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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 29 (2019), No. 2, 87-96

SELECTION PROPERTIES OF QUASI-UNIFORM SPACES USING IDEALS

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Abstract. In this paper we study the properties of Pre Hurewicz spaces modulo $\mathcal{I}$, $\mathcal{I}$-Hurewicz bounded spaces, Pre Menger spaces modulo $\mathcal{I}$ and $\mathcal{I}$-Menger bounded spaces. Relationship among such spaces are also being investigated there after.

1. Introduction

Ljubiša D. R. Kočinac [4] first introduced selection principles in uniform spaces and proved that these are different from the selection principles in topological spaces which was further studied in [5, 6, 7]. In that paper, the author used the concept of covers to define uniform selection principles. In [8], G. D. Maio and Lj. D. R. Kočinac applied the idea of statistical convergence to define selection principles using the ideal of asymptotic density zero sets of natural numbers. In this paper this idea is extended to arbitrary ideals. With the help of ideals, we study selection properties in quasi-uniform spaces. Concerning quasi-uniform spaces we refer to [2]. At first we give some basic informations about quasi-uniform spaces and ideals.

Keywords and phrases: Quasi-uniform spaces, Ideals, Pre Hurewicz spaces modulo $\mathcal{I}$, $\mathcal{I}$-Hurewicz bounded spaces, Pre Menger spaces modulo $\mathcal{I}$, $\mathcal{I}$-Menger bounded spaces.
(2010) Mathematics Subject Classification: 54D20, 54E99.

2. Preliminaries

Definition 2.1. [2] A quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ satisfying the properties :
(1) $\Delta \subseteq U$, for all $U \in \mathcal{U}$, where $\Delta = \{(x, x) : x \in X\}$ is the diagonal of X.
(2) For $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$, where $V \circ V = \{(x, y) : \exists\ z \in X$ with $(x, z) \in V, (z, y) \in V\}$.
The pair $(X,\mathcal{U})$ is called a quasi-uniform space.

A quasi-uniformity $\mathcal{U}$ that satisfies the condition $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$ is called a unifomity on X and $(X,\mathcal{U})$ is then called a uniform space.

If $(X,\mathcal{U})$ is a quasi-uniform space, then $(X,\mathcal{U}^{-1})$ is also a quasi-uniform space, where $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is called the conjugate of $\mathcal{U}$.

Definition 2.2. [2] Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}$ generates a topology $\tau_{\mathcal{U}}$ on X such that for each $x \in X$, the family $\{U[x] : U \in \mathcal{U}\}$ is a local base at x, where $U[x] = \{y \in X : (x, y) \in U\}$. Also, $U[A] = \bigcup_{a \in A} U[a]$.

Definition 2.3. The supremum of $\mathcal{U}$ and $\mathcal{U}^{-1}$, denoted by $\mathcal{U}^s$ is a unifomity on X, called the symmetrization of $\mathcal{U}$.

Definition 2.4. [3] Let X be a non-empty set. A family $\mathcal{I} \subset \mathcal{P}(X)$ of subsets of X is said to be an ideal on X if
(1) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$,
(2) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

Definition 2.5. [4] A uniform space $(X,\mathcal{U})$ is called uniformly Hurewicz if for each sequence $\{U_n : n \in \mathbb{N}\}$ of entourages of the diagonal, there is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $U_n[F_n]$, where $U_n[F_n] = \{y \in X : (x, y) \in U_n$ for some $x \in F_n\}$.

Definition 2.6. [4] A uniform space $(X,\mathcal{U})$ is called uniformly Menger if for each sequence $\{U_n : n \in \mathbb{N}\}$ of entourages of the diagonal, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of finite subsets of X such that $\bigcup_{n \in \mathbb{N}} U_n[A_n] = X$.

Definition 2.7. A mapping $q : X \times X \to [0, \infty)$ is called a quasi-metric on X, if for all $x, y, z \in X$ the following hold :
(i) $x = y \Leftrightarrow q(x, y) = q(y, x) = 0$
(ii) $q(x, z) \leq q(x, y) + q(y, z)$.
The pair (X, q) is called a quasi-metric space.

