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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 29 (2019), No. 1, 129-136

GENERAL RELATED FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS IN TWO METRIC SPACES

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Abstract. In this paper a general fixed point theorem for two pairs of mappings in two metric spaces, which generalize the results from [2] and [7], is proved.

1. Introduction

The following related fixed point theorem was proved by Fisher in [1].

Theorem 1.1 ([1]). *Let (X, d_1) and (Y, d_2) be complete metric spaces. If $T : X \to Y$ and $S : Y \to X$ are two mappings such that for all $x \in X$ and $y \in Y$*

$$d_1(Tx, TSy) \le c \max\{d_1(x, Sy), d_2(y, Tx), d_1(y, TSy)\}, \tag{1.1}$$

$$d_2(Sy, STx) \le c \max\{d_2(y, Tx), d_1(x, Sy), d_2(y, STx)\}, \tag{1.2}$$

where $0 \le c \le 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

The first present author proved the following theorem in [5].

Keywords and phrases: fixed point, related fixed point, metric space, implicit relation.
(2010) Mathematics Subject Classification: 54H25, 47H10.

Theorem 1.2. *Let (X, d_1) and (Y, d_2) be complete metric spaces. If $T : X \to Y$ and $S : Y \to X$ satisfy the inequalities*

$$d_1^2(Sy, STx) \le c_1 \max \left\{ \begin{array}{c} d_2(y, Tx) \cdot d_1(x, Sy), \\ d_2(y, Tx) \cdot d_1(x, STx), \\ d_1(x, Sy) \cdot d_1(x, STx) \end{array} \right\}, \tag{1.3}$$

$$d_2^2(Tx, TSy) \le c_2 \max \left\{ \begin{array}{c} d_1(x, Sy) \cdot d_2(y, Tx), \\ d_1(x, Sy) \cdot d_2(y, TSy), \\ d_2(y, Tx) \cdot d_2(y, TSy) \end{array} \right\}, \tag{1.4}$$

for all $x, y \in X$ and $0 \le c_1 c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Quite recently, the following theorem was proved in [7].

Theorem 1.3. *Let (X, d_1) and (Y, d_2) be complete metric spaces. Let $A, B : X \to Y$ and $C, D : Y \to X$ satisfying the inequalities*

$$d_1^2(Cy, DBx) \le c_1 \max \left\{ \begin{array}{c} d_2(y, Bx) \cdot d_1(x, Cy), \\ d_2(y, Bx) \cdot d_1(x, DBx), \\ d_1(x, Cy) \cdot d_1(x, DBx) \end{array} \right\}, \tag{1.5}$$

$$d_2^2(Bx, ADy) \le c_2 \max \left\{ \begin{array}{c} d_1(x, Dy) \cdot d_2(y, Bx), \\ d_1(x, Dy) \cdot d_2(y, ADy), \\ d_2(y, Bx) \cdot d_2(y, ADy) \end{array} \right\}, \tag{1.6}$$

for all $x \in X$ and $y \in Y$, where $0 \le c_1 c_2 < 1$. If one of the mappings A, B, C, D is continuous, then CA and DB have a unique fixed point z in X and BC and AD have a unique fixed point w in Y. Further, $Az = Bz = w$ and $Cw = Dw = z$.

Several clasical fixed point theorems and common fixed point theorems have been unified by an implicit relation in [3] and [4].

Several related fixed point theorems for pairs of mappings satisfying implicit relations are published in [2], [5] and [6].

The purpose of this paper is to prove a general related fixed point theorem for two pairs of mappings in two metric spaces, which generalize Theorems 1.2 and 1.3 using implicit relations.

2. Implicit relations

Let $\mathcal{F}_4$ be the family of lower semi - continuous functions $F : \mathbb{R}_+^4 \to \mathbb{R}$ satisfying the following conditions: for all $u, v \geq 0$, there exists $h \in (0,1)$ such that

$(F_1) : F(u, v, 0, u) \leq 0$ implies $u \leq hv$,

$(F_2) :\ F(u, v, u, 0) \leq 0$ implies $u \leq hv$.

