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A NOTE ABOUT TRUNCATED RHOMBICUBOCTAHEDRON AND TRUNCATED RHOMBICICOSIDODECAHEDRON SPACE

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Abstract. The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. If every points of a line segment that connects any two points of the set are in the same set, then it is convex. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. In this study, we give two new metrics to be their spheres a truncated rhombicuboctahedron and a truncated rhombicicosidodecahedron.

1. Introduction

Polyhedra have very interesting symmetries. For example, they have symmetries about a plane, a line and a point. Therefore they have attracted the attention of scientists and artists from past to present. Thus mathematicians, geometers, physicists, chemists, artists have studied and continue to study on polyhedra.

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Consequently, polyhedra take place in many studies with respect to different fields. As it is stated in [3] and [6], polyhedra have been used for explaining the world around us in philosophical and scientific way. There are only five regular convex polyhedra known as the platonic solids.
These regular polyhedra were known by the Ancient Greeks. They are generally known as the "Platonic" or "cosmic" solids because Plato mentioned them in his dialogue Timeous, where each is associated with one of the five elements - the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron with universe (or with ether, the material of the heavens). The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result.For more detailed knowledge, see [3] and [6].

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [17].

Some mathematicians have studied and improved metric space geometry. According to the mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. In [1, 2, 4, 5, 7, 8, 9, 10] the authors give some metrics which the spheres of the 3-dimensional analytical space furnished by these metrics are some of Platonic solids, Archimedian solids and Catalan solids. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. In this study, two new metrics are introduced, and showed that the spheres of the 3-dimensional analytical space furnished by these metrics are truncated rhombicuboctahedron and truncated rhombicosidodecahedron. Also some properties about these metrics are given.

2. Truncated Rhombicuboctahedron Metric and Some Properties

It has been stated in [16], there are many variations on the theme of the regular polyhedra. First one can meet the eleven which can be made by cutting off (truncating) the corners, and in some cases the edges, of the regular polyhedra so that all the faces of the faceted polyhedra obtained in this way are regular polygons. These polyhedra were first discovered by Archimedes (287-212 B.C.E.) and so they are often called Archimedean solids. Notice that vertices of the Archimedean polyhedra are all alike, but their faces, which are regular polygons, are of two or more different kinds. For this reason they are often called semiregular. Archimedes also showed that in addition to the eleven obtained by truncation, there are two more semiregular polyhedra: the snub cube and the snub dodecahedron.

Two of Archimedian solids, the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) apparently seem to be derived from truncating the two preceding ones. However, it is apparent from the above discussion on the percentage of truncation that one cannot truncate a solid with unequally shaped faces and end up with regular polygons as faces. Therefore, these two solids need be constructed with another technique. Actually, the can be built from the original platonic solids by a process called expansion. It consists on separating apart the faces of the original polyhedron with spherical symmetry, up to a point where they can be linked through new faces which are regular polygons. The name of the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and of the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) again seem to indicate that they can be derived from truncating the Cuboctahedron and the Icosidodecahedron. But, as reasoned above, this is not possible [18].

One of the solids which is obtained by truncating from another solid is the truncated rhombicuboctahedron. It has 6 regular octagonal faces, 12 bi-mirror-symmetric octagonal faces, 8 regular hexagonal faces, 24 square faces, 96 vertices and 144 edges. The truncated rhombicuboctahedron can be obtained by truncating operation from rhombicuboctahedron. Figure 1 shows the rhombicuboctahedron, the truncated rhombicuboctahedron, and the transparent truncated rhombicuboctahedron, respectively.

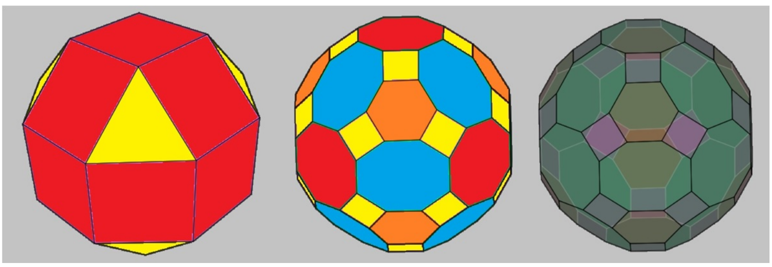

Figure 1: Rhombicuboctahedron, Truncated Rhombicuboctahedron

Before we give a description of the truncated rhombicuboctahedron distance function, we must agree on some impressions. Therefore U, V, W denote the maximum, the middle and the minimum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. The metric that unit sphere is the truncated rhombicuboctahedron is described as following:

