"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 29 (2019), No. 2, 79-86

λ-EDGE SPAN OF SOME ALMOST REGULAR GRAPHS

M. MURUGAN, P. SRIRAMAN AND M. SURIYA

Abstract. In this paper, we introduce, λ-edge span of a graph, where λ is the λ-number or $L(2,1)$ labeling number of a graph. Also, we introduce the concept of almost regularness for infinite graphs. An infinite graph is almost regular if it is regular except for a finite number of points. Here, we consider some important infinite graphs which are almost regular and find the λ-number and λ-edge span of them.

1. Introduction

For standard terminology and notation, we follow Bondy and Murty[1] or Murugan[2]. Unless or otherwise mentioned, let $G(V, E)$ be a simple, finite, connected, undirected graph without loops or multiple edges. Following the standard terminology, we use P_n to denote a path on n vertices, C_n to denote a cycle on n vertices, T to denote a Tree, Δ to denote the maximum degree of a graph, $\lceil x \rceil$ to denote the least integer greater than or equal to x, and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

Keywords and phrases: Distance two labeling, Channel assignment, λ-edge span, λ-number.
(2010) Mathematics Subject Classification: 05C78

The Channel assignment problem consists in efficiently assigning radio channels to transmitters at several locations, using non-negative integers to represent channels. Vertices correspond to transmitter locations and the labels to radio channels. These assignments have to be such that close locations receive different channels and channels for very close locations are at least two apart such that these channels would not interfere with each other. The mathematical abstraction of this concept is $L(2,1)$-labeling.

An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set to the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. A $k - L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \to \{0, \ldots, k\}$, and we are interested to find the minimum k among all possible assignments. This invariant, the minimum k, is known as the $L(2,1)$-labeling number or λ-number and is denoted by $\lambda(G)$. The generalization of this concept is as below.

For positive integers k, d_1, d_2, a $k - L(d_1, d_2)$-labeling of a graph G is a function $f : V(G) \to \{0, \ldots, k\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v in G, $d_G(u, v) = i$, for $i = 1, 2$. The $L(d_1, d_2)$-number of G, $\lambda_{d_1,d_2}(G)$, is the smallest k such that there exists a $k - L(d_1, d_2)$-labeling of G.

2. Some Existing Results

Many interesting results have been published on distance two labeling and received the attention of many researchers. Here we present some important existing results.

- In [3] Griggs and Yeh have discussed $L(2,1)$-labeling for path, cycle, tree and cube. They have derived results for the graphs of diameter 2. They have shown that the $\lambda(T)$ for tree graphs with maximum degree $\Delta \geq 1$, is either $\Delta + 1$ or $\Delta + 2$.
- Chang and Kuo [4] provided an algorithm to obtain $\lambda(T)$.
- Yeh [5] have discussed the $L(2,1)$-labeling on various class of graphs like trees, cycles, chordal graphs, Cartesian products of graphs etc.,
- Griggs and Yeh [3] proved that if a graph G contain three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$, where Δ is the maximum degree of G.

- Griggs and Yeh [3] posed a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G, and they proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ at the same time.
- Chang and Kuo [4] proved that $\lambda(G) \leq \Delta^2 + \Delta$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Kral and Skrekovski [6] proved that $\lambda(G) \leq \Delta^2 + \Delta - 1$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Goncalves [7] proved that $\lambda(G) \leq \Delta^2 + \Delta - 2$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Murugan [8] determined an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs such that G_2 has an injective $L(2,1)$-labeling and also he proved that the bound is attainable when G_1 and G_2 are complete. Also he determined an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any to graphs.
- Murugan [9] determined λ-number for some family of cycle dominated graphs.

In spite of all the efforts the conjecture posed by Griggs and Yeh is still open.

3. Almost Regular Graphs and λ-edge span

Regular graphs play an important role in modeling network configurations where equipment limitations impose a restriction on the maximum number of links emanating from a node. These limitation do not enforce strict regularity and here, non regular graphs that are in some sense close to regular are useful. In 1984, Alon et.al. [10], introduced the concept of almost regular graphs as a graph in which all vertex degrees are either t or $t+1$ and at least one vertex has each degree. Also they defined another type of almost regular graph as, a graph G is of type (k, s), $k < s$, if the degree of every vertex of it satisfies $k \leq d(v) \leq s$ and G is not k-regular.

Now, we define almost regular graph for infinite graphs: An infinite graph is said to be almost regular if it is regular except for a finite number of points.

A k-distant tree consists of a main path called the "spine" such that each vertex on the spine is joined by an edge to at most one path on k-vertices. Those paths are called "tails" (that is, each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length k, it is a uniform k-distant tree.

A uniform k-distant tree with odd number of vertices is called a uniform k-distant odd tree. A uniform k-distant tree with even number of vertices is called a uniform k-distant even tree.

