof. It is similar to the proof of Theorem 3.9. □

Theorem 2.9. If $f : (X, \tau) \to (Y, \beta)$ is a closed and surjective function such that $f^{-1}(\{y\})$ is compact for each $y \in Y$, and if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-Hausdorff, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{J}$-Hausdorff.

Proof. Suppose that $\{u, v\} \subseteq Y$ and $u \neq v$. Given that $f^{-1}(\{u\})$ and $f^{-1}(\{v\})$ are compact and disjoint sets then there is a $\{U, V\} \subseteq \tau$ such that $f^{-1}(\{u\}) \subseteq U$, $f^{-1}(\{v\}) \subseteq V$ and $U \cap V \in \mathcal{I}$, by Theorem 2.1. This implies that $u \in Y \backslash f(X \backslash U)$ and $v \in Y \backslash f(X \backslash V)$. On the other hand, $\{Y \backslash f(X \backslash U), Y \backslash f(X \backslash V)\} \subseteq \beta$ and $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)] = Y \backslash f[X \backslash (U \cap V)]$. Since f is surjective we have that $Y \backslash f[X \backslash (U \cap V)] \subseteq f(U \cap V) \in f(\mathcal{I})$, and so $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)] \in f(\mathcal{I})$. □

We end this section by presenting a characterization of the $\mathcal{J}$-Hausdorff spaces by means of filters.

If $(X,\tau,\mathcal{I})$ is an ideal space, $\mathcal{F}$ is a filter in X and $a \in X$, we say $\mathcal{F}$ *converge to* a *modulo* $\mathcal{I}$ (written $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} a$) if $\{U \backslash I : U \in \tau,\ a \in U \text{ and } I \in \mathcal{I}\} \subseteq \mathcal{F}$. It is observed that $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} a$ implies $\mathcal{F} \rightarrow a$.

Theorem 2.10. The ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-Hausdorff if and only if, for each filter $\mathcal{F}$ and each $\{a,b\} \subseteq X$, if $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} a$ and $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} b$, then $a = b$.

Proof. $(\rightarrow)$ If $(X,\tau,\mathcal{I})$ is not $\mathcal{J}$-Hausdorff, there is a $\{a,b\} \subseteq X$, with $a \neq b$, such that for each $\{U,V\} \subseteq \tau$, if $a \in U$ and $b \in V$ then $U \cap V \notin \mathcal{I}$. If $\mathcal{F}$ is the set of all $A \subseteq X$ such that there are $\{U,V\} \subseteq \tau$ and $I \in \mathcal{I}$, with $a \in U$, $b \in V$ and $(U \cap V) \backslash I \subseteq A$, then $\mathcal{F}$ is a filter in X, $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} a$ and $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} b$.

$(\leftarrow)$ Suppose that $\mathcal{F}$ is a filter in X, $\{a,b\} \subseteq X$, $a \neq b$, $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} a$ and $\mathcal{F} \overset{\mathcal{I}}{\rightarrow} b$. If $\{U,V\} \subseteq \tau$, $\{I,J\} \subseteq \mathcal{I}$, $a \in U$ and $b \in V$, then $(UI\backslash) \cap (V \backslash J) \in \mathcal{F}$ and so $(U \cap V) \setminus (I \cup J) \in \mathcal{F}$. This implies that $U \cap V \notin \mathcal{I}$. Thus $(X,\tau,\mathcal{I})$ is not $\mathcal{J}$-Hausdorff. $\square$

3. About $\mathcal{J}$-regular spaces

In this section we will present new properties of $\mathcal{J}$-regular spaces. We also introduce the $\mathcal{J}$-Urysohn spaces, an intermediate concept between the Urysohn spaces and the $\mathcal{J}$-Hausdorff spaces. We will establish a relationship between the $\mathcal{J}$-regular and the $\mathcal{J}$-Urysohn spaces.

