"Vasile Alecsandri" University of Bacău
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ON THE COINCIDENCE AMONG ORLICZ-SOBOLEV SPACES ON METRIC SPACES

MARCELINA MOCANU

Abstract. We generalize a coincidence result from the case of Sobolev-type spaces to the case of Orlicz-Sobolev spaces corresponding to a doubling Young function, in the setting of doubling metric measure spaces. We consider three types of Orlicz-Sobolev spaces: (i) a space of Newtonian type; (ii) a space associated to a generalized Poincaré inequality; (iii) a space defined as the closure of the class of Orlicz functions that are locally Lipschitz, under some norm involving an abstract differential operator.

1. Introduction

In the following, we consider that (X, d, μ) is a metric measure space, i.e. a metric space (X, d) equipped with a Borel regular outer measure μ, which positive and finite on balls [9]. Throughout the paper, we assume that the measure μ is doubling.

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Among the most important extensions of first order Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$, to a metric measure space (X, d, μ) are Hajłasz spaces $M^{1,p}(X)$ [6], Newtonian spaces $N^{1,p}(X)$[14], Cheeger spaces $H_{1,p}(X)$ [2], the spaces $P^{1,p}(X)$ [14], [5] and, for some special metric spaces X, the Sobolev spaces arising from vector fields $H^{1,p}(X)$ [4]. Sufficient conditions for inclusions between these Sobolev-type spaces are proved in several papers, such as [4], [14], [5].

For $1 \leq p < \infty$, the Hajłasz spaces $M^{1,p}(X)$ continuously embeds into the Newtonian space $N^{1,p}(X)$ [14, Theorem 4.8]. In the case $1 < p < \infty$, if X supports a weak $(1, q)-$Poincaré inequality for some $q \in (1, p)$, then $M^{1,p}(X) = N^{1,p}(X) = P^{1,p}(X)$ [14, Theorem 4.9]. Also, for $1 < p < \infty$, the spaces $N^{1,p}(X)$ and $H_{1,p}(X)$ are isometrically equivalent [14, Theorem 4.10].

Extensions to the metric setting of the Orlicz-Sobolev spaces $W^{1,\Phi}(\mathbb{R}^n)$, where Φ is a Young function, have been introduced by [1] through the generalization $M^{1,\Phi}(X)$ of Hajłasz spaces , respectively by [15] through the generalization $N^{1,\Phi}(X)$ of Newtonian spaces.

If Ψ is a doubling $N-$function, then $M^{1,\Psi}(X)$ continuously embeds into $N^{1,\Psi}(X)$ [15, Theorem 6.22]. In [11] sufficient conditions are provided for the existence of a continuous embedding of $N^{1,\Psi}(X)$ into $M^{1,\Psi}(X)$, using as a main tool the Hardy-Littlewood maximal operator. In [11] the Cheeger type Orlicz-Sobolev space $H_{1,\Psi}(X)$ is introduced as a natural generalization of the Cheeger space $H_{1,p}(X)$ [2]. It is shown that a continuous embedding $H_{1,\Psi}(X) \subset N^{1,\Psi}(X)$ holds whenever Ψ is a Young function, while $N^{1,\Psi}(X)$ embeds continuously into $H_{1,\Psi}(X)$ provided that the Banach space $L^{\Psi}(X)$ is reflexive.

In [10] the extensions $P^{1,\Phi}(X)$ and $H^{1,\Phi}(X)$ of $P^{1,p}(X)$ and $H^{1,p}(X)$, respectively, are introduced and the inclusions between $P^{1,\Phi}(X)$ and $H^{1,\Phi}(X)$ are investigated. Assume that (X, d, μ) is a doubling metric measure space, $\Phi : X \to [0, \infty)$ is a doubling Young function and D is an abstract differential operator on $LIP_{loc}(X)$. Under these three assumptions, we proved the following inclusions:

(1) $P^{1,\Phi}(X) \subset H^{1,\Phi}(X)$ if the complementary function of Φ is also doubling (equivalently, provided that $L^{\Phi}(X)$ is reflexive) [10, Theorem 4].

(2) $H^{1,\Phi}(X) \subset P^{1,\Phi}(X)$ if for each locally Lipschitz function u on X, the pair $(u, |Du|)$ satisfies the weak $(1, \Phi)-$Poincaré inequality with fixed constants [10, Theorem 5].

The main aim of this paper is to compare $N^{1,\Phi}(X)$ and $P^{1,\Phi}(X)$.

