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SOME TOPOLOGICAL ASPECTS IN m-METRIC SPACES

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Abstract. In this paper, we introduce a new class of open balls in an m-metric space (X, μ) which will form a base for a Hausdorff topology on X. This will facilitate the initiation of open and closed sets, neighbourhoods and other allied notions in m-metric spaces. Moreover, we discuss the regularity and first countability properties of m-metric spaces and prove Cantor's intersection theorem, Baire's category theorem, Urysohn's lemma in the setting of m-metric spaces.

1. Introduction

It is well known that convergence of sequences and continuity of functions are two important concepts in real or complex analysis. Our main task in metric spaces is to introduce an abstract formulation of the notion of distance between two points of an arbitrary nonempty set. It is interesting to note that most of the central concepts of real or complex analysis can be generalized in metric spaces. Several authors successfully extended the notion of metric spaces in different directions such as G-metric space [9, 15], cone metric space [4, 16], b-metric space [2, 3], C^*-algebra valued metric space [10, 11].

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In 1994, Matthews [12] introduced the concept of partial metric spaces as a generalization of metric spaces and proved the well-known Banach contraction theorem in this setting. Afterwards, a lot of articles have been dedicated to the improvement of fixed point theory in partial metric spaces(see [5, 6, 7] and references therein). Very recently, Asadi et al. [1] extended the notion of partial metric spaces to m-metric spaces and established Banach and Kannan fixed point theorems in this new framework. They showed that every partial metric is an m-metric, but the converse may not be hold, in general. Moreover, they introduced a class of open balls which generates a topology. Some fixed point results in m-metric spaces have been very recently obtained in [8, 13, 17]. In this work, we introduce a new class of open balls in an m-metric space (X, μ) which will generate a Hausdorff topology on X. We shall establish some topological properties of m-metric spaces and prove Cantor's intersection theorem, Baire's category theorem, Urysohn's lemma in the setting of m-metric spaces. We also prove that every m-metric space is a first countable topological space and hence continuity is equivalent to sequential continuity.

2. Some Basic Concepts

We begin with some basic notations, definitions, and necessary results in m-metric spaces.

Definition 2.1. [12] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all x, y, $z \in X$:*

(p1) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$,
(p2) $p(x, x) \leq p(x, y)$,
(p3) $p(x, y) = p(y, x)$,
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Definition 2.2. [1] *Let X be a nonempty set. A function $\mu : X \times X \to \mathbb{R}^+$ is called an m-metric if the following conditions are satisfied:*

(m1) $\mu(x, x) = \mu(y, y) = \mu(x, y) \Longleftrightarrow x = y$,
(m2) $m_{xy} \leq \mu(x, y)$,
(m3) $\mu(x, y) = \mu(y, x)$,
(m4) $(\mu(x, y) - m_{xy}) \leq (\mu(x, z) - m_{xz}) + (\mu(z, y) - m_{zy})$,

where $m_{xy} := min\,\{\mu(x,x),\, \mu(y,y)\}$. *Then the pair* (X,μ) *is called an* m*-metric space. The following notation is useful in the sequel:*

$$M_{xy} := max\,\{\mu(x,x),\, \mu(y,y)\}.$$

Example 2.3. [1] *Let* $X := [0,\infty)$. *Then* $\mu(x,y) = \frac{x+y}{2}$ *on* X *is an* m*-metric.*

It is valuable to note that μ is not a partial metric on X. In fact, if $x = 4,\ y = 2$ then $\mu(x,x) > \mu(x,y)$.

Example 2.4. *Let* $X := [0,\infty)$. *Then* $\mu(x,y) = \frac{x^2+y^2}{2}$ *on* X *is an* m*-metric.*

Example 2.5. [1] *Let* (X,d) *be a metric space. Then* $\mu(x,y) = ad(x,y) + b$ *where* $a,\, b > 0$ *is an* m*-metric on* X.

Remark 2.6. [1] *For every* $x,\, y \in X$,

1. $0 \leq M_{xy} + m_{xy} = \mu(x,x) + \mu(y,y)$;
2. $0 \leq M_{xy} - m_{xy} = \mid \mu(x,x) - \mu(y,y) \mid$;
3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

Lemma 2.7. [1] *Every* p*-metric is an* m*-metric.*

3. Open Balls and Topology

In this section, we first introduce a new class of open balls in m-metric spaces which will generate a topology τ_μ on (X,μ).

Definition 3.1. *Let* (X,μ) *be an* m*-metric space,* $x \in X$ *and* $r > 0$. *Then the set* $B(x,r) = \{y \in X : \mu(x,y) < 2m_{xy} - M_{xy} + r\}$ *is called an open ball with centered at* x *and radius* r. *A closed ball with centered at* x *and radius* r *is defined by the set* $B[x,r] = \{y \in X : \mu(x,y) \leq 2m_{xy} - M_{xy} + r\}$.

Theorem 3.2. *Let* $\mathcal{B} = \{B(x,r) : x \in X,\, r > 0\}$. *Then* $\mathcal{B}$ *is a base for a topology* τ_μ *on* (X,μ).

Proof. Clearly, $X = \displaystyle\bigcup_{x\in X} B(x,r)$. Let $B(x_1,r_1),\, B(x_2,r_2) \in \mathcal{B}$ and $u \in B(x_1,r_1) \cap B(x_2,r_2)$. Then, $\mu(x_1,u) < 2m_{x_1u} - M_{x_1u} + r_1$ and $\mu(x_2,u) < 2m_{x_2u} - M_{x_2u} + r_2$. Let $0 < r < min\{2m_{x_1u} - M_{x_1u} + r_1 - \mu(x_1,u),\, 2m_{x_2u} - M_{x_2u} + r_2 - \mu(x_2,u)\}$. We consider the open ball $B(u,r)$. It is sufficient to show that $B(u,r) \subseteq B(x_1,r_1) \cap B(x_2,r_2)$.

Let $z \in B(u,r)$. Then, $\mu(u,z) < 2m_{uz} - M_{uz} + r$.