Recall that an ideal space $(X,\tau,\mathcal{I})$ is said to be *$\mathcal{J}$-regular* [6] if (X,τ) is T_1 and, for each closed $F \subseteq X$ and each $x \in X \backslash F$, there exists $\{U,V\} \subseteq \tau$ such that $x \in U$, $F \subseteq V$ and $U \cap V \in \mathcal{I}$. If $A \subseteq X$ then A is *$\mathcal{J}$-regular* if the space $(A,\tau_A,\mathcal{I}_A)$ is $\mathcal{J}$-regular. It is noted that $T_3 \rightarrow \mathcal{J}$-regular.

Theorem 3.1. (1) If $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-regular then for each $U \in \tau$ and for each $x \in U$, there are $V \in \tau$ and $I \in \mathcal{I}$ such that $x \in V \subseteq \overline{V} \subseteq U \cup \overline{I}$.

(2) If $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-regular and $A \subseteq X$ then $(A,\tau_A,\mathcal{I}_A)$ is $\mathcal{J}$-regular.

(3) If $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-regular, $K \subseteq X$ is compact, $F \subseteq X$ is closed and $K \cap F = \varnothing$, then there is a $\{U,V\} \subseteq \tau$ such that $K \subseteq U$, $F \subseteq V$ and $U \cap V \in \mathcal{I}$.

Proof. (1) If $U \in \tau$ and $x \in U$ then there are $\{V, W\} \subseteq \tau$ and $I \in \mathcal{I}$ such that $x \in V$, $X \backslash U \subseteq W$ and $V \cap W = I$. Then $V \subseteq (X \backslash W) \cup I$ and so $\overline{V} \subseteq (X \backslash W) \cup \overline{I} \subseteq U \cup \overline{I}$.

(2) Suppose that $F \subseteq A$ is closed in A and that $b \in A \backslash F$. Since $F = \overline{F} \cap A$ then $b \notin \overline{F}$. There is a $\{U, V\} \subseteq \tau$ such that $b \in U$, $\overline{F} \subseteq V$ and $U \cap V \in \mathcal{I}$. Hence $b \in U \cap A$, $F = \overline{F} \cap A \subseteq V \cap A$ and $(U \cap A) \cap (V \cap A) = A \cap (U \cap V) \in \mathcal{I}_A$.

(3) For each $x \in K$ there exists $\{U_x, V_x\} \subseteq \tau$ such that $x \in U_x$, $F \subseteq V_x$ and $U_x \cap V_x \in \mathcal{I}$. Given that K is compact, there is a finite $A \subseteq K$ with $K \subseteq \bigcup_{x \in A} U_x$. If $V = \bigcap_{x \in A} V_x$ and $U = \bigcup_{x \in A} U_x$ then $F \subseteq V$ and $U \cap V = \bigcup_{x \in A} (V \cap U_x) \subseteq \bigcup_{x \in A} (V_x \cap U_x) \in \mathcal{I}$.

□

In the three theorems that follow we will review the functional behavior of the $\mathcal{J}$-regular spaces.

Theorem 3.2. If $f : (X, \tau) \to (Y, \beta)$ is a continuous, closed, open and surjective function, and if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-regular, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{J}$-regular.

Proof. Suppose that $F \subseteq Y$ is closed and that $b \in Y \backslash F$. Let $a \in X$ be such that $f(a) = b$. Since $a \notin f^{-1}(F)$, there exists $\{U, V\} \subseteq \tau$ such that $a \in U$, $f^{-1}(F) \subseteq V$ and $U \cap V \in \mathcal{I}$. Then $b \in f(U)$ and $F \subseteq Y \backslash f(X \backslash V)$. Moreover $\{f(U), Y \backslash [f(X \backslash V)]\} \subseteq \tau$, because f is closed and open. Let $I \in \mathcal{I}$ be such that $U \cap V = I$. Thus $U \subseteq (X \backslash V) \cup I$ and so $f(U) \subseteq f(X \backslash V) \cup f(I)$, $f(U) \cap [Y \backslash f(X \backslash V)] \subseteq f(I)$, and $f(U) \cap [Y \backslash f(X \backslash V)] \in f(\mathcal{I})$. Moreover (Y, β) is T_1 given that (X, τ) is T_1 and f is closed and surjective. □

Recall that a function $f : (X, \tau) \to (Y, \beta)$ is said to be a *perfect function* if f is closed, surjective and continuous, and if $f^{-1}(\{y\})$ is compact, for each $y \in Y$.