Definition 1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{TRC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated rhombicuboctahedron distance between P_1 and P_2 is defined by*

$$d_{TRC}(P_1, P_2) = \max\{U, k_1(U+V), k_2(2U+V+W), k_3(U+V+W)\}$$

where $k_1 = \dfrac{106 + 57\sqrt{2} + 44\sqrt{3} - 28\sqrt{6}}{46}$, $k_2 = \dfrac{2000 - 213\sqrt{2} - 56\sqrt{3} + 212\sqrt{6}}{5422}$ *and* $k_3 = \dfrac{73 - 3\sqrt{2} - 10\sqrt{3} + 12\sqrt{6}}{141}$.

According to truncated rhombicuboctahedron distance, there are four different paths from P_1 to P_2. These paths are

i) a line segment which is parallel to a coordinate axis.

ii) union of two line segments each of which is parallel to a coordinate axis.

iii) union of three line segments which one is parallel to a coordinate axis and other line segments makes $\arctan\left(\frac{3}{4}\right)$ angle with another coordinate axes.

iv) union of three line segments each of which is parallel to a coordinate axis.

Thus truncated rhombicuboctahedron distance between P_1 and P_2 is for (i) Euclidean length of mentioned the line segments, for (ii) k_1 times the sum of Euclidean lengths of mentioned two line segments, for (iii) k_2 times the sum of Euclidean lengths of mentioned three line segments, for (iv) k_3 times the sum of Euclidean lengths of mentioned three line segments.

In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 2 illustrates some of truncated rhombicuboctahedron way from P_1 to P_2

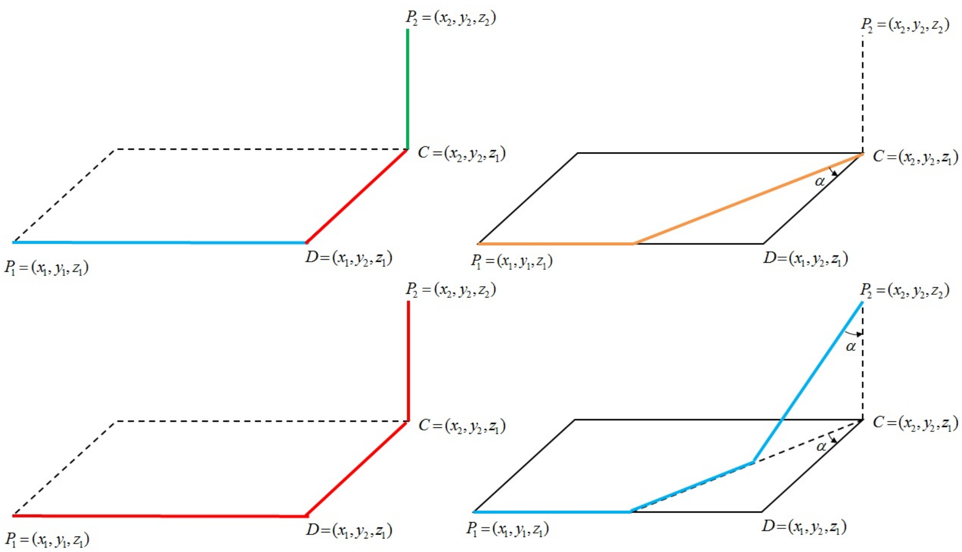

Figure 2: Some TRC way from P_1 to P_2

In [14], the authours introduce a metric and show that sphere of 3-dimensional analytical space furnished by this metric is the the rhombicuboctahedron. These metrics for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ are defined as follows:

$$d_{RC}(P_1, P_2) = \max\{U, \frac{7\sqrt{2}}{2}(U+V), \frac{2\sqrt{2}+1}{7}(U+V+W)\}.$$

