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A STRONG CONVERGENCE OF A MODIFIED KRASNOSELSKII-MANN ALGORITHM FOR A FINITE FAMILY OF DEMICONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we propose an iterative algorithm, which is based on the Krasnoselskii-Mann iterative algorithm for fixed point problems of a finite family of demicontractive mappings in the setting of real Banach spaces. We prove that the sequence generated by the proposed method converges strongly to a common fixed point of a finite family of demicontractive mappings which is also the solution of a variational inequality. The iterative algorithm and results presented in this paper generalize, unify and improve some previously known results of this area.

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1. Introduction

Let H be a real Hilbert space, K be a nonempty subset of H. A map $T : K \to K$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in K, \tag{1.1}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive. We denote by $Fix(T)$ the set of fixed points of the mapping T, that is $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. If T is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex (see, e.g., [3]). A map T is called quasi-nonexpansive if $\|Tx-p\| \leq \|x-p\|$ holds for all x in K and $p \in Fix(T)$. The mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

A mapping $T : K \to H$ is called k-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \ \forall x, y \in K.$$

If this inequality holds for $k = 1$ then T is called simply pseudocontractive.

A map T is called k-demi-contractive if $Fix(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \ \forall x \in K, \ \ p \in Fix(T). \tag{1.2}$$

We note that the following inclusions hold for the classes of the mappings:

firmly nonexpansive $\subset$ nonexpansive $\subset$ quasi-nonexpansive $\subset$ k-quasi-strictly pseudo-contractive $\subset$ k-demicontractive.

The following example shows that there exists a k-demi-contractive mapping which is not k-strictly pseudo-contractive mapping.

Example 1.1. Let $H = \mathbb{R}$ and $K = [-1, 1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \dfrac{2}{3}x\sin(\frac{1}{x}), & x \neq 0 \\ 0 \quad x = 0. \end{cases} \tag{1.3}$$

Clearly $Fix(T) = \{0\}$. For $x \in K$, we have

$$\begin{aligned} |Tx - 0|^2 &= |\frac{2}{3}x\sin(\frac{1}{x})|^2 \\ &\leq |\frac{2}{3}x|^2 \\ &\leq |x|^2 \\ &\leq |x-0|^2 + k|x - Tx|^2 \;\; \forall k \in [0,1). \end{aligned}$$

Thus T is k demi-contratcive for $k \in [0, 1)$. To see that T is not k strictly pseudo-contractive, choose $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, then

$$|Tx - Ty|^2 > |x - y|^2 + k|x - y - (Tx - Ty)|^2.$$

Hence, T is not k strictly pseudo-contractive mapping for $k \in [0, 1)$.

Example 1.2. (Example of a Demicontractive Function which is not Quasi-nonexpansive and is not Pseudocontractive). Let f be a real function defined by $f(x) = -x^2 - x$; it can be seen that $f : [-2, 1] \to [-2, 1]$. This function is demicontractive on $[-2, 1]$ and continuous. It is not quasi-nonexpansive and is not pseudocontractive on $[-2, 1]$ (check for instance the condition of pseudocontractivity for $x = -1.5$ and $y = -0.6$).

For nonexpansive mappings with fixed points, Mann iterative method [11] is a valuable tool to study them. However, only weak convergence is guaranteed in infinite dimensional spaces. Thus a natural question rises: could we obtain a strong convergence result by using the well-known Krasnoselskii-Mann method for non-expansive mappings? In this connection, in 1975, Genel and Lindenstrauss [7] gave a counterexample. Hence the modification is necessary in order to guarantee the strong convergence of Krasnoselskii-Mann's method. Lot of works have been done for the modification of the normal Mann's iteration so that strong convergence is guaranteed. See, e.g., [12, 13, 17, 9, 8] and the reference therein.

In 2010, Yonghong Yao and Yeol Je Cho [16], motivated by the fact that Krasnoselskii-Mann algorithm method is remarkably useful for finding fixed points of single-valued nonexpansive mapping, proved the following theorem.