Theorem 3.3. Let $(X, \tau, \mathcal{I})$ be a $\mathcal{J}$-regular space. If $f : (X, \tau) \to (Y, \beta)$ is a perfect function then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{J}$-regular.

Proof. Suppose that $F \subseteq Y$ is closed and that $b \in Y \backslash F$. Then $f^{-1}(\{b\}) \subseteq X \backslash f^{-1}(F) \in \tau$. Since $f^{-1}(\{b\})$ is compact and $f^{-1}(F)$ is closed, there exists $\{U, V\} \subseteq \tau$ such that $f^{-1}(\{b\}) \subseteq U$, $f^{-1}(F) \subseteq V$ and $U \cap V \in \mathcal{I}$, by Theorem 3.1.

Now, $b \in Y \backslash f(X \backslash U)$, $F \subseteq Y \backslash f(X \backslash V)$, $\{Y \backslash f(X \backslash U), Y \backslash f(X \backslash V)\} \subseteq \beta$ and $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)] = Y \backslash f(X \backslash (U \cap V))$. Given that f is surjective, $Y \backslash f(X \backslash (U \cap V)) \subseteq f(U \cap V)$. All this allows us to conclude that $[Y\backslash f(X\backslash U)] \cap [Y\backslash f(X\backslash V)] \in f(\mathcal{I})$. Moreover (Y,β) is T_1 given that (X,τ) is T_1 and f is closed and surjective. □

Theorem 3.4. (1) If $f : X \to Y$ is inyective and $(Y,\beta,\mathcal{J})$ is $\mathcal{J}$-regular, then $(X, f^{-1}(\beta), f^{-1}(\mathcal{J}))$ is $\mathcal{J}$-regular.

(2) If $(Y,\beta,\mathcal{J})$ is an ideal space, $f : X \to Y$ is a surjective function and $(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{J}})$ is $\mathcal{J}$-regular, then $(Y,\beta,\mathcal{J})$ is $\mathcal{J}$-regular.

Proof. (1) Suppose that $F \subseteq X$ is closed and $a \in X\backslash F$. There is a closed set $G \subseteq Y$ such that $F = f^{-1}(G)$. Given that $a \in f^{-1}(Y\backslash G)$, that is, $f(a) \in Y\backslash G$, there exists $\{U,V\} \subseteq \beta$ such that $f(a) \in U$, $G \subseteq V$ and $U \cap V \in \mathcal{J}$. Thus $a \in f^{-1}(U)$, $F = f^{-1}(G) \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \in f^{-1}(\mathcal{J})$. Now, since (Y,β) is T_1 and f is inyective, if $x \in X$ then $\{x\} = f^{-1}(\{f(x)\})$ is closed in X.

(2) If $G \subseteq Y$ is closed and $b = f(a) \in Y\backslash G$ then $a \in X\backslash f^{-1}(G)$ and so, since $(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{J}})$ is $\mathcal{J}$-regular, there is a $\{U,V\} \subseteq \beta$ such that $a \in f^{-1}(U)$, $f^{-1}(G) \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) \in \mathcal{I}_{f,\mathcal{J}}$. All this allows us to conclude that $b \in U$, $G \subseteq V$ and $U \cap V \in \mathcal{J}$, since f is surjective. Finally, given that $f : (X, f^{-1}(\beta)) \to (Y,\beta)$ is closed, we have that (Y,β) is T_1. □

Next we will analyze the products of $\mathcal{J}$-regular spaces.

Theorem 3.5. If $\{(X_i,\tau_i,\mathcal{I}_i) : i \in \Lambda\}$ is a non-empty collection of non-empty $\mathcal{J}$-regular spaces, then $\left(\prod_{i\in\Lambda} X_i, \prod_{i\in\Lambda} \tau_i, \bigotimes_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{J}$-regular.