If (i) is replaced by (i′) $q(x, x) = 0$ for each $x \in X$, then q is called a quasi-pseudometric on X.

3. Pre Hurewicz spaces modulo $\mathcal{I}$ and $\mathcal{I}$-Hurewicz bounded spaces

In this section we first define pre Hurewicz spaces modulo $\mathcal{I}$ and $\mathcal{I}$-Hurewicz bounded spaces and study some of their properties with $\mathcal{I}$ as a proper ideal on $\mathbb{N}$ ($\equiv$ the set of natural numbers).

Definition 3.1. A quasi-uniform space $(X, \mathcal{U})$ is said to be Pre Hurewicz modulo $\mathcal{I}$ if for each sequence $\{U_n : n \in \mathbb{N}\}$ of elements of $\mathcal{U}$, there is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin U_n[F_n]\} \in \mathcal{I}$.

Definiton 3.2. A quasi-uniform space $(X, \mathcal{U})$ is said to be $\mathcal{I}$-Hurewicz bounded if for each sequence $\{U_n : n \in \mathbb{N}\}$ of elements of $\mathcal{U}$, there is a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of finite collections of subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin U_n[F], F \in \mathcal{F}_n\} \in \mathcal{I}$ and for each $n \in \mathbb{N}, F \times F \subset U_n$, for each $F \in \mathcal{F}_n$.

Remark 3.3. Clearly a quasi-uniform space $(X, \mathcal{U})$ is $\mathcal{I}$-Hurewicz bounded if the uniform space $(X, \mathcal{U}^s)$ is pre Hurewicz modulo $\mathcal{I}$.

Theorem 3.4. Each $\mathcal{I}$-Hurewicz bounded quasi-uniform space $(X, \mathcal{U})$ is pre Hurewicz modulo $\mathcal{I}$.

Proof : Let $\{U_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{U}$. Since X is $\mathcal{I}$-Hurewicz bounded, there exists a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of finite collections of subsets of X such that for each $x \in X$ $\{n \in \mathbb{N} : x \notin U_n[F], F \in \mathcal{F}_n\} \in \mathcal{I}$ and for each $n \in \mathbb{N}, F \times F \subset U_n$, for each $F \in \mathcal{F}_n$. For each $n \in \mathbb{N}$ and for each $F \in \mathcal{F}_n$, choose $x_F \in F$ and set $V_n = \{x_F : F \in \mathcal{F}_n\}$. Then $\{n \in \mathbb{N} : x \notin U_n[V_n]\} \in \mathcal{I}$. Hence $(X, \mathcal{U})$ is pre Hurewicz modulo $\mathcal{I}$.

Theorem 3.5. For a quasi-uniform space $(X,\mathcal{U})$, the followings are equivalent :
(1) X is pre Hurewicz modulo $\mathcal{I}$.
(2) For each sequence $\{U_n : n \in \mathbb{N}\}$ in $\mathcal{U}$, there is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X and a sequence $n_1 < n_2 < ... < n_k < ...$ in $\mathbb{N}$ such that for each $x \in X$, $\{k \in \mathbb{N} : x \notin \bigcup U_i[F_i] : n_k \leq i < n_{k+1}\} \in \mathcal{I}$.

Proof : (1) $\Rightarrow$ (2) : Clear.

(2) $\Rightarrow$ (1) : Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of elements in $\mathcal{U}$ and let for each $n \in \mathbb{N}$, $V_n = \bigcap_{i \leq n} U_i$. By (2) to the sequence $\{V_n : n \in \mathbb{N}\} \subset \mathcal{U}$ and for each $n \in \mathbb{N}$, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X and a sequence $n_1 < n_2 < ... < n_k < ...$ in $\mathbb{N}$ such that for each $x \in X$, $\{k \in \mathbb{N} : x \in \bigcup V_i[F_i] : n_k \leq i < n_{k+1}\} \notin \mathcal{I}$. Now define,

$$A_n = \begin{cases} \cup\{F_i : i < n_1\}, \text{for } n < n_1 \\ \cup\{F_i : n_k \leq i < n_{k+1}\}, \text{for } n_k \leq n < n_{k+1}, k \in \mathbb{N} \end{cases}$$

Clearly for each $n \in \mathbb{N}$, A_n is a finite subset of X. Now for each $x \in X$, $\{n \in \mathbb{N} : x \in U_n[A_n] : n_k \leq i < n_{k+1}\} \supset \{k \in \mathbb{N} : x \in \bigcup V_i[F_i] : n_k \leq i < n_{k+1}\} \notin \mathcal{I}$ so that for each $x \in X$, $\{n \in \mathbb{N} : x \in U_n[A_n] : n_k \leq i < n_{k+1}\} \notin \mathcal{I}$. Hence X is pre Hurewicz modulo $\mathcal{I}$.