Example 2.1. *$F(t_1, ..., t_4) = t_1 - k \max\{t_2t_3, t_2t_4, t_3t_4\}$, where $k \in [0,1)$.*

Let $u, v \geq 0$ and $F(u, v, 0, u) = u^2 - kuv \leq 0$. If $u > v$, then $u^2 (1 - k) \leq 0$, a contradiction. Hence $u < v$ which implies $u \leq hv$, where $0 \leq h = k < 1$.

Similarly, $F(u, v, u, 0) \leq 0$ implies $u \leq hv$.

For the following examples, the proofs are similar to the proof of Example 2.1.

Example 2.2. *$F(t_1, ..., t_4) = t_1 - k \max\{t_2, t_3, t_4\}$, where $k \in [0,1)$.*

Example 2.3. *$F(t_1, ..., t_4) = t_1 - k \max\left\{t_2, \dfrac{t_3 + t_4}{2}\right\}$, where $k \in [0,1)$.*

Example 2.4. *$F(t_1, ..., t_4) = t_1 - \max\{t_3, t_4\} - c \max\{t_2, t_4\}$, where $c \in (0,1)$.*

Example 2.5. *$F(t_1, ..., t_4) = t_1^2 - (at_1t_2 + bt_1t_3 + ct_4^2)$, where $a, b, c \geq 0$ and $a + b + c < 1$.*

Example 2.6. *$F(t_1, ..., t_4) = t_1^3 - (at_1^2t_2 + bt_1t_3t_4 + ct_2t_3t_4)$, where $a, b, c \geq 0$ and $a + b + c < 1$.*

3. Main results

Theorem 3.1. *Let (X, d_1) and (Y, d_2) be complete metric spaces. Let $A, B : X \to Y$ and $C, D : Y \to X$ such that for all $x \in X$ and $y \in Y$*

$$G\left(d_1\left(Cy, DBx\right), d_2\left(y, Bx\right), d_1\left(x, Cy\right), d_1\left(x, DBx\right)\right) \leq 0, \tag{3.1}$$

$$H\left(d_2\left(Bx, ADy\right), d_1\left(x, Dy\right), d_2\left(y, Bx\right), d_2\left(y, ADy\right)\right) \leq 0, \tag{3.2}$$

for some $G, H \in \mathcal{F}_4$. If one of the mappings A, B, C, D is continuous, then CA and DB have a unique common fixed point $z \in X$ and BC and AD have a unique common fixed point $w \in Y$. Further, $Az = Bz = w$ and $Cw = Dw = z$.

Proof. Let x_0 be an arbitrary point of X. Let

$$Ax_0 = y_1, Cy_1 = x_1, Bx_1 = y_2, Dy_2 = x_2, Ax_2 = y_3, ...$$

and, in general, let

$$Cy_{n-1} = x_{n-1}, Bx_{n-1} = y_n, Dy_n = x_n, Ax_n = y_{n+1} \text{ for } n = 2, 3, ...$$

Using inequality (3.1) for $x = x_n$ and $y = y_n$ we obtain

$$G\left(d_1\left(Cy_n, DBx_n\right), d_2\left(y_n, Bx_n\right), d_1\left(x_n, Cy_n\right), d_1\left(x_n, DBx_n\right)\right) \leq 0,$$

$$G\left(d_1\left(x_n, x_{n+1}\right), d_2\left(y_n, y_{n+1}\right), 0, d_1\left(x_n, x_{n+1}\right)\right) \leq 0.$$