Theorem 1.3 (Yonghong Yao and Yeol Je Cho [16]). *Let H be a real Hilbert space $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$.*

Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in H$ by:

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n. \tag{1.4}$$

Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\quad (ii)$ $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$

(iii) $\lim_{n\to\infty} \lambda_n = 1$, $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$, *and* $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$

Then, the sequence $\{x_n\}$ generated by (1.4) converges strongly to $x^* \in Fix(T)$.

Let E be a Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $p \in E^*$, $\langle p, x\rangle$ is used to refer to $p(x)$. Let $\varphi : [0,+\infty) \to [0,\infty)$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{p \in E^* : \langle x, p\rangle = ||x||\varphi(||x||), ||p|| = \varphi(||x||)\}. \tag{1.5}$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{p \in E^* : \langle x, p\rangle = ||x||^2 = ||p||^2\}, \forall\, x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), \;\; x \neq 0.$$

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.
Let E be a normed space with dimE ≥ 2. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth.* Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p) \ or \ W_p^m \text{ is}$$

$$\begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases} \tag{1.6}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}.$$

J_2 is called the *normalized duality mapping* and is denoted by J. It is known that E is smooth if and only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E. Following Browder [2], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is single-valued and is weak-to-weak* sequentially continuous, i.e., if $(x_n) \subset E$, $x_n \stackrel{w}{\to} x$, then $J_\varphi(x_n) \stackrel{w^*}{\longrightarrow} J_\varphi(x)$. It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see e.g., [4] for more details on duality maps).

Remark 1.4. Note also that a duality mapping exists in each Banach space. We recall from [1] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.

(i) $l_p : \ Jx = \|x\|_{l_p}^{2-p} y \in l_q, \ \ x = (x_1, x_2, \cdots, x_n, \cdots),$
$y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots),$

(ii) $L_p : \ Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q,$

(iii) $W^{m,p} \ : \ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha \Big(|D^\alpha u|^{p-2} D^\alpha u \Big) \in W^{-m,q},$
where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

In [6], Chidume extended the condition (1.2) to arbitrary real Banach spaces X. If X is q-uniformly smooth, then the condition (1.2) becomes

$$(1.7)\quad \langle x-Tx, j_q(x-p)\rangle \geq \frac{(1-k)^{q-1}}{2^{q-1}}\|x-Tx\|^q,\ \ x\in X,\ \ p\in Fix(T).$$

Recently, Sow et al. [14] extended Theorem 1.3 from Hilbert spaces to Banach spaces, by proving the following theorem.

Theorem 1.5 (Sow et al. [14])**.** *Let E be a uniformly smooth real Banach space having a weakly continuous duality map and K a nonempty, closed and convex cone of E. Let $T: K\to K$ be a nonexpansive mapping with $Fix(T)\neq\emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0\in K$ by:*

$$(1.8)\qquad x_{n+1}=\alpha_n(\lambda_n x_n)+(1-\alpha_n)Tx_n.$$

Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty}\alpha_n=0$; *(ii)* $\sum\limits_{n=0}^{\infty}|\alpha_n-\alpha_{n+1}|<\infty$,

(iii) $\lim\limits_{n\to\infty}\lambda_n=1$, $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n=\infty$, *and* $\sum\limits_{n=0}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty$.

Then, the sequence $\{x_n\}$ generated by (1.8) converges strongly to $x^*\in Fix(T)$.

In this paper, motivated by above results, the fact that the class of demicontractive mappings properly includes that of quasi-nonexpansive, strictly pseudocontractive mappings and Krasnoselskii-Mann algorithm is remarkably useful for solving fixed point problems, we construct and study an explicit iterative method and prove strong convergence theorems by using the Krasnoselskii-Mann iteration for approximating a common fixed points of a finite family of demicontractive mappings in the setting of a real Banach space without any compactness assumption. Our technique of proof is of independent interest.