Theorem 3.6. A quasi-uniform space $(X,\mathcal{U})$ is hereditarily pre Hurewicz modulo $\mathcal{I}$ if and only if each $G\delta$-subset of $(X, \tau_{\mathcal{U}})$ is pre Hurewicz modulo $\mathcal{I}$.

Proof : One part is clear. For the next part, let Y be a subset of X and let $\{U_n : n \in \mathbb{N}\}$ be a sequence of elements in $\mathcal{U}$. For each $n \in \mathbb{N}$, choose $V_n \in \mathcal{U}$ such that $V_n \circ V_n \subset U_n$ and $V_n[x]$ is open in $(X, \tau_{\mathcal{U}})$ for each $x \in X$. Then the set $G = \bigcap_{n \in \mathbb{N}} V_n[Y]$ is a G_δ-set in $(X, \tau_{\mathcal{U}})$ and so is pre Hurewicz modulo $\mathcal{I}$. So there exist finite subsets $\{A_n : n \in \mathbb{N}\}$ of G such that $\{n \in \mathbb{N} : y \notin V_n[A_n]\} \in \mathcal{I}$. For each $n \in \mathbb{N}$, choose $B_n \subset Y$ such that $A_n \subset V_n[B_n]$. Hence $\{n \in \mathbb{N} : y \notin U_n[B_n]\} \in \mathcal{I}$, so that Y is pre Hurewicz modulo $\mathcal{I}$.

Theorem 3.7. The product $(X\times Y,\mathcal{U}\times\mathcal{V})$ of two pre Hurewicz modulo $\mathcal{I}$ quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is pre Hurewicz modulo $\mathcal{I}$.

Proof : Let $\{U_n\times V_n : n\in\mathbb{N}\}$ be a sequence in $\mathcal{U}\times\mathcal{V}$. Since $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are pre Hurewicz modulo $\mathcal{I}$, there exist sequences $\{E_n : n\in\mathbb{N}\}$ and $\{F_n : n\in\mathbb{N}\}$ of finite subsets of X and Y respectively, such that for each $x\in X$, $\{n\in\mathbb{N} : x\notin U_n[E_n]\}\in\mathcal{I}$ and for each $y\in Y$, $\{n\in\mathbb{N} : y\notin V_n[F_n]\}\in\mathcal{I}$. Let $(x,y)\in X\times Y$. Then $\{n\in\mathbb{N} : x\notin U_n[E_n]\}\in\mathcal{I}$ and $\{n\in\mathbb{N} : y\notin V_n[F_n]\}\in\mathcal{I}$ so that $\{n\in\mathbb{N} : (x,y)\notin(U_n\times V_n)[E_n\times F_n]\}=\{n\in\mathbb{N} : (x,y)\notin U_n[E_n]\times V_n[F_n]\}=\{n\in\mathbb{N} : x\notin U_n[E_n]\}\cup\{n\in\mathbb{N} : y\notin V_n[F_n]\}\in\mathcal{I}$. Hence $X\times Y$ is pre Hurewicz modulo $\mathcal{I}$.

4. Pre Menger spaces modulo $\mathcal{I}$ and $\mathcal{I}$-Menger bounded spaces

In this section we first define pre Menger spaces modulo $\mathcal{I}$ and $\mathcal{I}$-Menger bounded spaces and study some of their properties, where $\mathcal{I}$ has been chosen to be any ideal on X.