By using $(m4)$ and Remark 2.6, we have

$$\begin{aligned}
\mu(x_1,z)-2m_{x_1z}+M_{x_1z} &= (\mu(x_1,z)-m_{x_1z})+(M_{x_1z}-m_{x_1z})\\
&\le (\mu(x_1,u)-m_{x_1u}+\mu(u,z)-m_{uz})\\
&\quad +(M_{x_1u}-m_{x_1u}+M_{uz}-m_{uz})\\
&< \mu(x_1,u)-2m_{x_1u}+M_{x_1u}+r\\
&< \mu(x_1,u)-2m_{x_1u}+M_{x_1u}\\
&\quad +2m_{x_1u}-M_{x_1u}+r_1-\mu(x_1,u)\\
&= r_1.
\end{aligned}$$

This shows that $z\in B(x_1,r_1)$. By an argument similar to that used above, it follows that $z\in B(x_2,r_2)$. Consequently, $z\in B(x_1,r_1)\cap B(x_2,r_2)$. Therefore, $B(u,r)\subseteq B(x_1,r_1)\cap B(x_2,r_2)$. □

Remark 3.3. *The elements of τ_μ are called open sets and their complements in X are called closed sets. Obviously, a nonempty subset G of X is open if and only if it is a union of open balls.*

We now visualise the open balls in some particular cases.

Example 3.4. *Let $X := [0,1]$ and $\mu(x,y) = min\{x,\, y\}$ on X. Then (X,μ) is an m-metric space. In this case for $r>0$, we have*

$$\begin{aligned}
B(x,r) &= \{y\in X : \mu(x,y)<2m_{xy}-M_{xy}+r\}\\
&= \{y\in X : M_{xy}-m_{xy}<r\}\\
&= \{y\in X :\mid \mu(x,x)-\mu(y,y)\mid< r\}\\
&= \{y\in X :\mid y-x\mid< r\}\\
&= (x-r,x+r)\cap X.
\end{aligned}$$

Example 3.5. *Let $X := [0,\infty)$ and $\mu(x,y) = \frac{x+y}{2}$ on X. Then (X,μ) is an m-metric space. In this case for $r>0$, we have*

$$\begin{aligned}
B(x,r) &= \{y\in X : \mu(x,y)<2m_{xy}-M_{xy}+r\}\\
&= \{y\in X : \frac{x+y}{2}-m_{xy}+M_{xy}-m_{xy}<r\}\\
&= \{y\in X :\mid \frac{x-y}{2}\mid+\mid x-y\mid< r\}\\
&= \{y\in X :\mid y-x\mid< \frac{2}{3}r\}\\
&= (x-\frac{2}{3}r,x+\frac{2}{3}r)\cap X.
\end{aligned}$$

Theorem 3.6. *A closed ball $B[x,r]$ in an m-metric space (X,μ) is a closed set.*

Proof. It is sufficient to show that $X \setminus B[x,r]$ is open. Let $y \in X \setminus B[x,r]$. Then $\mu(x,y) > 2m_{xy} - M_{xy} + r$. Put $r_y = \mu(x,y) - 2m_{xy} + M_{xy} - r > 0$ and consider the open ball $B(y,r_y)$. Let $z \in B(y,r_y)$. So $\mu(y,z) < 2m_{yz} - M_{yz} + r_y$. By using $(m4)$ and Remark 2.6, we have

$$\begin{aligned}
\mu(x,y) - 2m_{xy} + M_{xy} &= (\mu(x,y) - m_{xy}) + (M_{xy} - m_{xy}) \\
&\leq (\mu(x,z) - m_{xz}) + (\mu(z,y) - m_{zy}) \\
&\quad +(M_{xz} - m_{xz}) + (M_{zy} - m_{zy}) \\
&< \mu(x,z) - 2m_{xz} + M_{xz} + r_y \\
&= \mu(x,z) - 2m_{xz} + M_{xz} \\
&\quad +\mu(x,y) - 2m_{xy} + M_{xy} - r
\end{aligned}$$

which gives that $\mu(x,z) > 2m_{xz} - M_{xz} + r$ and so, $z \in X \setminus B[x,r]$. This proves that $y \in B(y,r_y) \subseteq X \setminus B[x,r]$. As y runs over $X \setminus B[x,r]$, we have $X \setminus B[x,r] \subseteq \bigcup_{y \in X \setminus B[x,r]} B(y,r_y) \subseteq X \setminus B[x,r]$. This implies that $X \setminus B[x,r] = \bigcup_{y \in X \setminus B[x,r]} B(y,r_y)$. Consequently, it follows that $X \setminus B[x,r]$ is open. □

Theorem 3.7. *If $U \in \tau_\mu$ and $x \in U$, then there exists $r > 0$ such that $B(x,r) \subseteq U$.*

Proof. Since U is an open set containing x, there exists an open ball, say $B(y,\epsilon)$ such that $x \in B(y,\epsilon) \subseteq U$. Then $\mu(x,y) < 2m_{xy} - M_{xy} + \epsilon$. Let us choose $0 < r < 2m_{xy} - M_{xy} - \mu(x,y) + \epsilon$ and consider the open ball $B(x,r)$. Then it is easy to verify that $B(x,r) \subseteq B(y,\epsilon) \subseteq U$. □

Theorem 3.8. *(X,τ_μ) is a Hausdorff space.*

Proof. Let (X,μ) be an m-metric space and let x, $y \in X$ with $x \neq y$. Then, $\mu(x,y) - 2m_{xy} + M_{xy} > 0$. Otherwise, $\mu(x,y) - 2m_{xy} + M_{xy} = 0$ which gives that $\mu(x,y) - m_{xy} = 0$ and $M_{xy} - m_{xy} = 0$. Therefore, $\mu(x,y) = m_{xy}$ and $\mid \mu(x,x) - \mu(y,y) \mid = 0$ i.e., $\mu(x,x) = \mu(y,y)$. Thus, we get $\mu(x,y) = m_{xy} = \mu(x,x) = \mu(y,y)$ which implies that $x = y$, a contradiction.