2. Preliminaries

Let C be a nonempty subsets of a smooth real Banach space E. A mapping $Q_C: E\to C$ is said to be sunny if

$$Q_C(Q_Cx+t(x-Q_Cx))=Q_Cx$$

for each $x\in E$ and $t\geq 0$. A mapping $Q_C: E\to C$ is said to be a retraction if $Q_Cx=x$ for each $x\in C$.

Lemma 2.1. [8] *Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z)\rangle \leq 0$$

for all $z \in C$ and $y \in D$.

It is noted that Lemma 2.1 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

Remark 2.2. If K is a nonempty, closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.

Lemma 2.3 ([12], Proposition 2.1)**.** *Assume K is a closed convex subset of a Hilbert space H. Let $T : K \to K$ be a self-mapping of K. If T is a k-demicontractive mapping, then the fixed point set $Fix(T)$ is closed and convex.*

Theorem 2.4. [5] *Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) E is q-uniformly smooth.
(ii) There is a constant $d_q > 0$ such that for all $x, y \in E$

$$\|x + y\|^q \leq \|x\|^q + q\langle y\ ,\ J_q(x)\rangle + d_q\|y\|^q.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x - y\ ,\ J_q(x) - J_q(y)\rangle\ \leq c_1\|x - y\|^q\ \ \forall\ \ x, y \in E.$$

Lemma 2.5. [12] *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*
(i) If T is a k-strictly pseudo-contractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|.$$

(ii) If T is a k-quasi-strictly pseudo-contractive mapping, then the mapping $I - T$ is demiclosed at 0.

Lemma 2.6 ([10])**.** *Assume that a Banach space E has a weakly continous duality mapping J_φ with gauge φ.*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle. \tag{2.1}$$

For all $x, y \in E$, *where* $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma,\ t \geq 0$. *In particular,for the normilized duality mapping, we have the important special version of* (2.1)

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle,$$

for all $x, y \in E$.

Lemma 2.7 (Xu, [15]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$ *for all* $n \geq 0$, *where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}$ *is a sequence in* $\mathbb{R}$ *such that* (*a*) $\sum_{n=0}^{\infty}\alpha_n = \infty$, (*b*) $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\sigma_n\alpha_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

We now prove our main results.

Theorem 3.1. *Let* $q > 1$ *be a fixed real number and* E *be a* q*-uniformly smooth real Banach space having a weakly continuous duality map* J_φ *and* K *be a nonempty, closed convex cone of* E. *Let* $m \geq 1$ *be a fixed number, for* $i, 1 \leq i \leq m$, $T_i : K \to K$ *be a* k_i*-demicontractive mapping such that* $\Gamma := \bigcap_{i=1}^{m} Fix(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined iteratively from arbitrary* $x_0 \in K$ *by:*

$$\begin{cases} y_n = \lambda_0 x_n + \lambda_1 T_1 x_n + \cdots + \lambda_m T_m x_n, \\ \\ x_{n+1} = \alpha_n(\theta_n x_n) + (1-\alpha_n)y_n, \end{cases} \tag{3.1}$$

where $\lambda_i \in (0, \gamma)$,

$$\gamma := \min_{1\leq i\leq m}\Big\{1, \Big(\frac{q\beta_i^{q-1}}{2^{(m-1)q}d_q}\Big)^{\frac{1}{q-1}}\Big\}, \ \ \textit{with}\ \ \beta_i = \frac{1-k_i}{2}.$$

Suppose the following conditions hold:
$\{\theta_n\}$ *and* $\{\alpha_n\}$ *be sequences in* $(0,1)$ *satisfying:*
(*i*) $\lim_{n\to\infty}\alpha_n = 0$; (*ii*) $\sum_{n=0}^{\infty}(1-\theta_n)\alpha_n = \infty$, $\sum_{i=0}^{m}\lambda_i = 1$,
(*iii*) $\lim_{n\to\infty}\theta_n = 1$.
Assume that $I - T_i$ *is demiclosed at the origin.*
Then, the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $x^* \in$

Γ, *where* $x^* = Q_\Gamma(0)$ *with* Q_Γ *the sunny nonexpansive retraction of* K *onto* Γ.