We will consider the class $\mathcal{O}$ of the operators T which associate with each locally Lipschitz function $u : X \to \mathbb{R}$ a nonnegative function $T(u) : X \to [0, \infty)$ such that T satisfies the following conditions, for some constant $C = C(T) \geq 1$:

(T1) $T(u+v) \leq C\,(T(u) + T(v))$ and $T(\lambda u) \leq C\,|\lambda|\,T(u)$ a.e. in X, whenever $u, v \in LIP_{loc}(X)$ and $\lambda \in \mathbb{R}$.

(T2) If $u : X \to \mathbb{R}$ is $L-$Lipschitz, then $T(u) \leq CL$ a.e. in X.

(T3) If $u \in LIP_{loc}(X)$ is constant on an open set $\Omega \subset X$, then $T(u) = 0$ a.e. in Ω.

The following result extends [5, Theorem 10.4] from the case $\Phi(t) = t^p$, $1 \leq p < \infty$, to the case of a general doubling Young function Φ.

Theorem 1. *Assume that (X, d, μ) is a doubling metric measure space and that $\Phi : X \to [0, \infty)$ is a doubling Young function. Let T be an operator in the class $\mathcal{O}$.*

Assume that $W^{1,\Phi}(X)$ is a function space endowed with a norm $\|\cdot\|$, with the following properties:

(W1) $W^{1,\Phi}(X)$ contains every function $u \in LIP_{loc}(X) \cap L^{\Phi}(X)$ with $T(u) \in L^{\Phi}(X)$ and $\|u\| \leq C(\|u\|_{L^{\Phi}(X)} + \|u\|_{L^{\Phi}(X)})$ for some fixed constant $C > 0$;

(W2) If $(u_k)_{k\geq 1}$ is a sequence in $W^{1,\Phi}(X) \cap LIP_{loc}(X)$, convergent in $L^{\Phi}(X)$ to some function w, such that the sequence $(T(u_k))_{k\geq 1}$ is weakly convergent in $L^{\Phi}(X)$, then w has a representative in $W^{1,\Phi}(X)$.

Then $P^{1,\Phi}(X) \subset W^{1,\Phi}(X)$, in the sense that every function $P^{1,\Phi}(X)$ has a representative in $W^{1,\Phi}(X)$.

Using the above theorem and [10, Theorem 4, Theorem 5], we finally compare the three versions of Orlicz-Sobolev spaces $H^{1,\Phi}(X)$, $N^{1,\Phi}(X)$ and $P^{1,\Phi}(X)$.

Theorem 2. *Assume that (X, d, μ) is a doubling metric measure space and that $\Phi : X \to [0, \infty)$ is a doubling Young function. If (X, d, μ) supports a weak $(1, \Phi)-$Poincaré inequality, then $H^{1,\Phi}(X) = N^{1,\Phi}(X) = P^{1,\Phi}(X)$, in the sense that every function belonging to one of these spaces has a representative in each of the other spaces.*

2. Preliminaries

We use basic notions from the theory of Orlicz spaces [13].

In the following $\Phi : [0, \infty) \to [0, \infty]$ is always an Young function. Φ is called $N-$function if it is real-valued, continuous, vanishes only at the origin and for $a \in \{\infty, 0\}$ satisfies $\lim\limits_{t\to a} \frac{\Phi(t)}{t} = a$.

Φ is said to satisfy a Δ_2-condition if there is a constant $C_\Phi > 0$ such that $\Phi(2t) \leq C_\Phi \Phi(t)$ for every $t \in [0, \infty)$. A Young function satisfying a Δ_2-condition is called *doubling*. Every doubling Young function is real-valued, strictly increasing and continuous. The Δ_2-condition for an increasing Young function Φ implies the power growth estimate: $\Phi(\lambda t) \leq C_\Phi \lambda^{\log_2 C_\Phi} \Phi(t)$, for all $\lambda \geq 1, t \geq 0$ [15, Lemma 2.7].

Let $(X, \mathcal{A}, \mu)$ be a measure space with a complete and $\sigma-$finite measure μ and let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function. The Orlicz space $L^\Phi(X)$ associated to Φ consists of all measurable functions $u : X \to [-\infty, \infty]$ satisfying $\int\limits_X \Phi(\lambda |u|) d\mu < \infty$ for some $\lambda > 0$. The Orlicz space $L^\Phi(X)$ is a Banach space with the Luxemburg norm defined by

$$\|u\|_{L^\Phi(X)} = \inf \left\{ k > 0 : \int\limits_X \Phi(\frac{|u|}{k}) d\mu \leq 1 \right\}.$$

For every measurable function $u : X \to [-\infty, +\infty]$, denote $I_\Phi(u) = \int\limits_X \Phi(|u|) d\mu$. If $I_\Phi(u) < \infty$, then $u \in L^\Phi(X)$ and the converse is true provided that Φ is doubling.