We choose $\epsilon = \frac{1}{3}(\mu(x,y) - 2m_{xy} + M_{xy}) > 0$ and consider the open balls $B(x,\epsilon)$ and $B(y,\epsilon)$. We shall show that $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$. If possible, suppose that $B(x,\epsilon) \cap B(y,\epsilon) \neq \emptyset$. Let $z \in B(x,\epsilon) \cap B(y,\epsilon)$. Then, $\mu(x,z) < 2m_{xz} - M_{xz} + \epsilon$ and $\mu(y,z) < 2m_{yz} - M_{yz} + \epsilon$.

By using $(m4)$ and Remark 2.6, we have

$$\begin{aligned}
3\epsilon &= \mu(x,y) - 2m_{xy} + M_{xy} \\
&= (\mu(x,y) - m_{xy}) + (M_{xy} - m_{xy}) \\
&\leq (\mu(x,z) - m_{xz}) + (\mu(z,y) - m_{zy}) \\
&\quad +(M_{xz} - m_{xz}) + (M_{zy} - m_{zy}) \\
&= (\mu(x,z) - 2m_{xz} + M_{xz}) + (\mu(z,y) - 2m_{zy} + M_{zy}) \\
&< 2\epsilon
\end{aligned}$$

which gives that $3 < 2$, a contradiction. $\square$

Remark 3.9. *Let (X,μ) be an m-metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_μ if and only if $\lim_{n\to\infty}(\mu(x_n,x) - m_{x_nx}) = 0$ and $\lim_{n\to\infty}(M_{x_nx} - m_{x_nx}) = 0$.*

Let $x_n \to x$ w.r.t. τ_μ and $\epsilon > 0$. Then there exists a natural number n_0 such that $x_n \in B(x,\epsilon)$ for all $n \geq n_0$. This gives that $(\mu(x_n,x) - m_{x_nx}) + (M_{x_nx} - m_{x_nx}) < \epsilon$ for all $n \geq n_0$. Since $(\mu(x_n,x) - m_{x_nx}) \geq 0$ and $(M_{x_nx} - m_{x_nx}) \geq 0$, it follows that

$$\mid \mu(x_n,x) - m_{x_nx} \mid < \epsilon \textit{ and } \mid M_{x_nx} - m_{x_nx}) \mid < \epsilon \textit{ for all } n \geq n_0.$$

This proves that

$$\lim_{n\to\infty}(\mu(x_n,x) - m_{x_nx}) = 0 \textit{ and } \lim_{n\to\infty}(M_{x_nx} - m_{x_nx}) = 0.$$

Conversely, suppose that $\lim_{n\to\infty}(\mu(x_n,x) - m_{x_nx}) = 0$ and $\lim_{n\to\infty}(M_{x_nx} - m_{x_nx}) = 0$. We shall show that $x_n \to x$ w.r.t. τ_μ. Let $U \in \tau_\mu$ and $x \in U$. Then there exists $\epsilon > 0$ such that $x \in B(x,\epsilon) \subseteq U$. We note that

$$\mu(x_n,x) - 2m_{x_nx} + M_{x_nx} = (\mu(x_n,x) - m_{x_nx}) + (M_{x_nx} - m_{x_nx}).$$

By hypotheses, it follows that

$$\lim_{n\to\infty}(\mu(x_n,x) - 2m_{x_nx} + M_{x_nx}) = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n,x) - 2m_{x_nx} + M_{x_nx} < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B(x,\epsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_μ on X.

In view of the above remark, we propose the following definitions of convergence of a sequence and m-Cauchy sequence in m-metric spaces instead of that introduced by Asadi et al. [1].

Definition 3.10. *Let (X,μ) be an m-metric space. Then:*

1. *A sequence (x_n) in an m-metric space (X, μ) converges to a point $x \in X$ if (x_n) converges to x w.r.t. τ_μ i.e., if* $\lim_{n\to\infty}(\mu(x_n, x) - m_{x_n x}) = 0$ *and* $\lim_{n\to\infty}(M_{x_n x} - m_{x_n x}) = 0$.
2. *A sequence (x_n) in an m-metric space (X, μ) is called an m-Cauchy sequence if* $\lim_{n,m\to\infty}(\mu(x_n, x_m) - m_{x_n x_m}) = 0$ *and* $\lim_{n,m\to\infty}(M_{x_n x_m} - m_{x_n x_m}) = 0$.
3. *An m-metric space (X, μ) is said to be complete if every m-Cauchy sequence (x_n) in X converges to a point $x \in X$ w.r.t. τ_μ.*

4. Main Results

Definition 4.1. *Let (X, μ) be an m-metric space and $A \subseteq X$. The interior of A, denoted by A^0 or $Int(A)$ is the union of all open sets contained in A. Clearly, $Int(A)$ is always an open set. Moreover, A is open if and only if $A = Int(A)$.*

Definition 4.2. *Let (X, μ) be an m-metric space and $A \subseteq X$. The closure of A, denoted by $\overline{A}$ or $cl(A)$ is the intersection of all closed subsets of X which contains A. Clearly, $cl(A)$ is always a closed set. Moreover, A is closed if and only if $A = \overline{A}$.*

Theorem 4.3. *Let (X, μ) be an m-metric space, τ_μ be the topology defined above and A be any nonempty subset of X. Then,*

(i) *A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;*
(ii) *for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B(x, \epsilon) \cap A \neq \emptyset$.*

Proof. (i) Suppose that A is a closed subset of X. Let (x_n) be a sequence in A such that $x_n \to x$ as $n \to \infty$. We shall show that $x \in A$. If possible, suppose that $x \notin A$. So $x \in X \setminus A$ and $X \setminus A$ is open. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq X \setminus A$. Therefore, $B(x, \epsilon) \cap A = \emptyset$. Since $x_n \to x$ as $n \to \infty$, we have $\lim_{n\to\infty}(\mu(x_n, x) - m_{x_n x}) = 0$ and $\lim_{n\to\infty}(M_{x_n x} - m_{x_n x}) = 0$. Thus, $\lim_{n\to\infty}(\mu(x_n, x) - 2m_{x_n x} + M_{x_n x}) = 0$. So for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n, x) - 2m_{x_n x} + M_{x_n x} < \epsilon$, for all $n \geq n_0$. So, $x_n \in B(x, \epsilon)$, for all $n \geq n_0$. Hence $x_n \in B(x, \epsilon) \cap A$, for all $n \geq n_0$, which leads to a contradiction that $B(x, \epsilon) \cap A = \emptyset$. So, $x \in A$.