By (F_1) we obtain

$$d_1\left(x_n, x_{n+1}\right) \leq h d_2\left(y_n, y_{n+1}\right).$$

By (3.2) for $x = x_{n-1}$ and $y = y_n$ we obtain

$$H\left(d_2\left(Bx_{n-1}, ADy_n\right), d_1\left(x_{n-1}, Dy_n\right), d_2\left(y_n, Bx_{n-1}\right), d_2\left(y_n, ADy_n\right)\right) \leq 0,$$

$$H\left(d_2\left(y_n, y_{n+1}\right), d_1\left(x_{n-1}, x_n\right), 0, d_2\left(y_n, y_{n+1}\right)\right) \leq 0.$$

By (F_1) we obtain

$$d_2\left(y_n, y_{n+1}\right) \leq h d_1\left(x_{n-1}, x_n\right),$$

which implies

$$d_1\left(x_n, x_{n+1}\right) \leq h^2 d_1\left(x_{n-1}, x_n\right) \leq ... \leq h^{2n} d_1\left(x_0, x_1\right),$$

$$d_2\left(y_n, y_{n+1}\right) \leq h^2 d_2\left(y_{n-1}, y_n\right) \leq ... \leq h^{2n} d_2\left(y_1, y_2\right).$$

By a routine calculation we obtain that $\{x_n\}$ is a Cauchy sequence in X and $\{y_n\}$ is a Cauchy sequence in Y.

Since X and Y are complete, then $\{x_n\}$ and $\{y_n\}$ are convergent. Then, there exist $z \in X$ and $w \in Y$ such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} y_n = w$.

If A is continuous, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Az = \lim_{n\to\infty} y_{n+1} = w.$$

Hence $Az = w$.

By (3.1) for $x = x_n$ and $y = w$ we obtain

$$G\left(d_1\left(Cw, DBx_n\right), d_2\left(w, Bx_n\right), d_1\left(x_n, Cw\right), d_1\left(x_n, DBx_n\right)\right) \leq 0,$$

$$G\left(d_1\left(Cw, x_{n+1}\right), d_2\left(w, y_{n+1}\right), d_1\left(x_n, Cw\right), d_1\left(x_n, x_{n+1}\right)\right) \leq 0.$$

Letting n tend to infinity we obtain

$$G\left(d_1\left(Cw, z\right), 0, d_1\left(z, Cw\right), 0\right) \leq 0,$$

which implies by (F_2) for $v = 0$ that $d_1\left(z, Cw\right) = 0$. Hence

$$z = Cw. \tag{3.3}$$

By (3.2) for $x = z$ and $y = y_n$ we obtain

$$H\left(d_2\left(Bz, ADy_n\right), d_1\left(z, Dy_n\right), d_2\left(y_n, Bz\right), d_2\left(y_n, ADy_n\right)\right) \leq 0,$$

$$H\left(d_2\left(Bz, y_n\right), d_1\left(z, x_n\right), d_2\left(y_n, Bz\right), d_2\left(y_n, y_{n+1}\right)\right) \leq 0.$$

Letting n tend to infinity we obtain

$$H\left(d_2\left(Bz, w\right), 0, d_2\left(w, Bz\right), 0\right) \leq 0,$$

which implies by (F_2) for $v = 0$ that $d_2\left(Bz, w\right) = 0$. Hence

$$w = Bz. \tag{3.4}$$

By (3.3) and (3.4) we obtain

$$Cw = CBz = z \text{ and } z \text{ is a fixed point of } CB,$$

$$Bz = BCw = w \text{ and } w \text{ is a fixed point of } BC.$$

By (3.1) for $x = z$ and $y = w$ we obtain

$$G\left(d_1\left(Cw, DBz\right), d_2\left(w, Bz\right), d_1\left(z, Cw\right), d_1\left(z, DBz\right)\right) \leq 0,$$

$$G\left(d_1\left(z, DBz\right), 0, 0, d_1\left(z, DBz\right)\right) \leq 0,$$

which implies by (F_1) for $v = 0$ that $d_1\left(z, DBz\right) = 0$, which implies $z = DBz$ and z is a fixed point of DB.