Lemma 2. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12}, V_{12}, W_{12} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Then*

$$\begin{aligned}
d_{TRC}(P_1, P_2) &\geq U_{12} \\
d_{TRC}(P_1, P_2) &\geq k_1 (U_{12} + V_{12}) \\
d_{TRC}(P_1, P_2) &\geq k_2 (2U_{12} + V_{12} + W_{12}) \\
d_{TRC}(P_1, P_2) &\geq k_3 (U_{12} + V_{12} + W_{12})
\end{aligned}$$

Proof. Proof is trivial by the definition of maximum function. ∎

Theorem 3. *The distance function d_{TRC} is a metric. Also according to d_{TRC}, the unit sphere is a truncated rhombicuboctahedron in $\mathbb{R}^3$.*

Proof. Let $d_{TRC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ be the truncated rhombicuboctahedron distance function and $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ and $P_3 = (x_3, y_3, z_3)$ are distinct three points in $\mathbb{R}^3$. U_{12}, V_{12}, W_{12} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. To show that d_{TRC} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d_{TRC}(P_1, P_2) \geq 0$ and $d_{TRC}(P_1, P_2) = 0$ iff $P_1 = P_2$

M2) $d_{TRC}(P_1, P_2) = d_{TRC}(P_2, P_1)$

M3) $d_{TRC}(P_1, P_3) \leq d_{TRC}(P_1, P_2) + d_{TRC}(P_2, P_3)$.

Since absolute values is always nonnegative value $d_{TRC}(P_1, P_2) \geq 0$. If $d_{TRC}(P_1, P_2) = 0$ then
$\max\{U, k_1(U+V), k_2(2U+V+W), k_3(U+V+W)\} = 0$,
where U, V, W are the maximum, the middle and the minimum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$, respectively. Therefore, $U = 0$, $k_1(U+V) = 0$, $k_2(2U+V+W) = 0$ and $k_3(U+V+W) = 0$. Hence, it is clearly obtained by $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$. That is, $P_1 = P_2$. Thus it is obtained that $d_{TRC}(P_1, P_2) = 0$ iff $P_1 = P_2$.

Since $|x_1 - x_2| = |x_2 - x_1|$, $|y_1 - y_2| = |y_2 - y_1|$ and $|z_1 - z_2| = |z_2 - z_1|$, obviously $d_{TRC}(P_1, P_2) = d_{TRC}(P_2, P_1)$. That is, d_{TRC} is symmetric.

U_{13}, V_{13}, W_{13} and U_{23}, V_{23}, W_{23} denote the maximum, the middle and the minimum of quantities of $\{|x_1 - x_3|, |y_1 - y_3|, |z_1 - z_3|\}$ and $\{|x_2 - x_3|, |y_2 - y_3|, |z_2 - z_3|\}$, respectively.

$$\begin{aligned}
&d_{TTD}(P_1, P_3) \\
&= \max\{U_{13}, k_1(U_{13} + V_{13}), k_2(2U_{13} + V_{13} + W_{13}), k_3(U_{13} + V_{13} + W_{13})\} \\
&\leq \max\left\{\begin{array}{c} U_{12} + U_{23}, k_1(U_{12} + U_{23} + V_{12} + V_{23}), \\ k_2(2(U_{12} + U_{23}) + V_{12} + V_{23} + W_{12} + W_{23}), \\ k_3(U_{12} + U_{23} + V_{12} + V_{23} + W_{12} + W_{23}) \end{array}\right\} \\
&= I.
\end{aligned}$$

Therefore one can easily find that $I \leq d_{TRC}(P_1, P_2) + d_{TRC}(P_2, P_3)$ from Lemma 1. So $d_{TRC}(P_1, P_3) \leq d_{TRC}(P_1, P_2) + d_{TRC}(P_2, P_3)$. Consequently, the truncated rhombicuboctahedron distance is a metric in 3-dimensional analytical space.
Finally, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that truncated rhombicuboctahedron distance is 1 from $O = (0, 0, 0)$ is

$$\begin{aligned}
&S_{TRC} = \\
&\{(x, y, z) : \max\{U, k_1(U + V), k_2(2U + V + W), k_3(U + V + W)\} = 1\},
\end{aligned}$$

where U, V, W are the maximum, the middle and the minimum of quantities $\{|x|, |y|, |z|\}$, respectively. Thus the graph of S_{TRC} is as in the figure 3:

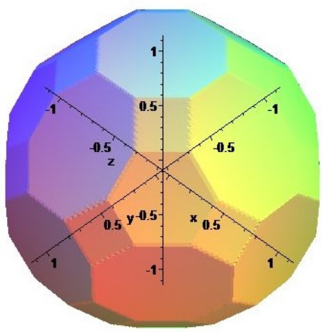

Figure 3 The unit sphere in terms of d_{TRC}: Truncated rhombicuboctahedron

∎

Corollary 4. *The equation of the truncated rhombicuboctahedron with center* (x_0, y_0, z_0) *and radius* r *is*

$$\max\{U_0, k_1(U_0 + V_0), k_2(2U_0 + V_0 + W_0), k_3(U_0 + V_0 + W_0)\} = r,$$

which is a polyhedron which has 50 faces and 96 vertices, where U_0, V_0, W_0 *are the maximum, the middle and the minimum of quantities* $\{|x-x_0|, |y-y_0|, |z-z_0|\}$*, respectively. Coordinates of the vertices are translation to* (x_0, y_0, z_0) *all permutations of the three axis components and all possible +/- sign changes of each axis component of* (C_1r, C_4r, r) *and* (C_0r, C_2r, C_3r)*, where* $C_0 = \frac{199-126\sqrt{2}+118\sqrt{3}-68\sqrt{6}}{167}$, $C_1 = \frac{51+81\sqrt{2}-52\sqrt{3}-4\sqrt{6}}{167}$, $C_2 = \frac{139-15\sqrt{2}+22\sqrt{3}-24\sqrt{6}}{167}$, $C_3 = \frac{79+96\sqrt{2}-74\sqrt{3}+20\sqrt{6}}{167}$ *and* $C_4 = \frac{111-30\sqrt{2}+44\sqrt{3}-48\sqrt{6}}{167}$.

Lemma 5. *Let* l *be the line through the points* $P_1=(x_1, y_1, z_1)$ *and* $P_2=(x_2, y_2, z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p, q, r)*, then*

$$d_{TRC}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\max\{U_d, k_1(U_d+V_d), k_2(2U_d+V_d+W_d), k_3(U_d+V_d+W_d)\}}{\sqrt{p^2+q^2+r^2}},$$

U_d, V_d, W_d *are the maximum, the middle and the minimum of quantities* $\{|p|, |q|, |r|\}$*, respectively.*

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{TRC}(P_1, P_2) = |\lambda| \left(\max\{U_d, k_1(U_d+V_d), k_2(2U_d+V_d+W_d), k_3(U_d+V_d+W_d)\}\right),$$

where U_d, V_d, W_d are the maximum, the middle and the minimum of quantities $\{|p|, |q|, |r|\}$, respectively, and $d_E(A, B) = |\lambda| \sqrt{p^2+q^2+r^2}$ which implies the required result. ∎

The above lemma says that d_{TRC}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 6. *If* P_1, P_2 *and* X *are any three collinear points in* $\mathbb{R}^3$, *then* $d_E(P_1, X) = d_E(P_2, X)$ *if and only if* $d_{TRC}(P_1, X) = d_{TRC}(P_2, X)$.

Corollary 7. *If* P_1, P_2 *and* X *are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TRC}(X, P_1) \;/\; d_{TRC}(X, P_2) = d_E(X, P_1) \;/\; d_E(X, P_2) \;.$$

That is, the ratios of the Euclidean and $d_{TRC}-$*distances along a line are the same.*

3. Truncated Rhombicosidodecahedron Metric and Some Properties

The truncated rhombicosidodecahedron can be obtained by truncating operation from rhombicosidodecahedron. The truncated rhombicosidodecahedron has 12 regular decagonal faces, 30 bi-mirror-symmetric octagonal faces, 20 regular hexagonal faces, 60 square faces, 240 vertices and 360 edges. Figure 4 show the rhombicosidodecahedron, the truncated rhombicosidodecahedron, and the transparent truncated rhombicosidodecahedron.