Proof. We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Let $p \in \Gamma$. Using (3.1), inequality (ii) of Theorem 2.4 and inequality (1.7), we have

$$\begin{aligned}
\|y_n - p\|^q &= \Big\|\lambda_0(x_n - p) + \sum_{i=1}^{m}\lambda_i(T_ix_n - p)\Big\|^q \\
&= \Big\|\lambda_0(x_n - p) + \sum_{i=1}^{m}\lambda_i(T_ix_n - x_n) + \sum_{i=1}^{m}\lambda_i(x_n - p)\Big\|^q \\
&= \Big\|x_n - p + \sum_{i=1}^{m}\lambda_i(T_ix_n - x_n)\Big\|^q.
\end{aligned}$$

Hence,

(3.2)
$$\|y_n-p\|^q \le \|x_n-p\|^q - q\sum_{i=1}^{m}\lambda_i\beta_i^{q-1}\|x_n-T_ix_n\|^q + d_q\Big\|\sum_{i=1}^{m}\lambda_i(T_ix_n-x_n)\Big\|^q.$$

Therefore,

$$\Big\|\sum_{i=1}^{m}\lambda_i(T_ix_n - x_n)\Big\|^q \le 2^{(m-1)q}\sum_{i=1}^{m}\lambda_i^q\Big\|T_ix_n - x_n\Big\|^q. \tag{3.3}$$

Combining inequalities (3.2) and (3.3), it then follows that :

$$\begin{aligned}
\Big\|y_n - p\Big\|^q &\le \Big\|x_n - p\Big\|^q - q\sum_{i=1}^{m}\lambda_i\beta_i^{q-1}\Big\|x_n - T_ix_n\Big\|^q + \\
&+ d_q2^{(m-1)q}\sum_{i=1}^{m}\lambda_i^q\Big\|T_ix_n - x_n\Big\|^q. \\
&= \Big\|x_n - p\Big\|^q - \sum_{i=1}^{m}\lambda_i\Big[q\beta_i^{q-1} - 2^{(m-1)q}d_q\lambda_i^{q-1}\Big]\Big\|x_n - T_ix_n\Big\|^q. \qquad (3.4)
\end{aligned}$$

Since $q\beta_i^{q-1} - 2^{(m-1)q}d_q\lambda_i^{q-1} > 0\ \ \forall\, i = 1, \cdots, m$, we obtain,

$$\|y_n - p\Big\| \le \|x_n - p\Big\|. \tag{3.5}$$

By inequality (3.5) and (3.1), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(\theta_n x_n)+(1-\alpha_n)y_n-p\| \\ &\leq \alpha_n\theta_n\|x_n-p\|+(1-\alpha_n)\|y_n-p\|+(1-\theta_n)\alpha_n\|p\| \\ &\leq \alpha_n\theta_n\|x_n-p\|+(1-\alpha_n)\|x_n-p\|+(1-\theta_n)\alpha_n\|p\| \\ &\leq [1-(1-\theta_n)\alpha_n]\|x_n-p\|+(1-\theta_n)\alpha_n\|p\| \\ &\leq \max\{\|x_n-p\|,\ \|p\|\}.\end{aligned}$$