Throughout this paper we deal with a metric measure space (X, d, μ), which is a metric space (X, d) equipped with a Borel regular outer measure μ. Assume that μ is finite and positive on balls.

Remark 1. *Since μ is finite on balls, for every doubling $N-$function $\Phi : [0, \infty) \to [0, \infty)$ we have $L^\Phi(X) \subset L^1_{loc}(X)$ [13, Proposition 3.1.7].*

For every open ball $B = B(x, r) = \{y \in X : d(y, x) < r\}$ and each $\lambda > 0$ we will denote $\lambda B := B(x, \lambda r)$.

Definition 1. *The measure μ on the metric space (X, d, μ) is said to be doubling if there is a constant $C_\mu \geq 1$ such that*

$$\mu(2B) \leq C_\mu \mu(B) \tag{2.1}$$

for every ball $B \subset X$.

In the following we will assume that the measure μ is doubling.

We will denote by $LIP(X)$ and $LIP_{loc}(X)$ the collections of all real-valued Lipschitz functions, respectively locally Lipschitz functions.

The infinitesimal behavior of a real function on a metric space $u : X \to \mathbb{R}$ at a point $x \in X$ is described by the upper and lower Lipschitz constants

$$Lip\ u(x) = \limsup_{r\to 0} \frac{L(x,u,r)}{r} \text{ and } lip\ u(x) = \liminf_{r\to 0} \frac{L(x,u,r)}{r},$$

where $L(x,u,r) = \sup\{|u(y) - u(x)| : d(x,y) \le r\}$.

A substitute for the norm of the gradient in analysis on metric measure spaces is the concept of upper gradient. Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0,+\infty]$ is said to be an *upper gradient* of u in X if

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\ ds, \tag{2.2}$$

for every compact rectifiable path $\gamma : [a,b] \to X$.

It is well-known that, for every $u \in LIP_{loc}(X)$ the upper Lipschitz constant $Lip\ u$ is an upper gradient of u in X [2].

Definition 2. [15] *Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0,+\infty]$ is called a $\Phi-$weak upper gradient of u if (2.2) holds for all compact rectifiable paths $\gamma : [a,b] \to X$ except for a path family with zero $\Phi-$modulus.*

The collection $\widetilde{N}^{1,\Phi}(X)$ of all functions $u \in L^\Phi(X)$ possessing a $\Phi-$weak upper gradient $g \in L^\Phi(X)$ is a vector space. For $u \in \widetilde{N}^{1,\Phi}(X)$ define $\|u\|_{1,\Phi} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)}$, where the infimum is taken over all $\Phi-$weak upper gradients $g \in L^\Phi(X)$ of u. Consider the equivalence relation $u \sim v \Leftrightarrow \|u - v\|_{1,\Phi} = 0$. Then $N^{1,\Phi}(X) = \widetilde{N}^{1,\Phi}(X)/\sim$ is a Banach space with the norm $\|u\|_{N^{1,\Phi}} := \|u\|_{1,\Phi}$ [15].

If $X = \Omega \subset \mathbb{R}^n$ is a domain and Φ is a doubling Young function, then $N^{1,\Phi}(X) = W^{1,\Phi}(\Omega)$ as Banach spaces and the norms are equivalent [15].

We recall the notion of weak $(1,\Phi)-$Poincaré inequality in an open set of a metric measure space.

Denote the mean value of a function $u \in L^1(A)$ over A by $u_A := \frac{1}{\mu(A)} \int\limits_B u d\mu$, where $0 < \mu(A) < \infty$.

Definition 3. [15, Definition 5.2] *Let $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing Young function and $\Omega \subset X$ an open set. We say that a function $u \in L^1_{loc}(\Omega)$ and a Borel measurable nonnegative function g on Ω satisfy a weak $(1,\Phi)-$Poincaré inequality in Ω if there exist some constants $C_P > 0$ and $\sigma \geq 1$ such that*

$$\frac{1}{\mu(B)} \int\limits_B |u - u_B| \, d\mu \leq C_P \; r \; \Phi^{-1} \left(\frac{1}{\mu(\sigma B)} \int\limits_{\sigma B} \Phi(g) d\mu \right). \tag{2.3}$$

for each ball $B = B(x,r)$ satisfying $\sigma B \subset \Omega$. It is said that Ω supports a weak $(1,\Phi)-$Poincaré inequality if the above inequality holds for each function $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants.