Proof. Let $X = \prod_{i\in\Lambda} X_i$ be. Suppose that $F \subseteq X$ is closed and that $x = (x_i)_{i\in\Lambda} \in X\backslash F$. There exists a finite $\Lambda_0 \subseteq \Lambda$ such that, for each $i \in \Lambda_0$, there exists a $U_i \in \tau_i$ with $x \in \bigcap_{i\in\Lambda_0} p_i^{-1}(U_i) \subseteq X\backslash F$. For each $i \in \Lambda_0$ we have that $x_i \notin X_i\backslash U_i$, and so there is a $\{V_i, W_i\} \subseteq \tau_i$ such that $x_i \in V_i$, $X_i\backslash U_i \subseteq W_i$ and $V_i \cap W_i \in \mathcal{I}_i$. Thus $x \in V = \bigcap_{i\in\Lambda_0} p_i^{-1}(V_i)$, $F \subseteq \bigcup_{i\in\Lambda_0} \left[X\backslash p_i^{-1}(U_i)\right] \subseteq W = \bigcup_{i\in\Lambda_0} p_i^{-1}(W_i)$ and $V \cap W = \bigcup_{i\in\Lambda_0} \left[V \cap p_i^{-1}(W_i)\right] \subseteq \bigcup_{i\in\Lambda_0} \left[p_i^{-1}(V_i \cap W_i)\right] \in \bigotimes_{i\in\Lambda} \mathcal{I}_i$. Hence $V \cap W \in \bigotimes_{i\in\Lambda} \mathcal{I}_i$. On the other hand, $\left(\prod_{i\in\Lambda} X_i, \prod_{i\in\Lambda} \tau_i\right)$ is T_1 since each space (X_i,τ_i) is T_1. □

Now we will consider the sums of $\mathcal{J}$-regular spaces.

Theorem 3.6. If $\{(X_i,\tau_i,\mathcal{I}_i) : i\in\Lambda\}$ is a non-empty collection of non-empty $\mathcal{J}$-regular spaces, with $X_i\cap X_j=\varnothing$ for each $i\neq j$, then the space $\left(\sum\limits_{i\in\Lambda} X_i, \sum\limits_{i\in\Lambda}\tau_i, \sum\limits_{i\in\Lambda}\mathcal{I}_i\right)$ is $\mathcal{J}$-regular.

Proof. Suppose that $F\subseteq \sum\limits_{i\in\Lambda} X_i$ is closed and that $a\in\left(\sum\limits_{i\in\Lambda} X_i\right)\setminus F$. Let $\alpha\in\Lambda$ be such that $a\in X_\alpha$. Since $F\cap X_\alpha$ is closed in X_α and $a\in X_\alpha\setminus(F\cap X_\alpha)$, there is a $\{U_\alpha,V_\alpha\}\subseteq\tau_\alpha\subseteq\sum\limits_{i\in\Lambda}\tau_i$ with $a\in U_\alpha$, $F\cap X_\alpha\subseteq V_\alpha$ and $U_\alpha\cap V_\alpha\in\mathcal{I}_\alpha$. If we do $V=\sum\limits_{i\in\Lambda} W_i$, where $W_i=X_i$ if $i\neq\alpha$, and $W_\alpha=V_\alpha$, then $V\in\sum\limits_{i\in\Lambda}\tau_i$, $F\subseteq V$ and $U_\alpha\cap V=U_\alpha\cap V_\alpha\in\mathcal{I}_\alpha\subseteq\sum\limits_{i\in\Lambda}\mathcal{I}_i$. Moreover $\left(\sum\limits_{i\in\Lambda} X_i,\sum\limits_{i\in\Lambda}\tau_i\right)$ is T_1 since each space (X_i,τ_i) is T_1.
□

We are going to introduce an intermediate concept between the Urysohn spaces and the $\mathcal{J}$-Hausdorff spaces.