Conversely, assume that the condition holds i.e., for any sequence (x_n) in A which converges to x, we have $x \in A$. Let us prove that A is closed. In fact, we have to show that $X \setminus A$ is open. So for any $x \in X \setminus A$, we need to prove that there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq X \setminus A$ i.e., $B(x,\epsilon) \cap A = \emptyset$. If possible, suppose that for any $\epsilon > 0$, we have $B(x,\epsilon) \cap A \neq \emptyset$. So for any $n \geq 1$, choose $x_n \in B(x, \frac{1}{n}) \cap A$. Then $x_n \in A$ for all $n \geq 1$ and $\mu(x_n, x) - 2m_{x_n x} + M_{x_n x} < \frac{1}{n}$ for all $n \geq 1$ i.e., $0 \leq \mu(x_n, x) - m_{x_n x} < \frac{1}{n}$ and $0 \leq M_{x_n x} - m_{x_n x} < \frac{1}{n}$ for all $n \geq 1$. Therefore, $\lim\limits_{n\to\infty} (\mu(x_n, x) - m_{x_n x}) = 0$ and $\lim\limits_{n\to\infty} (M_{x_n x} - m_{x_n x}) = 0$ i.e., $x_n \to x$ as $n \to \infty$ in (X, μ). Hence, by assumption $x \in A$, which is a contradiction. So for any $x \in X \setminus A$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq X \setminus A$ i.e., $X \setminus A$ is open and hence A is closed in X.

(ii) It follows from definition that $\overline{A}$ is the smallest closed subset which contains A. Set
$A^* = \{x \in X : \textit{for any } \epsilon > 0,\ \exists\ a \in A \textit{ such that } \mu(x,a) < 2m_{xa} - M_{xa} + \epsilon\}$. Obviously, $A \subseteq A^*$. Next we prove that A^* is closed. Let (x_n) be a sequence in A^* such that $x_n \to x$ as $n \to \infty$. We have to prove that $x \in A^*$. Since $x_n \to x$ as $n \to \infty$, we have $\lim\limits_{n\to\infty} (\mu(x_n, x) - m_{x_n x}) = 0$ and $\lim\limits_{n\to\infty} (M_{x_n x} - m_{x_n x}) = 0$. Therefore, $\lim\limits_{n\to\infty} (\mu(x_n, x) - 2m_{x_n x} + M_{x_n x}) = 0$.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n, x) - 2m_{x_n x} + M_{x_n x} < \frac{\epsilon}{2}$, for all $n \geq n_0$. As $x_n \in A^*$, there exists $a_n \in A$ such that $\mu(x_n, a_n) < 2m_{x_n a_n} - M_{x_n a_n} + \frac{\epsilon}{2}$. Hence,

$$\begin{aligned}
\mu(x, a_n) - 2m_{xa_n} + M_{xa_n} &\leq (\mu(x, x_n) - m_{xx_n}) + (\mu(x_n, a_n) - m_{x_n a_n}) \\
&\quad + (M_{xx_n} - m_{xx_n}) + (M_{x_n a_n} - m_{x_n a_n}) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,\ \textit{for all } n \geq n_0.
\end{aligned}$$

In particular, $\mu(x, a_{n_0}) - 2m_{xa_{n_0}} + M_{xa_{n_0}} < \epsilon$, which implies that $x \in A^*$. Therefore, by part (i), it follows that A^* is closed and contains A. The definition of $\overline{A}$ assures that $\overline{A} \subseteq A^*$, which implies the conclusion of (ii). □

Theorem 4.4. *Every closed subset of a complete m-metric space is complete.*

Proof. Let (X, μ) be a complete m-metric space and Y be a closed subset of X. Let (y_n) be an m-Cauchy sequence in (Y, μ_Y), where

$\mu_Y : Y \times Y \to \mathbb{R}^+$ is defined by $\mu_Y(u,v) = \mu(u,v)$ for all $u, v \in Y$. Then (y_n) is also an m-Cauchy sequence in (X,μ). As (X,μ) is a complete m-metric space, there exists $x \in X$ such that $y_n \to x$ as $n \to \infty$. By applying Theorem 4.3, it follows that $x \in Y$. Thus (y_n) converges in (Y,μ_Y). So, (Y,μ_Y) becomes a complete m-metric space. $\square$

Theorem 4.5. *$x \in \overline{A}$ iff every open set U containing x intersects A.*

Proof. We shall show that
$x \notin \overline{A} \iff$ there exists an open set U containing x which does not intersect A.
If $x \notin \overline{A}$, then the set $U = X \setminus \overline{A}$ is an open set containing x that does not intersect A, as desired.

Conversely, if there exists an open set U containing x which does not intersect A, then $X \setminus U$ is a closed set containing A. By definition of $\overline{A}$, it must be the case that $\overline{A} \subseteq X \setminus U$. Therefore, x can not be in $\overline{A}$. $\square$

Definition 4.6. *Let (X,μ) be an m-metric space, $A \subseteq X$ and $x \in X$. Then $\mu(x,A)$ is defined as follows:*

$$\mu(x,A) = inf\,\{\mu(x,a) - 2m_{xa} + M_{xa} : a \in A\}.$$

Obviously, $\mu(x,A) \geq 0$ and $\mu(x,A) = 0$ if $x \in A$.

Theorem 4.7. *Let (X,μ) be an m-metric space, $A \subseteq X$ and $x \in X$. Then $\mu(x,A) = 0$ if and only if $x \in \overline{A}$.*

Proof. Let $\mu(x,A) = 0$ and $U \in \tau_\mu$, $x \in U$. Then there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$. Since $\mu(x,A) = 0$, there exists $x_\epsilon \in A$ such that $\mu(x,x_\epsilon) - 2m_{xx_\epsilon} + M_{xx_\epsilon} < \epsilon$. Therefore, $x_\epsilon \in B(x,\epsilon) \subseteq U$ and $x_\epsilon \in A$. Hence, $U \cap A \neq \emptyset$. The last theorem ensures that $x \in \overline{A}$.