Remark 2. *If Φ is doubling, we may replace in the above definition upper gradients by $\Phi-$weak upper gradients.*

The weak $(1,p)-$Poincaré inequality is the weak $(1,\Phi)-$Poincaré inequality for $\Phi(t) = t^p$.

Definition 4. *The space $P^{1,\Phi}(X)$ consists of all functions $u \in L^\Phi(X)$ for which there exists $g \in L^\Phi(X)$ such that the pair (u,g) satisfies the weak $(1,\Phi)-$Poincaré inequality (2.3) for some constants $C_P > 0$ and $\sigma \geq 1$.*

The definition of a generalization of $H^{1,p}(X)$, the Orlicz-Sobolev space $H^{1,\Phi}(X)$, as the closure of the class of Orlicz functions in $L^\Phi(X)$ that are locally Lipschitz functions, under some norm involving an abstract differential operator requires a more specialized approach.

An *abstract differential operator* [4, Theorem 10] on $LIP_{loc}(X)$ is a linear operator D which associates with each $u \in LIP_{loc}(X)$ a measurable function $Du : X \to \mathbb{R}^N$, where N is a fixed positive integer, such that the following conditions are satisfied:

(D1) There exists a constant $C_D > 0$ such that $|Du| \leq C_D L$ $\mu-$a.e. whenever u is an $L-$Lipschitz function;

(D2) If $u \in LIP_{loc}(X)$ is constant in some measurable set $E \subset X$, then $Du = 0$ $\mu-$a.e. in E.

A remarkable example of abstract differential operator is the Cheeger's differential operator [2].

The set $V_\Phi(X) := \{u \in LIP_{loc}(X) \cap L^\Phi(X) : |Du| \in L^\Phi(X)\}$ is a vector space and the functional defined for $u \in V_\Phi(X)$ by

$$\|u\| = \|u\|_{L^\Phi(X)} + \||Du|\|_{L^\Phi(X)}$$

is a norm on this space. Then $H^{1,\Phi}(X)$ is defined as the closure of $V_\Phi(X)$ under the above norm.

Since $L^\Phi(X)$ is a Banach space, we see that each element of $H^{1,\Phi}(X)$ is represented by a pair (u, G), where $u \in L^\Phi(X)$ and $G : X \to \mathbb{R}^N$ is measurable with $|G| \in L^\Phi(X)$, for which there exists a sequence $(u_n)_{n\geq 1}$ in $V_\Phi(X)$ such that $u_n \to u$ in $L^\Phi(X)$ and $|Du_n - G| \to 0$ in $L^\Phi(X)$ as $n \to \infty$.

In order to approximate Orlicz-Sobolev functions by locally Lipschitz functions, we will use a discrete convolution operator for locally integrable functions on a doubling metric measure space. This operator was defined in [8] (see also [7]) using the notion of $(\varepsilon, \lambda)-$ cover of an open set and a Lipschitz partition of unity subordinated to an $(\infty, 2)-$ cover. In the following, X is a doubling metric measure space with a doubling constant C_μ and $\Omega \subset X$ is open.

Given $\varepsilon > 0$ and $\lambda \geq 1$, an $(\varepsilon, \lambda)-$ cover of Ω ([8], [7]) is a countable cover $\mathcal{F} = \{B_i = B(x_i, r_i) : i \geq 1\}$ of Ω with the following properties:

(C1) $r_i \leq \varepsilon$ for all i;

(C2) $\lambda B_i \subset \Omega$ for all i;

(C3) If λB_i meets λB_j, then $r_i \leq 2r_j$;

(C4) Each ball λB_i meets at most $C = C(C_\mu, \lambda)$ balls λB_j.