Definition 3.7. *The ideal space $(X,\tau,\mathcal{I})$ is said to be* $\mathcal{J}$-Urysohn *if for each $\{a,b\}\subseteq X$, with $a\neq b$, there exists $\{U,V\}\subseteq\tau$ such that $a\in U$, $b\in V$ and $\overline{U}\cap\overline{V}\in\mathcal{I}$.*

It is clear that Urysohn $\to$ $\mathcal{J}$-Urysohn $\to$ $\mathcal{J}$-Hausdorff. The reciprocal implications, in general, are false, as the following examples show. It is also easy to see that if τ and β are topologies in X, with $\tau\subseteq\beta$, and if $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-Urysohn, then $(X,\beta,\mathcal{I})$ is $\mathcal{J}$-Urysohn.

Example 3.8.
(1) *If $\mathcal{C}=\{\varnothing,\mathbb{R}\}\cup\{(r,\infty) : r\in\mathbb{R}\}$ then the space $(\mathbb{R},\mathcal{C},\mathcal{P}(\mathbb{R}))$ is $\mathcal{J}$-Urysohn, but $(\mathbb{R},\mathcal{C})$ is not Urysohn.*
(2) *If $X=\{0,1\}$, $\tau=\{\varnothing,\{0\},\{0,1\}\}$ and $\mathcal{I}=\{\varnothing,\{0\}\}$, then $(X,\tau,\mathcal{I})$ is $\mathcal{J}$-Hausdorff although $(X,\tau,\mathcal{I})$ is not $\mathcal{J}$-Urysohn.*
(3) *Let γ be the topology in $\mathbb{Z}^+$ given by: If $U\subseteq\mathbb{Z}^+$ then $U\in\gamma$ if and only if, when $2\in U$ then $\{2k : k\in\mathbb{Z}^+\}\subseteq U$. Let $\mathcal{I}$ be the ideal of all finite subsets of $\{2k : k\in\mathbb{Z}^+\}$. It is easy to prove that $(\mathbb{Z}^+,\gamma,\mathcal{I})$ is a $\mathcal{J}$-Urysohn space.*

In the next theorem we present some functional properties of the $\mathcal{J}$-Urysohn spaces.

Theorem 3.9.
(1) If $f : X\to Y$ is inyective and $(Y,\beta,\mathcal{J})$ is $\mathcal{J}$-Urysohn, then $(X,f^{-1}(\beta),f^{-1}(\mathcal{J}))$ is $\mathcal{J}$-Urysohn.

(2) If $(Y, \beta, \mathcal{J})$ is an ideal space, $f : X \to Y$ is surjective and if the space $\left(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{J}}\right)$ is $\mathcal{J}$-Urysohn, then $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-Urysohn.

Proof. (1) Suppose that $\{a, b\} \subseteq X$ and $a \neq b$. Given that $f(a) \neq f(b)$, there is a $\{V, W\} \subseteq \beta$ such that $f(a) \in V$, $f(b) \in W$ and $\overline{V} \cap \overline{W} \in \mathcal{J}$. Thus $a \in f^{-1}(V)$, $b \in f^{-1}(W)$ and, since $f : (X, f^{-1}(\beta)) \to (Y, \beta)$ is continuous, $\overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \subseteq f^{-1}\left(\overline{V}\right) \cap f^{-1}\left(\overline{W}\right) = f^{-1}\left(\overline{V} \cap \overline{W}\right) \in f^{-1}(\mathcal{J})$. Therefore $\overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \in f^{-1}(\mathcal{J})$.