By (3.2) for $x = x_{n-1}$ and $y = w$ we obtain

$$H\left(d_2\left(Bx_{n-1}, ADw\right), d_1\left(x_{n-1}, Dw\right), d_2\left(w, Bx_{n-1}\right), d_2\left(w, ADw\right)\right) \leq 0.$$

Letting n tend to infinity we obtain

$$H\left(d_2\left(w, ADw\right), d_1\left(z, Dw\right), 0, d_2\left(w, ADw\right)\right) \leq 0.$$

Since $z = DBz$, by (3.4) we obtain $z = Dw$, hence $d_1\left(z, Dw\right) = 0$. Therefore

$$H\left(d_2\left(w, ADw\right), 0, 0, d_2\left(w, ADw\right)\right) \leq 0,$$

which implies by (F_2) for $v = 0$ that $d_2\left(w, ADw\right) = 0$, which implies $w = ADw$ and w is a fixed point of AD.

Since $Az = w$, $CAz = Cw = z$. Then z is a fixed point of CA.

Hence z is a common fixed point of CA and DB and w is a common fixed point of BC and AD.

Suppose that DB has a second fixed point z'.

By (3.1) we have

$$G\left(d_1\left(CAz, DBz'\right), d_2\left(Az, Bz'\right), d_1\left(CAz, z'\right), d_1\left(z', DBz'\right)\right) \leq 0,$$

$$G\left(d_1\left(z, z'\right), d_2\left(Az, Bz'\right), d_1\left(z, z'\right), 0\right) \leq 0.$$

By (F_2) we obtain

$$d_1\left(z, z'\right) \leq hd_2\left(Az, Bz'\right).$$

By (3.2) we have

$$H\left(d_2\left(BDBz', ADw\right), d_1\left(DBz', Dw\right), d_2\left(w, BDBz'\right), d_2\left(w, ADw\right)\right) \leq 0,$$

$$H\left(d_2\left(Bz', Az\right), d_1\left(z, z'\right), d_2\left(Az, Bz'\right), 0\right) \leq 0.$$

By (F_2) we obtain

$$d_2\left(Az, Bz'\right) \leq hd\left(z, z'\right).$$

Therefore,

$$d\left(z, z'\right) \leq h^2 d\left(z, z'\right),$$

which implies $z = z'$.

Hence DB has a unique fixed point z. Similarly, CA has a unique fixed point and w is the unique common fixed point of BC and AD. $\square$

Remark 3.2. *i) By Theorem 3.1 and Example 2.1 we obtain a generalization of Theorem 1.3.*
ii) By Theorem 3.1 and Examples 2.2 - 2.6 we obtain new particular results.

References

[1] B. Fisher, **Fixed point on two metric spaces**, Glasnik Mat. 16 (30) (1961), 333 – 337.

[2] I. Kikina and K. Kikina, **Generalized fixed point theorems in two metric spaces under implicit relations**, Int. J. Math. Anal., 4 (2010), no. 30, 1483 – 1489.

[3] V. Popa, **Fixed point theorems for implicit contractive mappings**, Stud. Cerc. Ştiinţ., Ser. Mat., Univ. Bacău 7 (1997), 127 – 134.

[4] V. Popa, **Some fixed point theorems for compatible mappings satisfying an implicit relation**, Demonstr. Math. 32, 1 (1999), 157 – 163.

[5] V. Popa, **A general fixed point theorem for two pairs of mappings on two metric spaces**, Novi Sad J. Math. 35, 2 (2005), 79 – 83.

[6] V. Popa and M. Telci, **Fixed point theorems for mappings satisfying implicit relations on two metric spaces**, Math. Balkanika 202 (2006), 143 – 152.

[7] A. Saini, S. Deki and N. Gulat, **Existence and unique common fixed point for pair of mappings in complete metric spaces**, Intern. J. of Current Trends in Engineering and Research 3, 55 (2017), 7 – 12.

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