By induction, it is easy to see that

$$\|x_n-p\|\leq \max\{\|x_0-p\|,\ \|p\|\},\quad n\geq 1.$$

Hence $\{x_n\}$ is bounded and $\{y_n\}$ is also bounded.
Consequently, using inequality (3.4), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^q = \|\alpha_n(\theta_n x_n)+(1-\alpha_n)y_n-p\|^q &= \|y_n-p+\alpha_n((\theta_n x_n)-y_n)\|^q \\ &\leq \|y_n-p\|^q+q\alpha_n\langle(\theta_n x_n)-y_n, J_q(y_n-p)\rangle+d_q\Big\|\alpha_n((\theta_n x_n)-y_n)\Big\|^q \\ &\leq \|y_n-p\|^q+q\alpha_n\|(\theta_n x_n)-y_n\|\|y_n-p\|^{q-1}+d_q\alpha_n^q\Big\|(\theta_n x_n)-y_n\Big\|^q \\ &\leq \Big\|x_n-p\Big\|^q-\sum_{i=1}^m\lambda_i\Big[q\beta_i^{q-1}-2^{(m-1)q}d_q\lambda_i^{q-1}\Big]\Big\|x_n-T_ix_n\Big\|^q \\ &\quad +q\alpha_n\|(\theta_n x_n)-y_n\|\|y_n-p\|^{q-1}+d_q\alpha_n^q\Big\|(\theta_n x_n)-y_n\Big\|^q.\end{aligned}$$

Thus, for every $i, 1\leq i\leq m$, we get

$$\begin{aligned}\sum_{i=1}^m\lambda_i\Big[q\beta_i^{q-1}-2^{(m-1)q}d_q\lambda_i^{q-1}\Big]\Big\|x_n-T_ix_n\Big\|^q &\leq \Big\|x_n-p\Big\|^q-\Big\|x_{n+1}-p\Big\|^q \\ &\quad +q\alpha_n\|(\theta_n x_n)-y_n\|\|y_n-p\|^{q-1} \\ &\quad +d_q\alpha_n^q\|(\theta_n x_n)-y_n\|^q.\end{aligned}$$

Since $\{y_n\}$ and $\{(\theta_n x_n)\}$ are bounded, then there exists a constant $C>0$ such that for every $i, 1\leq i\leq m$,
(3.6)

$$\sum_{i=1}^m\lambda_i\Big[q\beta_i^{q-1}-2^{(m-1)q}d_q\lambda_i^{q-1}\Big]\|x_n-T_ix_n\|^q\leq \|x_n-p\|^q-\|x_{n+1}-p\|^q+\alpha_n C.$$

Now we prove that $\{x_n\}$ converges strongly to x^*.
We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n-p\|\}$ is monotonically decreasing. Then $\{\|x_n-p\|\}$ is convergent. Clearly, we have

$$\|x_n-p\|^q-\|x_{n+1}-p\|^q\to 0.$$

It then implies from (3.6) that

$$\lim_{n\to\infty}\sum_{i=1}^{m}\lambda_i\Big[q\beta_i^{q-1}-2^{(m-1)q}d_q\lambda_i^{q-1}\Big]\Big\|x_n-T_ix_n\Big\|^q=0. \tag{3.7}$$

Since $q\beta_i^{q-1}-2^{(m-1)q}d_q\lambda_i^{q-1}>0\ \forall\, i=1,\cdots,m$, we have

$$\lim_{n\to\infty}\Big\|x_n-T_ix_n\Big\|=0. \tag{3.8}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^*-x_n)\rangle\le 0$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^*-x_n)\rangle=\lim_{k\to+\infty}\langle x^*, J_\varphi(x^*-x_{n_k})\rangle.$$

From (3.8), taking into account that $I-T_i$ is demiclosed, we obtain $a\in\Gamma$. On other hand, by the assumption that the duality mapping J_φ is weakly continuous, the fact that $x^*=Q_\Gamma(0)$ and Lemma 2.1, we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^*-x_n)\rangle &= \lim_{k\to+\infty}\langle x^*, J_\varphi(x^*-x_{n_k})\rangle\\ &= \langle x^*, J_\varphi(x^*-a)\rangle\le 0.\end{aligned}$$