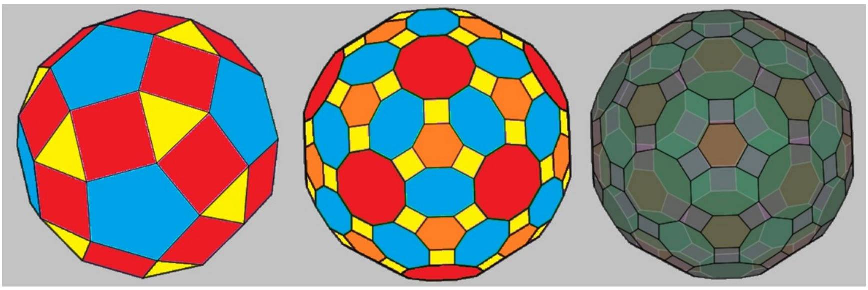

Figure 4: Rhombicosidodecahedron, Truncated rhombicosidodecahedron

Before we give a description of the truncated truncated dodecahedron distance function, we must agree on some impressions. Therefore U denote the maximum of quantities $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. Also, $X - Y - Z - X$ orientation and $Z - Y - X - Z$ orientation are called positive (+) direction and negative (−) directions, respectively. Accordingly, U^+ and U^- will display the next term in the respective direction according to U. For example, if $U = |y_1 - y_2|$, then $U^+ = |z_1 - z_2|$ and $U^- = |x_1 - x_2|$. The metric that unit sphere is the truncated rhombicosidodecahedron is described as following:

Definition 8. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{TRI} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated rhombicosidodecahedron distance between P_1 and P_2 is defined by*

$$d_{TRI}(P_1, P_2) = \max \left\{ \begin{array}{r} U, k_1 U + k_2 U^+, k_3 U + k_4 U^+ + k_5 U^-, \\ k_6 U + k_7 U^-, k_8 U + k_9 U^+ + k_{10} U^-, \\ k_{11} U + k_{12} U^-, k_{13} \left(U + U^+ + U^-\right), \\ k_{14} U + k_{15} U^+ + k_{16} U^- \end{array} \right\}$$

where $k_1 = \dfrac{355+21\sqrt{5}+\left(42-22\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{410}$,
$k_2 = \dfrac{-125+167\sqrt{5}+\left(32\sqrt{5}-76\right)\sqrt{25+10\sqrt{5}}}{410}$, $k_3 = \dfrac{\sqrt{5}+1}{4}$,
$k_4 = \dfrac{\sqrt{5}-1}{4}$, $k_5 = \frac{1}{2}$, $k_6 = \dfrac{277+81\sqrt{5}}{474} + \dfrac{\left(53\sqrt{5}-125\right)\sqrt{25+10\sqrt{5}}}{1185}$,
$k_7 = \dfrac{213-17\sqrt{5}}{474} + \dfrac{\left(142\sqrt{5}-320\right)\sqrt{25+10\sqrt{5}}}{1185}$,
$k_8 = \dfrac{1455-547\sqrt{5}}{244} + \dfrac{\left(391-175\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{122}$,
$k_9 = \dfrac{355\sqrt{5}-774}{122} + \dfrac{\left(512\sqrt{5}-1145\right)\sqrt{25+10\sqrt{5}}}{305}$,
$k_{10} = \dfrac{1001-419\sqrt{5}}{244} + \dfrac{\left(1415-633\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{610}$,
$k_{11} = \dfrac{291\sqrt{5}-547}{122}$, $k_{12} = 2k_9$, $k_{13} = \dfrac{32+49\sqrt{5}}{237} + \dfrac{\left(195-89\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{1185}$,
$k_{14} = \dfrac{227-64\sqrt{5}}{122} + \dfrac{\left(270-121\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{305}$,
$k_{15} = \dfrac{1001\sqrt{5}-2095}{244} + \dfrac{\left(283\sqrt{5}-633\right)\sqrt{25+10\sqrt{5}}}{122}$ *and*
$k_{16} = \dfrac{291\sqrt{5}-547}{244} + \dfrac{\left(391\sqrt{5}-875\right)\sqrt{25+10\sqrt{5}}}{610}$.