Conversely, suppose that $x \in \overline{A}$ and $\epsilon > 0$ is arbitrary. Then, $A \cap B(x,\epsilon) \neq \emptyset$. Let $a \in A \cap B(x,\epsilon)$. So, $\mu(x,A) \leq \mu(x,a) - 2m_{xa} + M_{xa} < \epsilon$, for arbitrary $\epsilon > 0$. Hence $\mu(x,A) = 0$. $\square$

We now prove the regularity property of m-metric spaces.

Theorem 4.8. *Let (X,μ) be an m-metric space. Then for each $x \in X$ and each open neighbourhood U of x, there is an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.*

Proof. Since $U \in \tau_\mu$ and $x \in U$, there exists $r > 0$ such that $B(x,r) \subseteq U$. Let us put $V = B(x, \frac{r}{2})$. Then V is open and $x \in V$. As $B[x, \frac{r}{2}]$ is a closed set containing V, it follows that $\overline{V} \subseteq B[x, \frac{r}{2}] \subseteq B(x,r) \subseteq U$. Thus, $x \in V \subseteq \overline{V} \subseteq U$. □

Next we prove the property of first countability of m-metric spaces.

Theorem 4.9. *Let (X, μ) be an m-metric space and $x \in X$ be arbitrary. Then there exists a countable collection $\{B_n\}_{n=1}^{\infty}$ of open neighbourhoods of x such that for any neighbourhood U of x, there exists $m \in \mathbb{N}$ with $B_m \subseteq U$.*

Proof. For each $n \in \mathbb{N}$, we consider $B_n = B(x, \frac{1}{n})$. Clearly, $\{B_n : n \in \mathbb{N}\}$ is a countable family of open balls centered at x. Let U be any neighbourhood of x. Then there exists $r > 0$ such that $B(x,r) \subseteq U$. We choose $m \in \mathbb{N}$ such that $\frac{1}{m} < r$. Then, $B_m = B(x, \frac{1}{m}) \subseteq B(x,r) \subseteq U$. □

Definition 4.10. *An m-metric space (X, μ) is said to be second countable if it has a countable open base.*

Definition 4.11. *An m-metric space (X, μ) is said to be separable if there exists a countable subset A of X such that $\overline{A} = X$.*

Theorem 4.12. *Every separable m-metric space (X, μ) is second countable.*

Proof. Let (X, μ) be a separable m-metric space. So there exists a countable subset A of X such that $\overline{A} = X$. We consider the collection $\mathbb{B} = \{B(x,r) : x \in A, r \in \mathbb{Q}, r > 0\}$. Then $\mathbb{B}$ becomes a countable collection of open sets in (X, μ). We now show that $\mathbb{B}$ is a base for the topology τ_μ. Let $U \in \tau_\mu$ and $x \in U$. Then there exists $r \in \mathbb{Q}$ with $r > 0$ such that $B(x,r) \subseteq U$. Since $\overline{A} = X$, it follows that $B(x, \frac{r}{2}) \cap A \neq \emptyset$. Suppose that $a \in B(x, \frac{r}{2}) \cap A$. Then $a \in A$ and $a \in B(x, \frac{r}{2})$. Let us put $B = B(a, \frac{r}{2})$. Clearly, $B \in \mathbb{B}$. We prove that $x \in B \subseteq U$. As $a \in B(x, \frac{r}{2})$, we have $\mu(x,a) < 2m_{xa} - M_{xa} + \frac{r}{2}$ which shows that $x \in B(a, \frac{r}{2}) = B$. If $y \in B$, then $\mu(a,y) < 2m_{ay} - M_{ay} + \frac{r}{2}$. Therefore by $(m4)$, we get

$$\begin{aligned}
\mu(x,y) - 2m_{xy} + M_{xy} &\leq (\mu(x,a) - m_{xa}) + (\mu(a,y) - m_{ay}) \\
&\quad +(M_{xa} - m_{xa}) + (M_{ay} - m_{ay}) \\
&< \frac{r}{2} + \frac{r}{2} \\
&= r
\end{aligned}$$

which implies that $y \in B(x, r) \subseteq U$.

Thus for a given open set U and any $x \in U$, there exists $B_x(say) \in \mathbb{B}$ such that $x \in B_x \subseteq U$. As x runs over U, we have $U = \bigcup_{x \in U} B_x$. This shows that every open set U of (X, μ) is expressible as a union of some members of $\mathbb{B}$. Hence $\mathbb{B}$ is a countable open base for τ_μ. This proves that X is second countable. $\square$

Definition 4.13. *Let (X, μ) be an m-metric space and $A \subseteq X$. The diameter of A, denoted by $diam(A)$, is defined by*

$$diam(A) = sup\,\{\mu(x, y) - 2m_{xy} + M_{xy} : x,\, y \in A\}.$$

Clearly, $0 \leq diam(A) \leq \infty$. The subset A is said to be bounded if $diam(A)$ is finite. Otherwise, A is said to be unbounded.

It follows from the above definition that if $A \subseteq B$, then $diam(A) \leq diam(B)$. Hence, it is worth mentioning that $diam(A) \leq diam(\overline{A})$. However, we have the following result.

Theorem 4.14. *Let (X, μ) be an m-metric space and $A \subseteq X$. Then $diam(A) = diam(\overline{A})$.*

Proof. It is sufficient to prove that $diam(\overline{A}) \leq diam(A)$. Let $x,\, y \in \overline{A}$ be arbitrary and $\epsilon > 0$ be given. Then, $B(x, \frac{\epsilon}{2}) \cap A \neq \emptyset$ and $B(y, \frac{\epsilon}{2}) \cap A \neq \emptyset$. Let $x_1 \in B(x, \frac{\epsilon}{2}) \cap A$ and $y_1 \in B(y, \frac{\epsilon}{2}) \cap A$. So, $\mu(x, x_1) - 2m_{xx_1} + M_{xx_1} < \frac{\epsilon}{2}$ and $\mu(y, y_1) - 2m_{yy_1} + M_{yy_1} < \frac{\epsilon}{2}$.