Finally, we show that $x_n\to x^*$. In fact, since $\Phi(t)=\int_0^t\varphi(\sigma)d\sigma,\ \forall t\ge 0$, and φ is a gauge function, then for $1\ge k\ge 0,\ \Phi(kt)\le k\Phi(t)$. From (3.1) and Lemma 2.6, we get that

$$\begin{aligned}\Phi(\|x_{n+1}-x^*\|) &= \Phi(\|\alpha_n(\theta_nx_n)+(1-\alpha_n)y_n-x^*\|)\\ &\le \Phi(\|\alpha_n\theta_n(x_n-x^*)+(1-\alpha_n)(y_n-x^*)\|)\\ &\quad +(1-\theta_n)\alpha_n\langle x^*, J_\varphi(x^*-x_{n+1})\rangle\\ &\le \Phi(\alpha_n\theta_n\|x_n-x^*\|+\|(1-\alpha_n)(y_n-x^*)\|)\\ &\quad +(1-\theta_n)\alpha_n\langle x^*, J_\varphi(x^*-x_{n+1})\rangle\\ &\le \Phi(\alpha_n\theta_n\|x_n-x^*\|+(1-\alpha_n)\|x_n-x^*\|)\\ &\quad +(1-\theta_n)\alpha_n\langle x^*, J_\varphi(x^*-x_{n+1})\rangle\\ &\le \Phi((1-(1-\theta_n)\alpha_n)\|x_n-x^*\|)+(1-\theta_n)\alpha_n\langle x^*, J_\varphi(x^*-x_{n+1})\rangle\\ &\le [1-(1-\theta_n)\alpha_n]\Phi(\|x_n-x^*\|)+(1-\theta_n)\alpha_n\langle x^*, J_\varphi(x^*-x_{n+1})\rangle.\end{aligned}$$

From Lemma 2.7, it follows that $x_n\to x^*$.

Case 2. Assume that the sequence $\{\|x_n-x^*\|\}$ is not monotonically decreasing sequence. Set $B_n=\|x_n-x^*\|$ and $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\ge n_0$ (for some n_0 large enough) by $\tau(n)=\max\{k\in\mathbb{N}: k\le n,\ \ B_k\le B_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. Let $i \in \mathbb{N}^*$, from (3.6), we have

$$\sum_{i=1}^{m} \lambda_i \Big[q\beta_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1}\Big] \Big\|x_{\tau(n)} - T_i x_{\tau(n)}\Big\|^q \leq \alpha_{\tau(n)} C.$$

Furthermore, we have

$$\lim_{n\to\infty} \sum_{i=1}^{m} \lambda_i \Big[q\beta_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1}\Big] \Big\|x_{\tau(n)} - u^i_{\tau(n)}\Big\|^q = 0.$$

Since $q\beta_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1} > 0 \ \forall\, i = 1, \cdots, m$, we have

$$\lim_{n\to\infty} \Big\|x_{\tau(n)} - T_i x_{\tau(n)}\Big\|^q = 0. \tag{3.9}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ is bounded in K and $\limsup_{\tau(n)\to+\infty} \langle x^*, J_\varphi(x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$\begin{aligned} 0 \leq \Phi(\|x_{\tau(n)+1} - x^*\|) - \Phi(\|x_{\tau(n)} - x^*\|) \ &\leq \ \Big(1 - \theta_{\tau(n)}\Big)\alpha_{\tau(n)}[-\Phi(\|x_{\tau(n)} - x^*\|) \\ &\quad + \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle], \end{aligned}$$

which implies that

$$\Phi(\|x_{\tau(n)} - x^*\|) \leq \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As consequence, we have for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Remark 3.2. In our theorem, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0, 1)$ and $x \in K$. Therefore, in the case where E is a real Hilbert space or $E = l_q,\ 1 < p < \infty$, our results can be used to approximated a common fixed points of a finite family of demicontractive mappings from the closed unit ball to itself.