Every open set $\Omega \subset X$ admits an $(\varepsilon, \lambda)-$ cover, whenever $\varepsilon > 0$ and $\lambda \geq 1$, as follows from [3, Theorem III.1.3] and [12, Lemma 2.9], see [8, Lemma 5.1] and [7, Lemma 3.1]. If $0 < \varepsilon \leq \varepsilon' \leq \infty$ and $1 \leq \lambda' \leq \lambda < \infty$, then every $(\varepsilon, \lambda)-$ cover of Ω is also an $(\varepsilon', \lambda')-$ cover of Ω

Let $\mathcal{F} = \{B_i = B(x_i, r_i) : i \geq 1\}$ be an $(\infty, 2)-$ cover of Ω. By [12, Lemma 2.16], as it is shown in [8, Lemma 5.2] and [7, Lemma 3.2], there exists a collection of real functions $\varphi = \{\varphi_i : i \geq 1\}$ defined on Ω such that

(P1) each φ_i is L_i-Lipschitz, where $L_i := \frac{C(C_\mu)}{r_i}$;

(P2) $0 \leq \varphi_i \leq 1$ for all i;

(P3) $\varphi_i = 0$ on $X \setminus 2B_i$ for all i;

(P4) $\sum\limits_{i\geq 1} \varphi_i = 1$ on X.

A collection $\varphi = \{\varphi_i : i \geq 1\}$ as above is called a *Lipschitz partition of unity* with respect to $\mathcal{F}$.

Given an $(\infty, 2)-$ cover $\mathcal{F}$ of Ω and a Lipschitz partition of unity φ with respect to $\mathcal{F}$, the corresponding discrete convolution of $u \in L^1_{loc}(\Omega)$ is defined by

$$u_{\mathcal{F}}(x) = \sum_{i \geq 1} u_{B_i} \varphi_i(x),\ x \in \Omega.$$

Note that, for each $x \in \Omega$, there are at most $C(C_\mu, 2)$ non-zero terms in the series defining $u_{\mathcal{F}}(x)$.

By [8, Lemma 5.3] (see also [7, Lemma 3.3]), $u_{\mathcal{F}}$ is locally Lipschitz. Moreover, for every set J of positive integers, the function $\sum\limits_{i \in J} u_{B_i} \varphi_i$ is locally Lipschitz.

Discrete convolutions constructed as above are used to approximate Orlicz functions on a doubling metric measure space, as it is shown in [7, Lemma 3.3] (see also [8, Lemma 5.3] for the case of $p-$integrable functions).

Lemma 1. [7, Lemma 3.3] *Assume that (X, d, μ) is a doubling metric measure space and Φ is a Young function. Let $u \in L^1_{loc}(\Omega)$, where $\Omega \subset X$ is open. For each $(\infty, 2)-$ cover $\mathcal{F}$ of Ω and any partition of unity φ with respect to $\mathcal{F}$, we consider the corresponding discrete convolution $u_{\mathcal{F}}$.*

(1) $u_{\mathcal{F}}$ is locally Lipschitz and for each $B \in \mathcal{F}$

$$Lip\ u_{\mathcal{F}} \leq C(C_\mu) \frac{1}{r(B)} \frac{1}{\mu(5B)} \int\limits_{5B} |u - u_{5B}|\, d\mu \text{ in } B. \tag{2.4}$$

(2) Let Φ be doubling and $u \in L^\Phi(X)$. If $\mathcal{F}_k$ is an $(\varepsilon_k, 2)-$cover of Ω, for each $k \geq 1$ and if $\varepsilon_k \to 0$ as $k \to \infty$, then $u_{\mathcal{F}_k} \to u$ in $L^\Phi(\Omega)$.

We prove an inequality analogous to (2.4) for nonlinear operators more general than *Lip*.

Lemma 2. *Let T be an operator as in Theorem 1. Let $\mathcal{F} = \{B_i : i \geq 1\}$ be an $(\varepsilon, 2)-$cover of X, where $\varepsilon > 0$ and let $\varphi = \{\varphi_i : i \geq 1\}$ a Lipschitz partition of unity with respect to $\mathcal{F}$. For $u \in LIP_{loc}(X)$ denote by $u_{\mathcal{F}}$ the corresponding discrete convolution of u. There exists some constant C', depending only on C_μ and on T, such that for every $B \in \mathcal{F}$, of radius $r(B)$, we have*

$$T(u_{\mathcal{F}}) \leq C' \frac{1}{r(B)} \frac{1}{\mu(5B)} \int\limits_{5B} |u - u_{5B}|\, d\mu \text{ a.e. in } B.$$

Proof. Fix $B \in \mathcal{F}$. The set $I := \{i \geq 1 : 2B_i \cap 2B \neq \emptyset\}$ is finite, having at most $C_1 = C(C_\mu, 2)$ elements. Denote by J the complement of I with respect to the set of positive integers.