By using $(m4)$ and Remark 2.6, we have

$$\begin{aligned}
\mu(x, y) - 2m_{xy} + M_{xy} &\leq (\mu(x, x_1) - m_{xx_1}) + (\mu(x_1, y) - m_{x_1y}) \\
&\quad +(M_{xx_1} - m_{xx_1}) + (M_{x_1y} - m_{x_1y}) \\
&< \frac{\epsilon}{2} + (\mu(x_1, y_1) - m_{x_1y_1}) + (\mu(y_1, y) - m_{y_1y}) \\
&\quad +(M_{x_1y_1} - m_{x_1y_1}) + (M_{y_1y} - m_{y_1y}) \\
&< \epsilon + (\mu(x_1, y_1) - 2m_{x_1y_1} + M_{x_1y_1}) \\
&\leq \epsilon + diam(A).
\end{aligned}$$

Since $x,\, y \in \overline{A}$ are arbitrary,

$$diam(\overline{A}) \leq \epsilon + diam(A).$$

Since $\epsilon > 0$ is arbitrary, it follows that $diam(\overline{A}) \leq diam(A)$. $\square$

We now prove Cantor's intersection theorem in m-metric spaces.

Theorem 4.15. *An m-metric space (X, μ) is complete if and only if every descending sequence (A_n) of nonempty closed sets with $diam(A_n) \to 0$ as $n \to \infty$, the intersection $A = \bigcap_{n=1}^{\infty} A_n$ consists of exactly one point.*

Proof. Let (X, μ) be a complete m-metric space and (A_n) be a descending sequence of nonempty closed sets with $diam(A_n) \to 0$ as $n \to \infty$. As each A_n is nonempty, we choose a point $x_n \in A_n$, for each $n \in \mathbb{N}$. We shall show that (x_n) is m-Cauchy in (X, μ). For m, $n \in \mathbb{N}$ with $m > n$, we have $A_m \subseteq A_n$ which gives that x_m, $x_n \in A_n$. Therefore,

$$\mu(x_n, x_m) - m_{x_n x_m} \leq \mu(x_n, x_m) - 2m_{x_n x_m} + M_{x_n x_m} \leq diam(A_n) \to 0,$$

as $n \to \infty$, that is, $\lim_{n,m\to\infty} (\mu(x_n, x_m) - m_{x_n x_m}) = 0$.
Moreover,

$$M_{x_n x_m} - m_{x_n x_m} \leq \mu(x_n, x_m) - 2m_{x_n x_m} + M_{x_n x_m} \leq diam(A_n) \to 0,$$

as $n \to \infty$, that is, $\lim_{n,m\to\infty} (M_{x_n x_m} - m_{x_n x_m}) = 0$.

This shows that (x_n) is an m-Cauchy sequence in (X, μ). Then by hypothesis, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$ i.e., $\lim_{n\to\infty} (\mu(x_n, x) - m_{x_n x}) = 0$ and $\lim_{n\to\infty} (M_{x_n x} - m_{x_n x}) = 0$ which imply that $\lim_{n\to\infty} (\mu(x_n, x) - 2m_{x_n x} + M_{x_n x}) = 0$. We prove that $x \in \bigcap_{n=1}^{\infty} A_n$. Let $U \in \tau_\mu$ and $x \in U$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. As $\lim_{n\to\infty} (\mu(x_n, x) - 2m_{x_n x} + M_{x_n x}) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n, x) - 2m_{x_n x} + M_{x_n x} < \epsilon$, for all $n \geq n_0$. Therefore, $x_n \in B(x, \epsilon) \subseteq U$, for all $n \geq n_0$. Again, $x_m \in A_n$, for all $m \geq n$ as $x_m \in A_m \subseteq A_n$, for all $m \geq n$. So, $U \cap A_n \neq \emptyset$, for all $n \in \mathbb{N}$. This proves that $x \in \overline{A_n} = A_n$, $\forall$ n, A_n being closed. Hence $x \in \bigcap_{n=1}^{\infty} A_n$.

Now, let $y \in \bigcap_{n=1}^{\infty} A_n$ with $y \neq x$. Then for each $n \in \mathbb{N}$, we have x, $y \in A_n$. Therefore,

$$0 \leq \mu(x, y) - 2m_{xy} + M_{xy} \leq diam(A_n) \to 0 \text{ as } n \to \infty$$

which gives that $\mu(x, y) - 2m_{xy} + M_{xy} = 0$. Since $\mu(x, y) - m_{xy} \geq 0$ and $M_{xy} - m_{xy} =| \mu(x, x) - \mu(y, y) |\geq 0$, it follows that $\mu(x, y) - m_{xy} = 0$

and $M_{xy} - m_{xy} = \mid \mu(x,x) - \mu(y,y) \mid = 0$. This shows that $\mu(x,x) = \mu(y,y) = m_{xy} = \mu(x,y)$ and so, $x = y$, a contradiction. This proves that A contains exactly one point.

Conversely, suppose that the given condition holds. Let (x_n) be an m-Cauchy sequence in (X, μ). For each $n \in \mathbb{N}$, we define

$$A_n = range\ of\ the\ sequence\ \{x_n,\, x_{n+1},\, \cdots\}.$$

Then, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ and so $\overline{A_1} \supseteq \overline{A_2} \supseteq \overline{A_3} \supseteq \cdots$. As (x_n) is m-Cauchy, we have $\lim\limits_{n,m\to\infty} (\mu(x_n, x_m) - m_{x_n x_m}) = 0$ and $\lim\limits_{n,m\to\infty} (M_{x_n x_m} - m_{x_n x_m}) = 0$. Therefore, $\lim\limits_{n,m\to\infty} (\mu(x_n, x_m) - 2m_{x_n x_m} + M_{x_n x_m}) = 0$. This implies that $diam(A_n) \to 0$ as $n \to \infty$ and hence $diam(\overline{A_n}) \to 0$ as $n \to \infty$. Our assumption ensures that $\bigcap\limits_{n=1}^{\infty} \overline{A_n}$ consists of exactly one point x, say in X. Since x, $x_n \in \overline{A_n}$, we have

$$\mu(x_n, x) - 2m_{x_n x} + M_{x_n x} \leq diam(\overline{A_n}) \to 0\ as\ n \to \infty.$$

