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ISAC'S CONES FOR THE SET FUNCTIONS WITH BOUNDED *p*-VARIATION

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Abstract. This research work is devoted to present original Isac's Cones for the set functions with bounded *p*-variation, $p \geq 1$, having main applications in the General Efficiency and also in the Best Approximation.

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1. INTRODUCTION

Until now, the basis on which was developed our general concept of the Efficiency in the Ordered Linear Spaces and its Applications, supplied by the Ordered Locally Convex Spaces, was and it is represented about the vast class of the Convex Cones introduced by Professor Isac in 1981 [1] and published in 1983 [2]. Firstly, this notion was set forth as a synthesis in 1994, being justified by the immediate implications and applications in Pareto type Optimization [4]. It was called by us and, officially recognized, as "Isac's Cone" in 2009 [6], after the acceptance of this last agreed denomination by Professor Isac. Following the integral characterization of the countable additive set functions with bounded *p*-variation, $p \geq 1$, on separable subspaces of the metric spaces established by us in [5], we present a significant natural class of Isac's cones, studies in [3] and [7], which has strong connections with the General Efficiency and, in particular, with the Best Approximation.

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2. SET FUNCTIONS OF BOUNDED p – VARIATION CHARACTERIZED BY THE INTEGRAL REPRESENTATIONS

In accordance with [5], let X be a metric space, $Y \subseteq X$ a separable subspace, $B(Y)$ the σ - algebra of all Borel subsets in Y and $\mu : B(Y) \to R_+$ a measure. One denotes by $BV_p{}^c(Y, \mu)$ *the class of all countable additive set functions* $F : B(Y) \to R$ with $p \geq 1$ *bounded variation* on Y with respect to μ .

Theorem 2. [5]. *A function* F *belongs to* $BV_p{}^c(Y, \mu)$ *whenever* $p > 1$ *if and only if there exists a function* $f \in L^p(Y)$ *such that*

$$F(A) = \int_A f d\mu, \forall A \in B(Y) .$$

In all these cases, the total p – variation of the set function F *is*

$$V_p(F, Y, \mu) = (\int_Y |f|^p d\mu)^{1/p} .$$

Moreover, the linear space $BV_2^C(Y, \mu)$ *of all countable set functions with bounded 2 – variation on* Y *with respect to* μ *is a Hilbert space, being endowed with the topology induced by the scalar product defined by* $(F, G) = \int_Y f \cdot g d\mu$, *for every* $F, G \in BV_2^c(Y, \mu)$ *given as* $F(A) = \int_A f d\mu,$ $G(A) = \int_A g d\mu,$ $\forall A \in B(Y)$ *with* $f, g \in L^2(Y),$

Simple examples show that, if the set function F is not countable additive or $p = 1$, then the above theorem loses its validity, in the sense that an integral representation as above cannot be used for every real extended values set function defined on an arbitrary σ *-algebra* Ω and having *p-bounded variation* on a non-empty set $T \in \Omega$.

If $p = 2$, the above characterization is essential for *the Best Approximation* of this class containing set functions and for its applications in *the General Efficiency* by the corresponding *splines* (see, for example, [6], [7] and their references).

3. PROPER ISAC'S CONES

Using the above *integral representation*, the *real linear space* $BV_p^{\,c}(Y,\mu)$ can be organized as *a genuine Orderd Hausdorff Locally Convex Space in this way:* let us consider the *natural generated family* $P = \ \ p_A : A \in B(Y)\ \ $ of *the seminorms introduced* as follows:

$$p_A(F) = \int_A |f|\, d\mu\,,\ \forall F \in BV_p^{\,c}(Y,\mu) \text{ with } F(A) = \int_A f d\mu, \text{ and } A \in B(Y).$$

Thus, the *usual ordering convex* and *pointed cone* is $K_P = \ \ F \in BV_p^c(Y,\mu) : F(A) \geq 0, \forall A \in B(Y)\ \ $ which is an *Isac's Cone* if and only if $p = 1$, being *well based* [6] in this case by the set $B_1 = \ \ G \in K_1 : G(Y) = 1\ \ $. Indeed, if $p > 1$, then the sequence (F_n) of the set functions defined by

$$F_n\ \ A\ \ = \begin{cases} n^{1/p}\mu(Y),\ \mu(A) \in [0, \dfrac{\mu(Y)}{2n}] \\ 0,\ \mu(A) \in (\dfrac{\mu(Y)}{2n}, \mu(Y)] \end{cases} ,\ A \in B(Y),\ n \in \mathbf{N}^*$$

converges to zero in *the weak topology* on X identified as the smallest locally convex topology for which every $x^* \in X^*$ is continuous, being generated by the family of the seminorms $\ \ p_{x^*} : x^* \in X^*\ \ $ where $p_{x^*}(x) = \left|x^*(x)\right|$, $\forall x^* \in X^*, x \in X.$, but does not converge in the usual topology.

Therefore, by virtue of the characterization for the nuclear (supernormal) cones given in [2], K_p is not an Isac's cone. However, for every $p > 1$, K_p has as a base the set $B_p = \ \ G \in K_p : G(Y) = 1\ \ $ which, unfortunately, is unbounded.

Clearly, any convex cone generated by every closed, convex and bounded set $B_t = \ \ G \in K_p : G(Y) \leq t\ \ $ with $t \geq 0$ is certainly an Isac's cone in $BV_p^{\,c}(Y,\mu)$. Let us now examine the immediate connections with the normality of these usual ordering cones K_p.

We recall that, in this context, a pointed convex cone $K \subset BV_p^{\,c}(Y,\mu)$ is normal with respect to the topology defined by the family of the seminorms P if it fulfils one of the next equivalent assertions:

(i) there exists at a base B *of the neighborhoods for the origin denoted by* θ *in* X *such that* $V=(V+K)\cap(V-K), \forall V\in \mathrm{B}$*;*

(ii) $p_A(F)\le p_A(G), \forall F,G\in X$ *with* $G-F\in K$ *and for all* $p_A\in P$*;*

(iii) for any two nets $\{F_i\}_{i\in I}$, $\{G_i\}_{i\in I}\subset K$ *with* $G_i-F_i\in K, \forall i\in I$ *and* $\lim\limits_{i\in I} G_i=\theta$ *it follows that there exists* $\lim\limits_{i\in I} F_i=\theta$ In particular, a convex cone K is normal in a non-trivial normed linear space $(E,\|\cdot\|)$ if and only if there exists $t\in(0,\infty)$ such that $x,y\in E$ and $y-x\in K$ implies $\|x\|\le t\|y\|$.

By the structures of the family P of the seminorms and of the convex cone K_p it is easy to see that K_p is a normal cone, with empty topological interior, for every $p\ge 1$.

Following [4], a convex cone is a weakly Isac's cone in $BV_p^{\,c}(Y,\mu)$ if and only if it is weakly normal.

Finally, we remark that another appropriate family of the seminorms which can be defined on the real linear space $BV_p^c(Y,\mu)$ is

$$\widetilde{P}=\{\widetilde{p}_A : A\in B(Y)\},$$

where $\widetilde{p}_A(F)=\left(\int\limits_A |f|^p\, d\mu\right)^{1/p}$ if $F\in BV_p^c(Y,\mu)$ is represented as $F(A)=\int\limits_A f d\mu$ for every $A\in B(Y)$.

The conclusions are the same concerning the above quality to be an Isac's cone for K_p.

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