Corollary 3.3. *Assume that $E = l_q,\ 1 < q < \infty$ or E is a real Hilbert space. Let $\mathbb{B}$ be the closed unit ball of E. Let $m \geq 1$ be a fixed number, for $i, 1 \leq i \leq m$, $T_i : \mathbb{B} \to \mathbb{B}$ be a k_i- demicontractive mapping such that $\Gamma := \bigcap_{i=1}^{m} Fix(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in \mathbb{B}$ by:*

$$
\begin{cases} y_n = \lambda_0 x_n + \lambda_1 T_1 x_n + \cdots + \lambda_m T_m x_n, \\ \\ x_{n+1} = \alpha_n(\theta_n x_n) + (1-\alpha_n) y_n, \end{cases} \tag{3.10}
$$

where $\lambda_i \in (0, \gamma)$,

$$
\gamma := \min_{1 \leq i \leq m} \Big\{ 1, \Big(\frac{q\beta_i^{q-1}}{2^{(m-1)q} d_q} \Big)^{\frac{1}{q-1}} \Big\}, \ \ \textit{with } \beta_i = \frac{1-k_i}{2}.
$$

Suppose the following conditions hold:
$\{\theta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=0}^{\infty}(1-\theta_n)\alpha_n = \infty$, $\sum_{i=0}^{m} \lambda_i = 1$,
(iii) $\lim_{n\to\infty} \theta_n = 1$.
Assume that $I - T_i$ is demiclosed at the origin.
Then, the sequence $\{x_n\}$ generated by (3.10) converges strongly to $x^* \in \Gamma$, where $x^* = Q_\Gamma(0)$ with Q_Γ the sunny nonexpansive retraction of $\mathbb{B}$ onto Γ.

Now, we give some remarks on our results as follows:

(1) The proof methods of our result are very different from the ones of Sow et al. [14] for finding fixed points of nonexpansive mapping. Further, we remove the following conditions: $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ in Theorem 1.3 of [14].

(2) Our results improve many recent results using Mann's method to approximate fixed points of nonexpansive mappings, quasi-nonexpansive, strictly pseudo-contractive in Banach spaces.

References

[1] Ya. Alber, *Metric and generalized Projection Operators in Banach space:properties and applications in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,*(A. G Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.

[2] F. E. Browder, *convergenge theorem for sequence of nonlinear operator in Banach spaces,* 100 (1967) 201-225.

[3] R. E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Transactions of the American Mathematical Society, 179, 251-262.

[4] I. Cioranescu, *Geometry of Banach space, duality mapping and nonlinear problems,* Kluwer, Dordrecht, (1990).

[5] Chidume, C. E., *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965,(2009), ISBN 978-1-84882-189.

[6] C.E. Chidume, The solution by iteration of nonlinear equations in certain Banach spaces, J. Nigerian Math. Soc., 3 (1984), 57-62.

[7] A. Genel, and J. Lindenstrauss, An example concerning fixed points, Israel J. Math., 22 (1975), No. 1, 81-86.

[8] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc., 3 (1967), 957-961.

[9] S. Ishikawa, Fixed points and iteration of nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc., 73 (1976), 61-71.

[10] T.C. Lim, H.K. Xu, *Fixed point theorems for asymptoticaly nonexpansive mapping,* Nonliear Anal 22(1994), no. 11, 1345-1355.

[11] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953) 506-510.

[12] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces* , J. Math. Math. Appl., 329(2007), 336-346.

[13] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000), 46-55.

[14] T.M.M. Sow, N. Djitté, and C.E. Chidume, *A path convergence theorem and construction of fixed points for nonexpansive mappings in certain Banach spaces*, Carpathian J.Math.,32(2016),No.2,217-226,2016.

[15] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. **66** (2002), no. 2, 240 - 256.

[16] Y. Yao and Y. Je. Cho *Strong convergence of modified Krasnoselskii-Mann iterative algorithm for nonexpansive mappings in Hilbert Spaces,* Mathematical Modelling and Analysis Vol 15, Number 2, 2010, pages 265-274.

[17] Y. Yao, H. Zhou, Y. C. Liou, *Strong convergence of modified Krasnoselskii-Mann iterative algorithm for nonexpansive mappings,* J. Math. Anal. Appl. Comput. 29 (2009) 383-389.

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