According to truncated rhombicosidodecahedron distance, there are eight different paths from P_1 to P_2. These paths are

i) a line segment which is parallel to a coordinate axis.

ii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{1}{2}\right)$ angle with another coordinate axis.

iii) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{\sqrt{5}}{2})$ and $\arctan\left(\frac{1}{2}\right)$ angles with one of the other coordinate axes .

iv) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{\sqrt{5}}{2}\right)$ angle with another coordinate axis.

v) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{15+6\sqrt{5}}{10})$ and $\arctan\left(\frac{10+3\sqrt{5}}{10}\right)$ angles with one of the other coordinate axes .

vi) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{1}{2}\right)$ angle with another coordinate axis.

vii) union of three line segments each of which is parallel to a coordinate axis.

$viii$) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{15-6\sqrt{5}}{10})$ and $\arctan\left(\frac{1}{2}\right)$ angles with one of the other coordinate axes .

Thus truncated rhombicosidodecahedron distance between P_1 and P_2 is for (i) Euclidean length of mentioned the line segments, for (ii) k_1 times the sum of Euclidean lengths of mentioned two line segments, for (iii) k_3 times the sum of Euclidean lengths of mentioned three line segments, for (iv) k_6 times the sum of Euclidean lengths of mentioned two line segments, for (v) k_8 times the sum of Euclidean lengths of mentioned three line segments, for (vi) k_{11} times the sum of Euclidean lengths of mentioned two line segments, for (vii) k_{13} times the sum of Euclidean lengths of mentioned three line segments, and for $(viii)$ k_{14} times the sum of Euclidean lengths of mentioned three line segments. In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 5 shows that some of the $TRI-$path between P_1 and P_2.

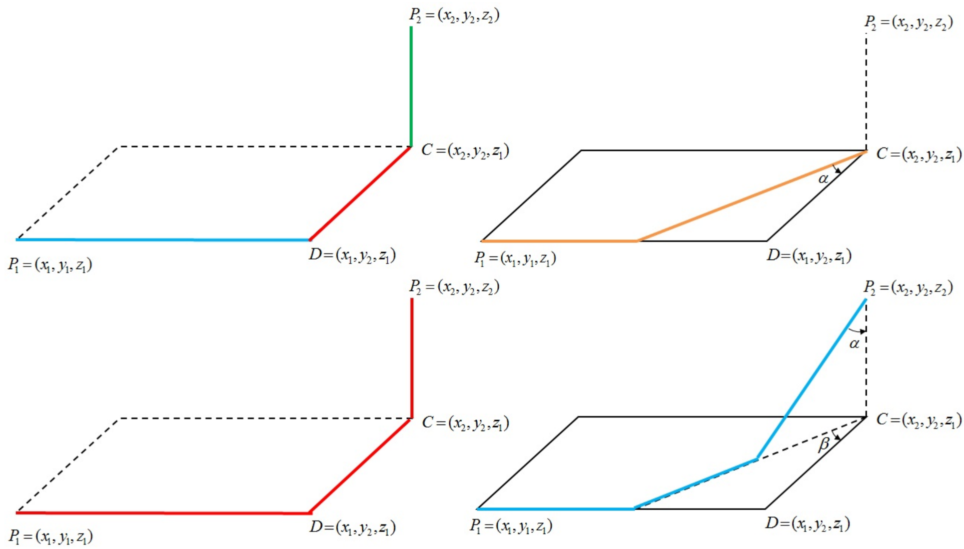

Figure 5: TRI way from P_1 to P_2

In [14], the authours introduce a metric and show that spheres of 3-dimensional analytical space furnished by these metric are the rhomicosidodecahedron. This metric for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is defined as follows:

$$d_{RI}(P_1, P_2) = \max \left\{ \begin{array}{c} U, \frac{9+5\sqrt{5}}{22}U + \frac{1+3\sqrt{5}}{22}U^-, \frac{3+\sqrt{5}}{6}U + \frac{1+\sqrt{5}}{6}U^+, \\ \frac{\sqrt{5}+1}{4}U + \frac{1}{2}U^- + \frac{\sqrt{5}-1}{4}U^+, \frac{\sqrt{5}+4}{11}\left(U + U^+ + U^-\right) \end{array} \right\}.$$