(2) Suppose that $\{a, b\} \subseteq X$ and $f(a) \neq f(b)$. Given that $a \neq b$, there is a $\{V, W\} \subseteq \beta$ such that $a \in f^{-1}(V)$, $b \in f^{-1}(W)$ and $\overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \in \mathcal{I}_{f,\mathcal{J}}$. There exists $J \in \mathcal{J}$ such that $\overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \subseteq f^{-1}(J)$. Since $f : (X, f^{-1}(\beta)) \to (Y, \beta)$ is closed and continuous, we have that $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$, for each $B \subseteq Y$. Hence $f(a) \in V$, $f(b) \in W$ and $f^{-1}\left(\overline{V} \cap \overline{W}\right) \subseteq f^{-1}(J)$. Given that f is surjective, $\overline{V} \cap \overline{W} \in \mathcal{J}$. □

Now we analyze the product and sum of $\mathcal{J}$-Urysohn spaces.

Theorem 3.10. If $\{(X_i, \tau_i, \mathcal{I}_i) : i \in \Lambda\}$ is a non-empty collection of non-empty $\mathcal{J}$-Urysohn spaces then:

(1) $\left(\prod\limits_{i\in\Lambda} X_i, \prod\limits_{i\in\Lambda} \tau_i, \bigotimes\limits_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{J}$-Urysohn.

(2) If $X_i \cap X_j = \varnothing$ for each $i \neq j$, the space $\left(\sum\limits_{i\in\Lambda} X_i, \sum\limits_{i\in\Lambda} \tau_i, \sum\limits_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{J}$-Urysohn.

Proof. (1) Let $X = \prod\limits_{i\in\Lambda} X_i$ be. Suppose that $\{a, b\} \subseteq X$, $a \neq b$, $a = (a_i)_{i\in\Lambda}$ and $b = (b_i)_{i\in\Lambda}$. There exists $j \in \Lambda$ with $a_j \neq b_j$. Since $(X_j, \tau_j, \mathcal{I}_j)$ is $\mathcal{J}$-Urysohn, there is a $\{U_j, V_j\} \subseteq \tau_j$ such that $a_j \in U_j$, $b_j \in V_j$ and $\overline{U_j} \cap \overline{V_j} \in \mathcal{I}_j$. Therefore $a \in p_j^{-1}(U_j)$, $b \in p_j^{-1}(V_j)$ and $\overline{p_j^{-1}(U_j)} \cap \overline{p_j^{-1}(V_j)} \subseteq p_j^{-1}\left(\overline{U_j}\right) \cap p_j^{-1}\left(\overline{V_j}\right) = p_j^{-1}\left(\overline{U_j} \cap \overline{V_j}\right)$. Hence $\overline{p_j^{-1}(U_j)} \cap \overline{p_j^{-1}(V_j)} \in \bigotimes\limits_{i\in\Lambda} \mathcal{I}_i$.

(2) Note that each X_i is open and closed in $\sum\limits_{i\in\Lambda} X_i$, and that if $A \subseteq X_i$ then $adh_{\tau_i}(A) = \overline{A}$, the adherence of A in $\sum\limits_{i\in\Lambda} X_i$.

Now, if $\{a, b\} \subseteq \sum\limits_{i\in\Lambda} X_i$ and $a \neq b$, there is a $\{j, k\} \subseteq \Lambda$ such that $a \in X_j$ and $b \in X_k$. If $j \neq k$ then $\overline{X_j} \cap \overline{X_k} = X_j \cap X_k = \varnothing$

$\in \sum_{i\in\Lambda} \mathcal{I}_i$. If $j = k$ then there exists $\{U, V\} \subseteq \tau_j \subseteq \sum_{i\in\Lambda} \tau_i$ with $a \in U$, $b \in V$ and $adh_{\tau_j}(U) \cap adh_{\tau_j}(V) \in \mathcal{I}_j \subseteq \sum_{i\in\Lambda} \mathcal{I}_i$. Therefore $\overline{U} \cap \overline{V} \in \sum_{i\in\Lambda} \mathcal{I}_i$. □

Next present a relationship between the $\mathcal{J}$-regular and $\mathcal{J}$-Urysohn spaces.

If $(X, \tau, \mathcal{I})$ is an ideal space then the ideal $\overline{\mathcal{I}}$ [5] is the set of all $A \subseteq X$ such that there is $I \in \mathcal{I}$ with $A \subseteq \overline{I}$.