Definition 4.1. A quasi-uniform space $(X,\mathcal{U})$ is said to be pre Menger modulo $\mathcal{I}$ if for each sequence $\{U_n : n\in\mathbb{N}\}$ of elements of $\mathcal{U}$, there is a sequence $\{F_n : n\in\mathbb{N}\}$ of finite subsets of X such that $X\setminus\bigcup\limits_{n\in\mathbb{N}}U_n[F_n]\in\mathcal{I}$.

Definiton 4.2. A quasi-uniform space $(X,\mathcal{U})$ is said to be $\mathcal{I}$-Menger bounded if for each sequence $\{U_n : n\in\mathbb{N}\}$ of elements of $\mathcal{U}$, there is a sequence $\{\mathcal{C}_n : n\in\mathbb{N}\}$ of finite collections of subsets of X such that $X\setminus\bigcup\limits_{n\in\mathbb{N}}\mathcal{C}_n\in\mathcal{I}$ and for each $n\in\mathbb{N}$, $C\times C\subset U_n$, for each $C\in\mathcal{C}_n$.

Remark 4.3. Clearly a quasi-uniform space $(X,\mathcal{U})$ is $\mathcal{I}$-Menger bounded if the uniform space $(X,\mathcal{U}^s)$ is pre Menger modulo $\mathcal{I}$.

Theorem 4.4. Each $\mathcal{I}$-Menger bounded quasi-uniform space $(X,\mathcal{U})$ is pre Menger modulo $\mathcal{I}$.

Proof : Let $\{U_n : n\in\mathbb{N}\}$ be a sequence in $\mathcal{U}$. Since X is $\mathcal{I}$-Menger bounded, there is a sequence $\{\mathcal{C}_n : n\in\mathbb{N}\}$ of finite collections of subsets of X such that $X \setminus \bigcup_{n\in\mathbb{N}} \mathcal{C}_n \in \mathcal{I}$ and for each $n \in \mathbb{N}$, $C \times C \subset U_n$, for each $C \in \mathcal{C}_n$. For each $n \in \mathbb{N}$ and for each $C \in \mathcal{C}_n$, choose a point $x_C \in C$ and set $F_n = \{x_C : C \in \mathcal{C}_n\}$. Obviously for each $n \in \mathbb{N}$, F_n's are finite subsets of X and $X \setminus \bigcup_{n\in\mathbb{N}} U_n[F_n] \in \mathcal{I}$, as for $C \in \mathcal{C}_n$, $U_n[x_C] \supset C$.

Theorem 4.5. For a quasi-uniform space $(X,\mathcal{U})$, the followings are equivalent :
(1) X is pre Menger modulo $\mathcal{I}$.
(2) For each sequence $\{U_n : n \in \mathbb{N}\}$ in $\mathcal{U}$, there is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X and a sequence $n_1 < n_2 < ... < n_k < ...$ in $\mathbb{N}$ such that $X \setminus \bigcup\{U_i[F_i] : n_k \leq i < n_{k+1}\} \in \mathcal{I}$.

Proof : (1) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (1) : Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of elements in $\mathcal{U}$. For each $n \in \mathbb{N}$, let $V_n = \bigcap_{i\leq n} U_n$. By (2), there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X and a sequence $n_1 < n_2 < ... < n_k < ...$ in $\mathbb{N}$ such that $X \setminus \bigcup\{V_i[F_i] : n_k \leq i < n_{k+1}\} \in \mathcal{I}$, for some $k \in \mathbb{N}$. Define,

$$A_n = \begin{cases} \cup\{F_i : i < n_1\}, \text{for } n < n_1 \\ \cup\{F_i : n_k \leq i < n_{k+1}\}, \text{for } n_k \leq n < n_{k+1}, k \in \mathbb{N} \end{cases}$$

Then each A_n is a finite subset of X. Now for $k \in \mathbb{N}$, $X \setminus \bigcup\{U_i[A_i] : n_k \leq i < n_{k+1}\} \subset X \setminus \bigcup\{V_i[F_i] : n_k \leq i < n_{k+1}\} \in \mathcal{I}$. Hence X is pre Menger modulo $\mathcal{I}$.

Theorem 4.6. A quasi-uniform space $(X,\mathcal{U})$ is hereditarily pre Menger modulo $\mathcal{I}$ if and only if each $G\delta$-subset of $(X, \tau_{\mathcal{U}})$ is pre Menger modulo $\mathcal{I}$.