In this paper, we consider k-distant trees with $k = 1, 2, \ldots, \infty$.

An infinite twig is a graph consisting of a path P with vertices v_i, $i = 1, 2, \ldots, \infty$ and two non-adjacent vertices u_i and w_i such that u_i and w_i are adjacent to v_i, $i = 1, 2, \ldots, \infty$.

A ladder is a graph obtained by the Cartesian product of a path P_n with P_2, that is, $P_n \times P_2$, where P_n consist of vertices v_i, $i = 1, 2, \ldots, n$. If the path P_n consist of vertices v_i, $i = 1, 2, \ldots, \infty$ then it is called an infinite ladder.

In [11], Roger K. Yeh defined the edge span of a distance two labeling f, $\beta(G, f) = \max\{|f(x) - f(y)| : \{x, y\} \in E(G)\}$. The $L(2, 1)$ edge span of G, $\beta(G)$, is min $\beta(G, f)$. In this paper, we introduce λ- edge span of a graph, where λ is the λ-number or $L(2, 1)$ labeling number of a graph. The λ-edge span of a $L(2, 1)$ labeling f with $L(2, 1)$ labeling number λ of graph G is, $\beta_\lambda(G, f) = \max\{|f(x) - f(y)| : \{x, y\} \in E(G)\}$. The λ-edge span of G, $\beta_\lambda(G)$, is min $\beta_\lambda(G, f)$. Clearly, $2 \leq \beta_\lambda(G) \leq \lambda$. Here, we consider infinite k-distant tree, infinite twig, infinite ladder and find the λ-number and λ-edge span of these classes of graphs.

4. Results

Theorem 4.1. *The λ-number of an infinite k-distant tree is 5, $k = 1, 2, \ldots, \infty$ and the λ-edge span of T is $\beta_\lambda(T) = 4$.*

Proof. Consider an infinite k-distant tree T. Let the vertices on the spine are v_i, $i = 1, 2, \ldots, \infty$ and the vertices of the i^{th} tail are $v_{i,j}$, $i = 1, 2, \ldots, \infty$ and $j = 1, 2, \ldots, \infty$ and v_i is adjacent with $v_{i,1}$, $i = 1, 2, \ldots, \infty$.

Define $f : V(T) \to \mathbb{N} \cup \{0\}$ such that

$$f(v_i) = \begin{cases} 0 & \text{if} \quad i \bmod 3 = 1 \\ 2 & \text{if} \quad i \bmod 3 = 2 \\ 4 & \text{if} \quad i \bmod 3 = 0 \end{cases}$$

and

$$f(v_{i,j}) = \begin{cases} 3 & \text{if} \quad i \bmod 3 = 1 \quad \text{and} \quad j \bmod 3 = 1 \\ 1 & \text{if} \quad i \bmod 3 = 1 \quad \text{and} \quad j \bmod 3 = 2 \\ 5 & \text{if} \quad i \bmod 3 = 1 \quad \text{and} \quad j \bmod 3 = 0 \\ 5 & \text{if} \quad i \bmod 3 = 2 \quad \text{and} \quad j \bmod 3 = 1 \\ 3 & \text{if} \quad i \bmod 3 = 2 \quad \text{and} \quad j \bmod 3 = 2 \\ 1 & \text{if} \quad i \bmod 3 = 2 \quad \text{and} \quad j \bmod 3 = 0 \\ 1 & \text{if} \quad i \bmod 3 = 0 \quad \text{and} \quad j \bmod 3 = 1 \\ 3 & \text{if} \quad i \bmod 3 = 0 \quad \text{and} \quad j \bmod 3 = 2 \\ 5 & \text{if} \quad i \bmod 3 = 0 \quad \text{and} \quad j \bmod 3 = 0 \end{cases}$$

Consider the vertices on the spine. If $d(v_i, v_j) = 1$ or 2, we have $|f(v_i) - f(v_j)| = 2$ or 4, using the labels on the spine.

Consider the vertices on the tail. If $d(v_{i,j}, v_{i,k}) = 1$ or 2, we have $|f(v_{(i,j)}) - f(v_{(i,k)})| = 2$ or 4, $i = 1, 2, \ldots, \infty$ using the labels on the tails.

Now consider v_i and $v_{i,1}$, $i = 1, 2, \ldots, \infty$. Clearly $|f(v_i) - f(v_{(i,1)})| = 3$, $i = 1, 2, \ldots, \infty$, by f. Hence f is a distance two labeling and $\lambda(T) \leq 5$.

If G is a graph with maximum degree Δ, $\Delta \geq 2$ and contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$. [3]. Therefore, $\lambda(T) \geq 5$.

Hence $\lambda(T) = 5$.

Now we find λ-edge span of T.