By (T1) and (T3), $T(u_\mathcal{F}) \leq CT(u_\mathcal{F} - u_B)$ a.e. in X.

By (T1), $T(u_\mathcal{F} - u_B) \leq CT\left(\sum_{i\in I}(u_{B_i} - u_B)\varphi_i\right) + CT\left(\sum_{i\in J}(u_{B_i} - u_B)\varphi_i\right)$.

The function $\sum_{i\in J}(u_{B_i} - u_B)\varphi_i$ is locally Lipschitz and is zero on B, hence $T\left(\sum_{i\in J}(u_{B_i} - u_B)\varphi_i\right) = 0$ a.e. in B.

It follows that $T(u_\mathcal{F}) \leq C^2 T\left(\sum_{i\in I}(u_{B_i} - u_B)\varphi_i\right)$ a.e. in B.

But, as follows by induction from (T1) and taking account of (T2),

$$\begin{aligned} T\left(\sum_{i\in I}(u_{B_i} - u_B)\varphi_i\right) &\leq C^{C_1}\sum_{i\in I} T\left((u_{B_i} - u_B)\varphi_i\right) \\ &\leq C^{1+C_1}\sum_{i\in I}|u_{B_i} - u_B|\, T(\varphi_i) \end{aligned}$$

a.e. in X.

Therefore,

$$T(u_\mathcal{F}) \leq C^{3+C_1}\sum_{i\in I}|u_{B_i} - u_B|\, T(\varphi_i) \text{ a.e. in } B. \tag{2.5}$$

If B_j, $B_k \in \mathcal{F}$ satisfy $2B_j \cap 2B_k \neq \emptyset$, a standard argument [10, Lemma 3] shows that

$$|u_{B_i} - u_{B_k}| \leq \left(C_\mu^5 + C_\mu^3\right)\frac{1}{\mu(5B_k)}\int_{5B_k}|u - u_{5B_k}|\, d\mu.$$

In particular, for each $i \in I$,

$$|u_{B_i} - u_B| \leq \left(C_\mu^5 + C_\mu^3\right)\frac{1}{\mu(5B)}\int_{5B}|u - u_{5B}|\, d\mu. \tag{2.6}$$

By the property (P1) of the Lipschitz partition of unity φ and by (T2), for each $i \geq 1$ we have

$$T(\varphi_i) \leq C\frac{C(C_\mu)}{r_i} \text{ a.e. in } X.$$

For $i \in I$, this latter inequality and (C3) imply

$$T(\varphi_i) \le 2C \frac{C(C_\mu)}{r(B)}. \tag{2.7}$$

From inequalities (2.5), (2.6) and (2.7) we get

$$T(u_{\mathcal{F}}) \le C' \frac{1}{r(B)} \frac{1}{\mu(5B)} \int_{5B} |u - u_{5B}| \, d\mu,$$

where $C' = 2C^{4+C_1} \cdot C(C_\mu)$. ∎

3. Examples and proofs of the main results

Example 1. *If $T(u) = Lip\ u$, then T satisfies (T1) and (T2) with $C = 1$ and T also satisfies (T3).*

Example 2. *If D is an abstract differential operator on $LIP_{loc}(X)$, then $T(u) = |Du|$ defines an operator which associates with each locally Lipschitz function $u : X \to \mathbb{R}$ a nonnegative function $T(u)$ on X and this T satisfies conditions (T1) and (T3) with $C = 1$. Moreover, if D is Cheeger's differential operator, then T also satisfies (T2) with some C depending only on the dimension of the strong differentiable structure* [2], [5].

Example 3. *The operator $T(u) = Lip\ u$ and the Orlicz-Sobolev space $W^{1,\Phi}(X) = N^{1,\Phi}(X)$ (with the usual norm) satisfy the conditions (W1) and (W2) from Theorem 1. Since $Lip\ u$ is an upper gradient of $u \in LIP_{loc}(X)$* [2], *(W1) holds with $C = 1$. If $u_j \to u$ in $L^\Phi(X)$ and $Lip\ u_j \to g$ weakly in $L^\Phi(X)$, then by a Mazur-type theorem* [15, Theorem 4.17] *(see also* [7, Lemma 2.3]*), g is a $\Phi-$weak upper gradient of a representative $\widetilde{u}$ of u, therefore $\widetilde{u} \in N^{1,\Phi}(X)$ and so (W2) holds.*

Proof of Theorem 1. Let $u \in P^{1,\Phi}(X)$.