Proof : One part is clear. For the next part, let Y be a subset of X and let $\{U_n : n \in \mathbb{N}\}$ be a sequence of elements in $\mathcal{U}$. For each $n \in \mathbb{N}$, choose $V_n \in \mathcal{U}$ such that $V_n \circ V_n \subset U_n$ and $V_n[x]$ is open in $(X, \tau_{\mathcal{U}})$ for each $x \in X$. Now the set $G = \bigcap_{n\in\mathbb{N}} V_n[Y]$ is a G_δ-set in $(X, \tau_{\mathcal{U}})$ and so is pre Menger modulo $\mathcal{I}$ by (2). Hence there exist finite subsets $\{A_n : n \in \mathbb{N}\}$ of G such that $G \setminus \bigcup_{n\in\mathbb{N}} V_n[A_n] \in \mathcal{I}$. For each $n \in \mathbb{N}$, find $B_n \subset Y$ such that $A_n \subset V_n[B_n]$. Then $Y \setminus \bigcup_{n\in\mathbb{N}} U_n[B_n] \subset G \setminus \bigcup_{n\in\mathbb{N}} V_n[A_n] \in \mathcal{I}$. Hence Y is pre Menger modulo $\mathcal{I}$.

5. Relations among the above spaces

In general pre Hurewicz spaces modulo $\mathcal{I}$ and pre Menger spaces modulo $\mathcal{I}$ are not comparable. But if we choose X as a subset of $\mathbb{R}$, then from the definitions, the relations among the above spaces are as follows :

$$\begin{array}{ccc} \text{Pre Hurewicz spaces modulo } \mathcal{I} & \Rightarrow & \text{Pre Menger spaces modulo } \mathcal{I} \\ \Uparrow & & \Uparrow \\ \mathcal{I}\text{-Hurewicz bounded spaces} & \Rightarrow & \mathcal{I}\text{-Menger bounded spaces} \end{array}$$

We now give examples to show that the reverse implications may not be true.

Remark 5.1. It is clear that to define the above selection properties in a quasi-metric space, we have to replace the entourages in case of a quasi-uniform space by an open ball in the case of a quasi-metric space.

Example 5.2. A space which is pre Menger modulo $\mathcal{I}$ but not pre Hurewicz modulo $\mathcal{I}$:

Consider $X = [0,\infty)$ and q be the following quasi-pseudometric on X.

$$q(x,y) = \begin{cases} 0, \text{if } x \geq y \\ y - x, \text{if } x < y \end{cases}$$

We show that (X, q) is pre Menger modulo $\mathcal{I}$ but not pre Hurewicz modulo $\mathcal{I}$, where we choose the ideal $\mathcal{I} = \{A \subseteq X : 2 \notin A\}$. Let us consider a sequence $\{\epsilon_n : n \in \mathbb{N}\}$ of positive reals. We have to find a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that $X \setminus$

$\bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} B_q(x,\epsilon_n)\in\mathcal{I}$. Let $F_n=\{1,2,...,n\}$. Then $B_q(x,\epsilon_n)=[0,x+\epsilon_n)\Rightarrow X\setminus\bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} B_q(x,\epsilon_n)=\phi\in\mathcal{I}$. Hence (X,q) is pre Menger modulo $\mathcal{I}$. We now show that (X,q) is not pre Hurewicz modulo $\mathcal{I}$. Consider the sequence $\{\frac{1}{2^n}:n\in\mathbb{N}\}$. Then for any choice of sequences $\{F_n:n\in\mathbb{N}\}$ of finite subsets, $\{n\in\mathbb{N}:10\notin\bigcup_{y\in F_n} B_q(y,\frac{1}{2^n})\}=\{1,2,...,9\}$ and since $2\in\{1,2,...,9\}$, $\{n\in\mathbb{N}:10\notin\bigcup_{y\in F_n} B_q(y,\frac{1}{2^n})\}\notin\mathcal{I}$. Hence (X,q) is not pre Hurewicz modulo $\mathcal{I}$.