Consequently, it follows that $\lim\limits_{n\to\infty} (\mu(x_n, x) - m_{x_n x}) = 0$ and $\lim\limits_{n\to\infty} (M_{x_n x} - m_{x_n x}) = 0$. Hence (x_n) converges to x, proving that (X, μ) is complete. □

Definition 4.16. *A subset A of an m-metric space (X, μ) is said to be nowhere dense if $Int(\overline{A}) = \emptyset$.*

Theorem 4.17. *A subset A of an m-metric space (X, μ) is nowhere dense if and only if every nonempty open set U contains a nonempty open set V such that $V \cap A = \emptyset$.*

Proof. Suppose that A is nowhere dense and U is any nonempty open set in X. Since $Int(\overline{A}) = \emptyset$, $U \nsubseteq \overline{A}$ and hence $U \cap (X \setminus \overline{A}) \neq \emptyset$. Let us put $V = U \cap (X \setminus \overline{A})$. Then V is a nonempty open set contained in U such that $V \cap A = \emptyset$.

Conversely, let U be any nonempty open set in X. By hypothesis, there exists a nonempty open set V such that $V \subseteq U$ and $V \cap A = \emptyset$. Therefore, $A \subseteq X \setminus V$ which implies that $\overline{A} \subseteq \overline{X \setminus V} = X \setminus V$ and hence $V \subseteq X \setminus \overline{A}$.

Now, $(X \setminus \overline{A}) \cap U \supseteq U \cap V = V \neq \emptyset$. This shows that $U \nsubseteq \overline{A}$. Therefore, $\overline{A}$ contains no nonempty open set and so $Int(\overline{A}) = \emptyset$. Consequently, A becomes nowhere dense. □

Definition 4.18. *A subset A of an m-metric space (X, μ) is said to be*

(i) *a set of the first category if A is expressible as a union of countably many nowhere dense sets.*
(ii) *a set of the second category if it is not a set of the first category.*

We now prove analogue of Baire's category theorem in m-metric spaces.

Theorem 4.19. *Every complete m-metric space (X, μ) is a set of second category.*

Proof. Let (X, μ) be a complete m-metric space and Y be a set of first category in X. It is sufficient to show that $Y \neq X$. As Y is a set of the first category, we can write $Y = \bigcup_{n=1}^{\infty} P_n$, where each P_n is nowhere dense in X. Let U be any nonempty open set in X. Since P_1 is nowhere dense, there exists a nonempty open set V such that $V \subseteq U$ and $V \cap P_1 = \emptyset$. As $V \neq \emptyset$, there exists $p_1 \in V$ and then by regularity property, there exists r_1 with $0 < r_1 < 1$ such that $B(p_1, r_1) = U_1(say) \subseteq \overline{U_1} \subseteq V$. Again, P_2 being nowhere dense and $U_1 \neq \emptyset$, there exists a nonempty open set $V_1 \subseteq U_1$ such that $V_1 \cap P_2 = \emptyset$. V_1 being nonempty, there exists $p_2 \in V_1$ and a positive real number $r_2 < \frac{1}{2}$ such that $B(p_2, r_2) = U_2(say) \subseteq \overline{U_2} \subseteq V_1$.

Now, $\overline{U_2} \subseteq V_1 \subseteq U_1 \subseteq \overline{U_1} \Longrightarrow \overline{U_2} \subseteq \overline{U_1}$ and $diam(\overline{U_1}) = diam(U_1) \leq 2r_1 < 2$, $diam(\overline{U_2}) = diam(U_2) \leq 2r_2 < 2.\frac{1}{2}$. Moreover, $\overline{U_1} \cap P_1 \subseteq V \cap P_1 = \emptyset$, $\overline{U_2} \cap P_2 \subseteq V_1 \cap P_2 = \emptyset$. Proceeding in this way, we obtain a descending sequence $(\overline{U_n})$ of nonempty closed sets such that $diam(\overline{U_n}) < 2.\frac{1}{2^{n-1}} \to 0$ as $n \to \infty$. Since (X, μ) is complete, by Cantor's intersection theorem, $\exists\, q \in \bigcap_{n=1}^{\infty} \overline{U_n}$. As $\overline{U_n} \cap P_n = \emptyset$ for each n, $q \notin \bigcup_{n=1}^{\infty} P_n = Y$ and so $Y \neq X$. □

Definition 4.20. *Let (X, μ_1) and (Y, μ_2) be two m-metric spaces. A function $f : (X, \mu_1) \to (Y, \mu_2)$ is said to be continuous at a point $a \in X$, if corresponding to every $\epsilon > 0$, $\exists\, \delta > 0$ such that*

$$x \in B_{\mu_1}(a, \delta) \Longrightarrow f(x) \in B_{\mu_2}(f(a), \epsilon).$$

f is said to be continuous on X if it is continuous at each point of X.

Obviously, the concept of continuity of a real valued function on an m-metric space turns out to be a special case of the above definition by considering $Y = \mathbb{R}$ and $\mu_2(y, z) =\mid y - z \mid$ for all y, $z \in \mathbb{R}$. For such real valued functions on an m-metric space, we can prove the following theorem, as exact duplicates of the corresponding proofs for real valued continuous functions on a metric space.