Lemma 9. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U denote the maximum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Then*

$$\begin{aligned}
&d_{TRI}(P_1, P_2) \geq U, \\
&d_{TRI}(P_1, P_2) \geq k_1 U + k_2 U^+, \\
&d_{TRI}(P_1, P_2) \geq k_3 U + k_4 U^+ + k_5 U^-, \\
&d_{TRI}(P_1, P_2) \geq k_6 U + k_7 U^-, \\
&d_{TRI}(P_1, P_2) \geq k_8 U + k_9 U^+ + k_{10} U^-, \\
&d_{TRI}(P_1, P_2) \geq k_{11} U + k_{12} U^-, \\
&d_{TRI}(P_1, P_2) \geq k_{13}\left(U + U^+ + U^-\right), \\
&d_{TRI}(P_1, P_2) \geq k_{14} U + k_{15} U^+ + k_{16} U^-.
\end{aligned}$$

Proof. Proof is trivial by the definition of maximum function. ∎

Theorem 10. *The distance function d_{TRI} is a metric. Also according to d_{TRI}, unit sphere is a truncated rhomicosidodecahedron in $\mathbb{R}^3$.*

Proof. One can easily show that the truncated rhomicosidodecahedron distance function satisfies the metric axioms by similar way in Theorem 1.

Consequently, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that truncated truncated icosahedron distance is 1 from $O = (0, 0, 0)$ is

$$S_{TRI} = \left\{ (x, y, z) : \max \left\{ \begin{array}{r} U, k_1 U + k_2 U^+, k_3 U + k_4 U^+ + k_5 U^-, \\ k_6 U + k_7 U^-, k_8 U + k_9 U^+ + k_{10} U^-, \\ k_{11} U + k_{12} U^-, k_{13} \left(U + U^+ + U^- \right), \\ k_{14} U + k_{15} U^+ + k_{16} U^- \end{array} \right\} = 1 \right\},$$

where U are the maximum of quantities $\{|x|, |y|, |z|\}$. Thus the graph of S_{TRI} is as in the figure 6:

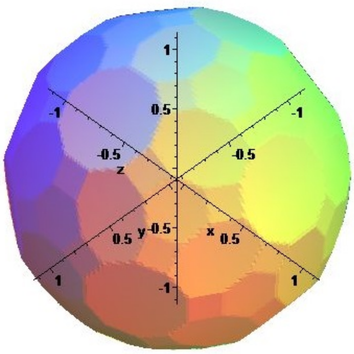

Figure 6 The unit sphere in terms of d_{TRI}: Truncated Rhomicosidodecahedron

∎

Corollary 11. *The equation of the truncated rhomicosidodecahedron with center (x_0, y_0, z_0) and radius r is*

$$\max \left\{ \begin{array}{r} U_0, k_1 U_0 + k_2 U_0^+, k_3 U_0 + k_4 U_0^+ + k_5 U_0^-, \\ k_6 U_0 + k_7 U_0^-, k_8 U_0 + k_9 U_0^+ + k_{10} U_0^-, \\ k_{11} U_0 + k_{12} U_0^-, k_{13} \left(U_0 + U_0^+ + U_0^- \right), \\ k_{14} U_0 + k_{15} U_0^+ + k_{16} U_0^- \end{array} \right\} = r,$$

which is a polyhedron which has 122 faces and 240 vertices, where U_0 are the maximum of quantities $\{|x - x_0|, |y - y_0|, |z - z_0|\}$. Coordinates of the vertices are translation to (x_0, y_0, z_0) all circular shift of the three axis components and all possible +/- sign changes of each axis component of $(C_2 r, C_{20} r, r)$, $(C_{20} r, C_0 r, r)$, $(C_{21} r, C_4 r, C_{18} r)$, $(C_5 r, C_1 r, C_{18} r)$, $(C_{20} r, C_8 r, C_{17} r)$, $(C_9 r, C_3 r, C_{16} r)$, $(C_{20} r, C_{12} r, C_{15} r)$, $(C_9 r, C_7 r, C_{13} r)$, $(C_{11} r, C_1 r, C_{14} r)$ and $(C_6 r, C_{10} r, C_{11} r)$, where $C_0 = \frac{740\sqrt{5} - 1455 + \left(531\sqrt{5} - 1155\right)\sqrt{25 + 10\sqrt{5}}}{2105}$,