By definition, there exists $g \in L^\Phi(X)$ such that the weak $(1,\Phi)-$Poincaré inequality (2.3) holds, for some constants $C_P > 0$ and $\sigma \ge 1$, possibly depending on u and g.

Assume for each $k \ge 1$ that $\mathcal{F}_k = \{B_{ki} : i \ge 1\}$ is an $\left(\frac{1}{k}, 5\sigma\right)-$cover of Ω and $\varphi_k = \{\varphi_{ki} : i \ge 1\}$ is a Lipschitz partition of unity with respect to $\mathcal{F}_k$.

We consider the corresponding discrete convolution $u_k := u_{\mathcal{F}_k}$, for $k \ge 1$.

By Lemma 1, $u_k \to u$ in $L^\Phi(X)$ as $k \to \infty$.

Fix $k \ge 1$.

Let $i \geq 1$. By Lemma 2,

$$T(u_k) \leq C' \frac{1}{r(B_{ki})} \frac{1}{\mu(5B_{ki})} \int_{5B_{ki}} |u - u_{5B_{ki}}| \, d\mu \text{ a.e. in } B_{ki}.$$

Using the weak $(1, \Phi)-$Poincaré inequality (2.3), this implies (3.1)

$$T(u_k) \leq C' C_P \Phi^{-1} \left(\frac{1}{\mu(5\sigma B_{ki})} \int_{5\sigma B_{ki}} \Phi(g) \, d\mu \right) \text{ a.e. in } B_{ki}.$$

The doubling property of the Young function Φ implies the power growth estimate $\Phi(\lambda t) \leq C_\Phi \lambda^{\log_2 C_\Phi} \Phi(t)$ for $\lambda \geq 1$ and $t \geq 0$. We recall that $\Phi(\lambda t) \leq \lambda \Phi(t)$, if $0 \leq \lambda \leq 1$, by convexity of Φ and $\Phi(0) = 0$, and that $\Phi(\Phi^{-1}(t)) \leq t$ for all $t \geq 0$. Then (3.1) implies

$$\Phi(T(u_k)) \leq C'' \frac{1}{\mu(5\sigma B_{ki})} \int_{5\sigma B_{ki}} \Phi(g) \, d\mu \text{ a.e. in } B_{ki}.$$

Here $C'' := \max \left\{ C' C_P, C_\Phi (C' C_P)^{\log_2 C_\Phi} \right\}$.

Integrating the previous inequality over B_{ki} we get

$$\int_{B_{ki}} \Phi(T(u_k)) \, d\mu \leq C'' \frac{\mu(B_{ki})}{\mu(5\sigma B_{ki})} \int_{5\sigma B_{ki}} \Phi(g) \, d\mu. \tag{3.2}$$

In particular,

$$\int_{B_{ki}} \Phi(T(u_k)) \, d\mu \leq C'' \int_{5\sigma B_{ki}} \Phi(g) \, d\mu.$$

But $\mathcal{F}_k$ is a cover of X, hence

$$\begin{aligned} \int_X \Phi(T(u_k)) \, d\mu &\leq \sum_{i \geq 1} \int_{B_{ki}} \Phi(T(u_k)) \, d\mu \\ &\leq C'' \sum_{i \geq 1} \chi_{5\sigma B_{ki}} \int_X \Phi(g) \, d\mu \\ &\leq C''' \int_X \Phi(g) \, d\mu \end{aligned}$$

where $C''' := C'' C(C_\mu, 5\sigma)$. In the last inequality we used the bounded overlap of the family of balls $\{5\sigma B_{ki} : i \geq 1\}$, guaranteed by (C4).

We proved that

$$\int_X \Phi\left(T(u_k)\right) d\mu \le C''' \int_X \Phi\left(g\right) d\mu \tag{3.3}$$

for each $k \ge 1$. Note that C''' depends only on the doubling constants C_μ, C_Φ, on the constants C_P and σ from the weak $(1,\Phi)-$Poincaré inequality (2.3) and on the constant $C = C(T)$ from the properties of the operator T.