Since the above f is a distance two labeling with $\lambda = 5$, the λ-edge span of T with respect to f is 4 by f. That is, the λ-edge span of T, $\beta_\lambda(T) \leq 4$.

Suppose $\beta_\lambda(T) \leq 3$, then there is a distance two labeling f_1 with $\lambda = 5$ such that the λ-edge span of T with respect to f_1 is 3. That is, the maximum edge difference with respect to f_1 is 3. That is the edge difference with respect to f_1 are 2 or 3. Therefore, the label 4 cannot occur on the spine since with 4, only the label 1 and 2 alone can occur on the spine which have edge differences 2 or 3. But the vertices on the spine are of degree 3 (except the initial vertex) and so 4 cannot occur on the spine. Similarly the label 5 cannot occur on the spine. Thus, labels 0, 1, 2, 3 alone can occur on the spine.

That is, λ-number of the spine $= \lambda(P_n) \leq 3$. This is a contradiction since $\lambda(P_n) = 4$ by [3].

Hence $\beta_\lambda(T) = 4$. □

Theorem 4.2. *The λ-number of an infinite twig T is 6 and the λ-edge span of T is $\beta_\lambda(T) = 5$.*

Proof. Consider an infinite twig T. Let the vertices on the path P be v_i, $i = 1, 2, \dots, \infty$ and let u_i, w_i be two distinct non-adjacent vertices adjacent to v_i, $i = 1, 2, \dots, \infty$.

Define $f : V(T) \to \mathbb{N} \cup \{0\}$ such that

$$f(v_i) = \begin{cases} 0 & \text{if} \quad i \bmod 3 = 1 \\ 2 & \text{if} \quad i \bmod 3 = 2 \\ 4 & \text{if} \quad i \bmod 3 = 0 \end{cases}$$

$$f(u_i) = \begin{cases} 3 & \text{if} \quad i \bmod 3 = 1 \\ 5 & \text{if} \quad i \bmod 3 = 2 \\ 1 & \text{if} \quad i \bmod 3 = 0 \end{cases}$$

$$f(w_i) = \begin{cases} 5 & \text{if} \quad i \bmod 3 = 1 \\ 6 & \text{if} \quad i \bmod 3 = 2 \\ 6 & \text{if} \quad i \bmod 3 = 0 \end{cases}$$

Consider the vertices on the path. If $d(v_i, v_j) = 1$ or 2, we have $|f(v_i) - f(v_j)| = 2$ or 4, $i = 1, 2, \dots, \infty$, using the labels on the path.

Clearly, $d(v_i, u_i) = 1$ and $|f(v_i) - f(u_i)| = 3$, $i = 1, 2, \dots, \infty$.

Also, $d(v_i, w_i) = 1$ and $|f(v_i) - f(w_i)| = 5$ or 4 or 2, $i = 1, 2, \dots, \infty$. by f.

Since any two labels on the vertices at a distance 2 are different, their label difference are greater than or equal to 1.

Hence f is a distance two labeling and $\lambda(T) \leq 6$.

If G is a graph with maximum degree Δ, $\Delta \geq 2$ and contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$ [3]. Therefore, $\lambda(T) \geq 6$.

Hence $\lambda(T) = 6$.

Now we find λ-edge span of T.

Since the above f is a distance two labeling with $\lambda = 6$, the λ-edge span of T with respect to f is 5 by f. That is, the λ-edge span of T, $\beta_\lambda(T) \leq 5$.

Suppose $\beta_\lambda(T) \leq 4$, then there is a distance two labeling f_1 with $\lambda = 6$ such that the λ-edge span of T with respect to f_1 is 4. That is, the maximum edge difference with respect to f_1 is 4. That is, the edge difference with respect to f_1 are 2, 3 or 4.

Here, the label 6 cannot occur on the vertices of the path P since the labels 2, 3, 4 alone can occur on the adjacent vertices of 6 keeping the edge differences as 2, 3 or 4. But 6 has 4 adjacent vertices (except for the origin of P) and one adjacent vertex of 6 cannot be labeled. Similarly the label 5, 0 cannot occur on the vertices of the path P. So the labels 1, 2, 3 and 4 alone can occur on the path P. That is,

f_1 is a distance two labeling of T in which the sub graph P, has labels from $\{1, 2, 3, 4\}$. Now redefine this labeling of P by subtracting 1, we get a new distance two labeling F with maximum label as 3. That is, $\lambda(P) \leq 3$. This is a contradiction since $\lambda(P_n) = 4$ by [3]. Hence $\beta_\lambda(T) = 5$. □

Theorem 4.3. *The λ-number of an infinite ladder L is 5 and the λ-edge span of L is $\beta_\lambda(L) = 5$.*

Proof. Consider an infinite ladder L. Let P_1 and P_2 be two paths of L such that the vertices of P_1 are v_i, $i = 1, 2, \ldots, \infty$ and of P_2 are u_i, $i = 1, 2, \ldots, \infty$ where v_i and u_i are adjacent.