Theorem 3.11. If $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-regular and (X, τ) is T_2 then $(X, \tau, \overline{\mathcal{I}})$ is $\mathcal{J}$-Urysohn.

Proof. If $\{a, b\} \subseteq X$ and $a \neq b$ then there exists $\{U, V\} \subseteq \tau$ such that $a \in U$, $b \in V$ and $U \cap V = \varnothing$. Given that $\overline{U} \cap V = \varnothing$ then $b \notin \overline{U}$, and so there is a $\{W, T\} \subseteq \tau$ such that $b \in W$, $\overline{U} \subseteq T$ and $W \cap T \in \mathcal{I}$. Given that $\overline{W} \cap T \subseteq \overline{W \cap T}$ we have that $\overline{W} \cap T \in \overline{\mathcal{I}}$, and so $\overline{U} \cap \overline{W} \in \overline{\mathcal{I}}$, because $\overline{U} \cap \overline{W} \subseteq T \cap \overline{W}$. □

Finally, we show a couple of properties of the compact sets in a $\mathcal{J}$-Urysohn space.

Theorem 3.12. If $(X, \tau, \mathcal{I})$ is a $\mathcal{J}$-Urysohn space, $K \subseteq X$ is compact and $a \in X\backslash K$, then there exists $\{U, V\} \subseteq \tau$ such that $K \subseteq U$, $a \in V$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

Proof. For each $x \in K$ there is a $\{U_x, V_x\} \subseteq \tau$ such that $x \in U_x$, $a \in V_x$ and $\overline{U_x} \cap \overline{V_x} \in \mathcal{I}$. There exists a finite $K_0 \subseteq K$ such that $K \subseteq U = \bigcup_{x\in K_0} U_x$. If we do $V = \bigcap_{x\in K_0} V_x$ then $a \in V$ and $\overline{U} \cap \overline{V} = \bigcup_{x\in K_0} \left(\overline{U_x} \cap \overline{V}\right) \subseteq \bigcup_{x\in K_0} \left(\overline{U_x} \cap \overline{V_x}\right) \in \mathcal{I}$. □

Corollary 3.13. If $(X, \tau, \mathcal{I})$ is a $\mathcal{J}$-Urysohn space, and if K and L are disjoint compact subsets of X, then there is a $\{U, V\} \subseteq \tau$ such that $K \subseteq U$, $L \subseteq V$ and $\overline{K} \cap \overline{L} \in \mathcal{I}$.

4. About $\mathcal{J}$-normal spaces

Recall that an ideal space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{J}$-normal* [6] if (X, τ) is T_1 and, for each pair of disjoint closed sets F and G, there exists $\{U, V\} \subseteq \tau$ such that $F \subseteq U$, $G \subseteq V$ and $U \cap V \in \mathcal{I}$. If $A \subseteq X$ then A is *$\mathcal{J}$-normal* if the space $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{J}$-normal. It is clear that $T_4 \rightarrow \mathcal{J}$-normal.

Theorem 4.1. (1) If $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-normal and $A \subseteq X$ is closed, then $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{J}$-normal.
(2) If $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-normal, $F \subseteq X$ is closed, $U \in \tau$ and $F \subseteq U$, then there are $V \in \tau$ and $I \in \mathcal{I}$ such that $F \subseteq V \subseteq \overline{V} \subseteq U \cup \overline{I}$.

Proof. (1) Suppose that $F \subseteq A$ and $G \subseteq A$ are closed in A. Since F and G are closed in X, there exists $\{U, V\} \subseteq \tau$ such that $F \subseteq U$, $G \subseteq V$ and $U \cap V \in \mathcal{I}$. Then $F \subseteq A \cap U$, $G \subseteq A \cap V$ and $(A \cap U) \cap (A \cap V) = A \cap (U \cap V) \in \mathcal{I}_A$. Further, (A, τ_A) is T_1.
(2) Given that F and $X \backslash U$ are closed and disjoint sets, there are $\{V, W\} \subseteq \tau$ and $I \in \mathcal{I}$ such that $F \subseteq V$, $X \backslash U \subseteq W$ and $V \cap W = I$. Hence $V \subseteq (X \backslash W) \cup I$ and so $\overline{V} \subseteq (X \backslash W) \cup \overline{I} \subseteq U \cup \overline{I}$. □

Now we present some functional properties of $\mathcal{J}$-normal spaces.