Since $g \in L^\Phi(X)$ and Φ is doubling, $\int_X \Phi(g)\, d\mu$ is finite. Denote $M = \max\left\{1, C''' \int_X \Phi(g)\, d\mu\right\}$. From inequality (3.3) we see that $T(u_k) \in L^\Phi(X)$. Moreover,

$$\int_X \Phi\left(\frac{T(u_k)}{M}\right) d\mu \le \frac{1}{M} \int_X \Phi\left(T(u_k)\right) d\mu \le 1,$$

hence $\|T(u_k)\|_{L^\Phi(X)} \le M$.

We proved that the sequence $\left(T(u_k)\right)_{k\ge1}$ is bounded in $L^\Phi(X)$.

If $L^\Phi(X)$ is reflexive, i.e. Φ and its complementary function are doubling, passing to a subsequence we may assume that $\left(T(u_k)\right)_{k\ge1}$ is weakly convergent in $L^\Phi(X)$. But $u_k \to u$ in $L^\Phi(X)$ as $k \to \infty$. Then using (W2) it follows that u has a representative in $W^{1,\Phi}(X)$.

If $L^\Phi(X)$ is not reflexive, we need a more elaborate approach. Since Φ is assumed to be doubling, we can use [7, Lemma 2.2], where it is proved that every bounded sequence $(f_k)_{k\ge1}$ in $L^\Phi(X)$ has a weakly convergent subsequence, provided that

$$\lim_{\mu(A)\to 0}\left(\sup_{k\ge1} \int_A \Phi\left(|f_k|\right) d\mu\right) = 0. \tag{3.4}$$

It remains to prove that the sequence $f_k := T(u_k)$, $k \ge 1$, satisfies (3.4).

Let $A \subset X$ be measurable. Going back to (3.2), we see that

$$\int_A \Phi\left(|T(u_k)|\right) d\mu \le C'' \sum_{i\ge1} \frac{\mu\left(A \cap B_{ki}\right)}{\mu\left(5\sigma B_{ki}\right)} \int_{5\sigma B_{ki}} \Phi\left(g\right) d\mu.$$

As in the proof of [7, Theorem 1.2, (3.31)], it follows that

$$\lim_{\mu(A)\to 0}\left(\sum_{i\ge1} \frac{\mu\left(A \cap B_{ki}\right)}{\mu\left(5\sigma B_{ki}\right)} \int_{5\sigma B_{ki}} \Phi\left(g\right) d\mu\right) = 0.$$

The latter two inequalities show that the sequence $f_k := T(u_k)$ satisfies (3.4), which completes the proof. ∎

Remark 3. *We can remove the assumption of the reflexivity of* $L^{\Phi}(X)$ *from* [10, Theorem 4] *and the proof follows for a general doubling Young function* Φ *using* [7, Lemma 2.2], *in a way analogous to the final part of the proof of Theorem 1.*

Proof of Theorem 2. We have $P^{1,\Phi}(X) \subset H^{1,\Phi}(X)$ by [10, Theorem 4] and Remark 3.

Since (X, d, μ) supports a weak $(1, \Phi)-$Poincaré inequality and Φ is a doubling Young function, by [15, Theorem 5.7] it follows that (X, d, μ) supports a weak $(1, p)-$Poincaré inequality whenever $\log_2 C_\Phi \leq p < \infty$. By Cheeger's fundamental result for $1 < p < \infty$ [2], X admits a non-degenerate strong measurable differentiable structure of some dimension N. So, the Cheeger differential operator D is well defined on $LIP_{loc}(X)$ and $|Du|$ is a $\Phi-$weak upper gradient of $u \in LIP_{loc}(X)$ [5]. Then for each $u \in LIP_{loc}(X)$, the pair $(u, |Du|)$ satisfies the $(1, \Phi)-$Poincaré inequality with fixed constants, hence $H^{1,\Phi}(X) \subset P^{1,\Phi}(X)$, by [10, Theorem 5].

We get the coincidence $P^{1,\Phi}(X) = H^{1,\Phi}(X)$.

By the definition of $P^{1,\Phi}(X)$, if (X, d, μ) supports a weak $(1, \Phi)-$Poincaré inequality, then $N^{1,\Phi}(X) \subset P^{1,\Phi}(X)$.

On the other hand, by Theorem 1, taking account of Example 1 and Example 3 it follows that $P^{1,\Phi}(X) \subset N^{1,\Phi}(X)$.

We conclude that $N^{1,\Phi}(X) = P^{1,\Phi}(X)$. ∎

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"Vasile Alecsandri" University of Bacău,
Department of Mathematics and Informatics,
Calea Mărăşeşti 157, Bacău 600115,
Romania
email:mmocanu@ub.ro