$C_1 = \frac{685\sqrt{5}+175+\left(62\sqrt{5}-230\right)\sqrt{25+10\sqrt{5}}}{4210}$, $C_2 = \frac{48\sqrt{5}-83+\left(73-27\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{421}$,
$C_3 = \frac{1625-255\sqrt{5}+\left(270-146\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_4 = \frac{1225\sqrt{5}-1285+\left(916\sqrt{5}-2040\right)\sqrt{25+10\sqrt{5}}}{4210}$,
$C_5 = \frac{710\sqrt{5}-1140+\left(115-31\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_6 = \frac{795+225\sqrt{5}+\left(1000-216\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{4210}$, $C_7 = \frac{387-97\sqrt{5}+\left(177-77\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{421}$,
$C_8 = \frac{34+97\sqrt{5}+\left(77\sqrt{5}-177\right)\sqrt{25+10\sqrt{5}}}{421}$,
$C_9 = \frac{1165\sqrt{5}-655+\left(500-208\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{4210}$, $C_{10} = \frac{5495-1225\sqrt{5}+\left(2040-916\sqrt{5}\right)\sqrt{3+2\sqrt{5}}}{4210}$,
$C_{11} = \frac{485+455\sqrt{5}+\left(385-177\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_{12} = \frac{480+255\sqrt{5}+\left(146\sqrt{5}-270\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_{13} = \frac{4045-285\sqrt{5}+\left(1540-708\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{4210}$,
$C_{14} = \frac{4035-685\sqrt{5}+\left(230-62\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{4210}$, $C_{15} = \frac{925\sqrt{5}-240+\left(135-73\sqrt{5}\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_{16} = \frac{2585+255\sqrt{5}+\left(146\sqrt{5}-270\right)\sqrt{25+10\sqrt{5}}}{4210}$,
$C_{17} = \frac{1195\sqrt{5}-970+\left(354\sqrt{5}-770\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_{18} = \frac{1135+1195\sqrt{5}+\left(354\sqrt{5}-770\right)\sqrt{25+10\sqrt{5}}}{2105}$, $C_{19} = 1$, $C_{20} = \frac{470\sqrt{5}-725+\left(104\sqrt{5}-250\right)\sqrt{25+10\sqrt{5}}}{2105}$ *and*
$C_{21} = \frac{940\sqrt{5}-1450+\left(208\sqrt{5}-500\right)\sqrt{25+10\sqrt{5}}}{2105}$.

Lemma 12. *Let l be the line through the points $P_1=(x_1, y_1, z_1)$ and $P_2=(x_2, y_2, z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p, q, r), then*

$$d_{TRI}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\left\{\begin{array}{r} U_d, k_1U_d + k_2U_d^+, k_3U_d + k_4U_d^+ + k_5U_d^-, \\ k_6U_d + k_7U_d^-, k_8U_d + k_9U_d^+ + k_{10}U_d^-, \\ k_{11}U_d + k_{12}U_d^-, k_{13}\left(U_d + U_d^+ + U_d^-\right), \\ k_{14}U_d + k_{15}U_d^+ + k_{16}U_d^- \end{array}\right\}}{\sqrt{p^2+q^2+r^2}},$$

U_d are the maximum of quantities $\{|p|, |q|, |r|\}$.

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{TRI}(P_1, P_2) = |\lambda| \left(\max \left\{ \begin{array}{r} U_d, k_1 U_d + k_2 U_d^+, k_3 U_d + k_4 U_d^+ + k_5 U_d^-, \\ k_6 U_d + k_7 U_d^-, k_8 U_d + k_9 U_d^+ + k_{10} U_d^-, \\ k_{11} U_d + k_{12} U_d^-, k_{13} \left(U_d + U_d^+ + U_d^- \right), \\ k_{14} U_d + k_{15} U_d^+ + k_{16} U_d^- \end{array} \right\} \right),$$

where U_d are the maximum of quantities $\{|p|, |q|, |r|\}$, and $d_E(A, B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. ∎

The above lemma says that d_{TRI}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 13. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{TRI}(P_1, X) = d_{TRI}(P_2, X)$.*

Corollary 14. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TTI}(X, P_1) \; / \; d_{TTI}(X, P_2) = d_E(X, P_1) \; / \; d_E(X, P_2) \; .$$

That is, the ratios of the Euclidean and $d_{TRI}-$distances along a line are the same.

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