Theorem 4.21. *Let f and g be real valued functions on an m-metric space (X, μ). If f and g are continuous at a point $a \in X$ and $g(x) \neq 0$ for all $x \in X$, then so are $f \pm g$, fg, αf (for any $\alpha \in \mathbb{R}$) and $\frac{f}{g}$.*

Theorem 4.22. *Let (X, μ_1) and (Y, μ_2) be two m-metric spaces. Then a function $f : (X, \mu_1) \to (Y, \mu_2)$ is continuous at a point $a \in X$ if and only if for each sequence (x_n) in X converging to a in (X, μ_1), the sequence $(f(x_n))$ in Y converges to $f(a)$ in (Y, μ_2).*

Proof. Suppose that f is continuous at $a \in X$. Then for a given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$x \in B_{\mu_1}(a, \delta) \Longrightarrow f(x) \in B_{\mu_2}(f(a), \epsilon).$$

Since (x_n) converges to a in (X, μ_1), we have $\lim\limits_{n\to\infty} (\mu_1(x_n, a) - m_{x_n a}) = 0$ and $\lim\limits_{n\to\infty} (M_{x_n a} - m_{x_n a}) = 0$. Therefore, $\lim\limits_{n\to\infty} (\mu_1(x_n, a) - 2m_{x_n a} + M_{x_n a}) = 0$. So, there exists $n_0 \in \mathbb{N}$ such that $\mu_1(x_n, a) - 2m_{x_n a} + M_{x_n a} < \delta$, for all $n \geq n_0$. This shows that $x_n \in B_{\mu_1}(a, \delta)$, for all $n \geq n_0$. By hypothesis, it follows that $f(x_n) \in B_{\mu_2}(f(a), \epsilon)$, for all $n \geq n_0$. Then $\mu_2(f(x_n), f(a)) - 2m_{f(x_n)f(a)} + M_{f(x_n)f(a)} < \epsilon$, for all $n \geq n_0$. This gives that $\mu_2(f(x_n), f(a)) - m_{f(x_n)f(a)} < \epsilon$ and $M_{f(x_n)f(a)} - m_{f(x_n)f(a)} < \epsilon$, for all $n \geq n_0$. Consequently, it follows that $\lim\limits_{n\to\infty} (\mu_2(f(x_n), f(a)) - m_{f(x_n)f(a)}) = 0$ and $\lim\limits_{n\to\infty} (M_{f(x_n)f(a)} - m_{f(x_n)f(a)}) = 0$. Therefore, $(f(x_n))$ converges to $f(a)$ in (Y, μ_2).

Conversely, suppose the condition holds but f is not continuous at $a \in X$. Then there exists $\epsilon > 0$ such that for each $\delta > 0$, $\exists\ x_\delta \in X$ with $x_\delta \in B_{\mu_1}(a, \delta)$ but $f(x_\delta) \notin B_{\mu_2}(f(a), \epsilon)$. In particular, for each $n \in \mathbb{N}$, $\exists\ x_n \in X$ with $x_n \in B_{\mu_1}(a, \frac{1}{n})$ but $f(x_n) \notin B_{\mu_2}(f(a), \epsilon)$. It then follows that $x_n \to a$ but $\mu_2(f(x_n), f(a)) - 2m_{f(x_n)f(a)} + M_{f(x_n)f(a)} \geq \epsilon$, for all $n \in \mathbb{N}$ i.e., $(f(x_n))$ does not converge to $f(a)$ in (Y, μ_2). This contradicts the assumed hypothesis. $\square$

Theorem 4.23. *Let A be a subset of an m-metric space (X, μ). Then the function $f : (X, \mu) \to \mathbb{R}$, defined by $f(x) = \mu(x, A)$, is a continuous function.*

Proof. Let $x_0 \in X$ be arbitrary and $\epsilon > 0$ be given. We shall show that

$$x \in B_\mu(x_0, \frac{\epsilon}{2}) \Longrightarrow \mid \mu(x, A) - \mu(x_0, A) \mid < \epsilon.$$

Suppose $x \in B_\mu(x_0, \frac{\epsilon}{2})$ and $a \in A$ be arbitrary. Then, $\mu(x_0, x) - 2m_{x_0x} + M_{x_0x} < \frac{\epsilon}{2}$. By using $(m4)$ and Remark 2.6, we have

$$\begin{aligned} \mu(x,a) - 2m_{xa} + M_{xa} &\leq (\mu(x,x_0) - m_{xx_0}) + (\mu(x_0,a) - m_{x_0a}) \\ &\quad + (M_{xx_0} - m_{xx_0}) + (M_{x_0a} - m_{x_0a}) \\ &< \frac{\epsilon}{2} + (\mu(x_0,a) - 2m_{x_0a} + M_{x_0a}). \end{aligned}$$

This gives that

$$\inf_{a\in A} \{\mu(x,a) - 2m_{xa} + M_{xa}\} \leq \frac{\epsilon}{2} + \inf_{a\in A} \{\mu(x_0,a) - 2m_{x_0a} + M_{x_0a}\}.$$

i.e., $\mu(x, A) \leq \frac{\epsilon}{2} + \mu(x_0, A)$. Interchanging the roles of x and x_0, we get $\mu(x_0, A) \leq \frac{\epsilon}{2} + \mu(x, A)$. Thus, we have $\mid \mu(x, A) - \mu(x_0, A) \mid \leq \frac{\epsilon}{2} < \epsilon$. □

We now present Urysohn's lemma in m-metric spaces.

Theorem 4.24. *For any two nonempty disjoint closed subsets U, V of an m-metric space (X, μ), there exists a continuous function $f : X \to \mathbb{R}$ such that $f(U) = \{0\}$, $f(V) = \{1\}$ and $0 \leq f(x) \leq 1$ for all $x \in X$.*

Proof. For $A \subseteq X$ and $x \in X$, we use the notation $\mu_A(x)$ for the function $\mu(x, A)$. We now show that the function $f(x) = \frac{\mu_U(x)}{\mu_U(x)+\mu_V(x)}$ is the desired function. If $\mu_U(x) + \mu_V(x) = 0$ for some $x \in X$, then $\mu_U(x) = \mu_V(x) = 0$ and hence $x \in \overline{U} = U$ and $x \in \overline{V} = V$, which contradicts the fact that $U \cap V = \emptyset$. Therefore, f is well defined. Moreover, it follows from Theorems 4.21 and 4.23 that f is continuous. Obviously, $0 \leq f(x) \leq 1$ for all $x \in X$. Now, $x \in U \Longrightarrow \mu_U(x) = 0 \Longrightarrow f(x) = 0$ and $x \in V \Longrightarrow \mu_V(x) = 0 \Longrightarrow f(x) = 1$. □

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