Theorem 4.2. (1) If $f : (X, \tau) \to (Y, \beta)$ is a continuous, closed and surjective function and if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-normal, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{J}$-normal.
(2) If $f : X \to Y$ is surjective and $(Y, \beta, \mathcal{J})$ is an ideal space such that $(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{J}})$ is $\mathcal{J}$-normal, then $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-normal.

Proof. (1) Suppose that F and G are disjoint closed subsets of Y. There are $\{U, V\} \subseteq \tau$ and $I \in \mathcal{I}$ such that $f^{-1}(F) \subseteq U$, $f^{-1}(G) \subseteq V$ and $U \cap V = I$. Then $F \subseteq Y \backslash f(X \backslash U)$, $G \subseteq Y \backslash f(X \backslash V)$, $\{Y \backslash f(X \backslash U), Y \backslash f(X \backslash V)\} \subseteq \beta$ and $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)] = Y \backslash f[X \backslash I]$. Since f is surjective, $Y \backslash f[X \backslash I] \subseteq f(I)$ and so $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)] \in f(\mathcal{I})$. Further (Y, β) is T_1, because f is closed and surjective.
(2) Suppose that F and G are disjoint closed subsets of Y. Since $f^{-1}(F)$ and $f^{-1}(G)$ are disjoint closed subsets of X, there are $\{U, V\} \subseteq \beta$ and $J \in \mathcal{J}$ such that $f^{-1}(F) \subseteq f^{-1}(U)$, $f^{-1}(G) \subseteq f^{-1}(V)$ and $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(J)$. Given that f is surjective we have that $F \subseteq U$, $G \subseteq V$ and $U \cap V \in \mathcal{J}$. □

Theorem 4.3. If $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-Hausdorff and (X, τ) is T_1 and compact then $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-normal.

Proof. This is an immediate consequence of Theorem 2.1 □

We end up analyzing the sum of $\mathcal{J}$-normal spaces.

Theorem 4.4. If $\{(X_i, \tau_i, \mathcal{I}_i) : i \in \Lambda\}$ is a non-empty collection of non-empty $\mathcal{J}$-normal spaces, with $X_i \cap X_j = \varnothing$ for each $i \neq j$, then $\left(\sum_{i\in\Lambda} X_i, \sum_{i\in\Lambda} \tau_i, \sum_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{J}$-normal.

Proof. Suppose that F and G are disjoint closed subsets of $\sum_{i\in\Lambda} X_i$. Given that, for each $i \in \Lambda$, $F \cap X_i$ and $G \cap X_i$ are disjoint closed subsets of X_i, there is a $\{U_i, V_i\} \subseteq \tau_i$ such that $F \cap X_i \subseteq U_i$ and $G \cap X_i \subseteq V_i$ and $U_i \cap V_i \in \mathcal{I}_i$. Then $F = \sum_{i\in\Lambda} (F \cap X_i) \subseteq \sum_{i\in\Lambda} U_i \in \sum_{i\in\Lambda} \tau_i$ and $G = \sum_{i\in\Lambda} (G \cap X_i) \subseteq \sum_{i\in\Lambda} V_i \in \sum_{i\in\Lambda} \tau_i$, with $\left(\sum_{i\in\Lambda} U_i\right) \cap \left(\sum_{i\in\Lambda} V_i\right) = \sum_{i\in\Lambda} (U_i \cap V_i) \in \sum_{i\in\Lambda} \mathcal{I}_i$. On the other hand, $\left(\sum_{i\in\Lambda} X_i, \sum_{i\in\Lambda} \tau_i\right)$ is T_1, given that each space (X_i, τ_i) is T_1. $\square$

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