Define $f : V(L) \to \mathbb{N} \cup \{0\}$ such that

$$f(v_i) = \begin{cases} 2 & \text{if} \quad i \bmod 3 = 1 \\ 0 & \text{if} \quad i \bmod 3 = 2 \\ 4 & \text{if} \quad i \bmod 3 = 0 \end{cases}$$

$$f(u_i) = \begin{cases} 5 & \text{if} \quad i \bmod 3 = 1 \\ 3 & \text{if} \quad i \bmod 3 = 2 \\ 1 & \text{if} \quad i \bmod 3 = 0 \end{cases}$$

Consider the vertices on the path P_1. If $d(v_i, v_j) = 1$ or 2, we have $|f(v_i) - f(v_j)| = 2$ or 4, $i = 1, 2, \ldots, \infty$, using the labels on the path.

Consider the vertices on the path P_2. If $d(u_i, u_j) = 1$ or 2, we have $|f(u_i) - f(u_j)| = 2$ or 4, $i = 1, 2, \ldots, \infty$, using the labels on the path.

Now, $|f(v_i) - f(u_i)| = 3$, $i = 1, 2, \ldots, \infty$ Also, if $d(v_i, u_j) = 2$, we have $|f(v_i) - f(u_j)| = 1$ or 5, using the labels on the path.

Hence f is a distance two labeling and $\lambda(L) \leq 5$.

If G is a graph with maximum degree Δ, $\Delta \geq 2$ and contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$ [3]. Therefore, $\lambda(L) \geq 5$.

Hence $\lambda(L) = 5$.

Now we find λ-edge span of L.

Since the above f is a distance two labeling with $\lambda = 5$, the λ-edge span of L with respect to f is 4 by f. That is, the λ-edge span of L, $\beta_\lambda(L) \leq 4$.

Suppose $\beta_\lambda(L) \leq 3$, then there is a distance two labeling f_1 such that the λ-edge span of L with respect to f_1 is 3. That is, the maximum edge difference with respect to f_1 is 3. That is, the edge difference with respect to f_1 are 2 or 3. Hence, labels 2 and 3 alone can occur on the vertices of P_1 or P_2 (except the initial vertex of the paths), keeping the edge differences as 2 or 3. This is a contradiction, since, 2 and 3 cannot be adjacent in a distance two labeling. Hence, $\beta_\lambda(L) = 4$. □

5. Conclusion

This work is an effort to study about the λ-edge span of a graph and we believe that it will create an interest in researchers towards this topic and find the family of graphs which attain the lower or upper bound of the λ-edge span.

References

[1] Bondy J.A, and Murty U.S.R, **Graph Theory**, Springer, 2008.
[2] Murugan M, **Topics in Graph Theory and Algorithms**, Muthali Publishing House, Chennai, Revised Edition, 2018.
[3] Griggs J.R. and Yeh R.K. **Labeling graphs with a condition at distance 2**, SIAM J. Discrete Math. 5 (1992), 586-595.
[4] Chang G.J. and Kuo D. **The $L(2,1)$-Labeling problem on graphs**, SIAM J. Discrete Math 9 (2) (1996), 309-316.
[5] Yeh R.K. **A survey on labeling graphs with a condition at distance two, Discrete Mathematics**, 306 (2006), 1217-1231.
[6] Kral D. and Skrekovski R. **A theorem about channel assignment problem**, SIAM J. Discr. Math. (2003), 426-437.
[7] Goncalves D. **On the $L(p,1)$-labeling of graphs**, Discr.Math.308 (2008), 1405 - 1414.
[8] Murugan M. **Distance Two Labeling on Special Family of Graphs**, Le Matematiche, Vol.LXX (2015) - Fasc.II,pp.35-48.
[9] Murugan M. **$L(2,1)$-Labeling for Subdivisions of Some Cycle Dominated Graphs**, Bull.Comput.Appl.Math., Vol.2, No.2, (2014).
[10] Alon N. and Friedland S. and Kalai G. **Regular Subgraphs of Almost Regular Graphs**, Journal of Combinatorial Theory, Series B 37, 79-91 (1984).
[11] Yeh K. Roger, **The Edge Span of Distance Two Labellings of Graphs**, Taiwanese Journal of Mathematics, Vol.4, No.4, pp.675-683 (2000).

M. Murugan
Department of Mathematics, School of Science,
Tamil Nadu Open University, Chennai, INDIA.
⟨muruganganesan@yahoo.in⟩
P. Sriraman
Research Scholar, Department of Mathematics, School of Science,
Tamil Nadu Open University, Chennai, INDIA,
and
M. Suriya
B.Tech (Mechanical), Second Year,
SRM Institute of Science and Technology, Chennai, INDIA.