Example 5.3. A space which is pre Menger modulo $\mathcal{I}$ but not $\mathcal{I}$-Menger bounded :

Consider $X=[0,\infty)$ and q be the following quasi-pseudometric on X.

$$q(x,y)=\begin{cases}0,\text{ if } x\geq y\\ y-x,\text{ if } x<y\end{cases}$$

Then $q^s(x,y)=|x-y|$ is the usual metric on X.
We show that (X,q) is pre Menger modulo $\mathcal{I}$ but not $\mathcal{I}$-Menger bounded, where we choose the ideal $\mathcal{I}=\{A\subseteq X: A \text{ is finite }\}$. Let $\{\epsilon_n:n\in\mathbb{N}\}$ be a sequence of positive reals. We have to find a sequence $\{F_n:n\in\mathbb{N}\}$ of finite subsets of X such that $X\setminus\bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} B_q(x,\epsilon_n)\in\mathcal{I}$. Let $F_n=\{n,...,n^2\}$. Then $B_q(x,\epsilon_n)=[0,x+\epsilon_n)\Rightarrow X\setminus\bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} B_q(x,\epsilon_n)=\phi\in\mathcal{I}$. Hence (X,q) is pre Menger modulo $\mathcal{I}$. We now show that (X,q^s) is not pre Menger modulo $\mathcal{I}$ and that will prove that (X,q) is not $\mathcal{I}$-Menger bounded. Consider the sequence $\{\frac{1}{2^n}:n\in\mathbb{N}\}$. Then for any choice of sequences $\{F_n:n\in\mathbb{N}\}$ of finite subsets, $X\setminus\bigcup_{n\in\mathbb{N}}\bigcup_{x\in F_n} B_{q^s}(x,\frac{1}{2^n})\notin\mathcal{I}$. Hence (X,q) is not $\mathcal{I}$-Menger bounded.

Example 5.4. A space which is $\mathcal{I}$-Menger bounded but not $\mathcal{I}$-Hurewicz bounded :

In example 5.3, we choose the ideal as $\mathcal{I} = \{A \subseteq X : 3 \notin A\}$ and show that (X, q^s) is pre Menger modulo $\mathcal{I}$ but not pre Hurewicz modulo $\mathcal{I}$. Consider a sequence $\{\epsilon_n : n \in \mathbb{N}\}$ of positive reals. We have to find a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that $X \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{x \in F_n} B_{q^s}(x, \epsilon_n) \in \mathcal{I}$. Let $F_n = \{n-1, ..., n^2\}$. Then $B_{q^s}(x, \epsilon_n) = (x - \epsilon_n, x + \epsilon_n) \Rightarrow X \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{x \in F_n} B_{q^s}(x, \epsilon_n) \in \mathcal{I}$. Hence (X, q) is $\mathcal{I}$-Menger bounded. We now claim that (X, q^s) is not pre Hurewicz modulo $\mathcal{I}$. Consider the sequence $\{\frac{1}{2^n} : n \in \mathbb{N}\}$. Then for any choice of sequences $\{F_n : n \in \mathbb{N}\}$ of finite subsets, $\{n \in \mathbb{N} : 0 \notin \bigcup_{y \in F_n} B_{q^s}(y, \dfrac{1}{2^n})\} = \{2, 3, ...\}$ and $3 \in \{2, 3, ...\}$ so that $\{n \in \mathbb{N} : 0 \notin \bigcup_{y \in F_n} B_{q^s}(y, \dfrac{1}{2^n})\} \notin \mathcal{I}$. Hence (X, q) is not $\mathcal{I}$-Hurewicz bounded.

Example 5.5. A space which is pre Hurewicz modulo $\mathcal{I}$ but not $\mathcal{I}$-Hurewicz bounded :

In Example 5.3, we consider the ideal as $\mathcal{I} = \{A \subseteq X : A$ is countable$\}$ and show that (X, q) is pre Hurewicz modulo $\mathcal{I}$ but not $\mathcal{I}$-Hurewicz bounded. Let $\{\epsilon_n : n \in \mathbb{N}\}$ be a sequence of positive reals. We have to find a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup_{n \in \mathbb{N}} \bigcup_{y \in F_n} B_q(y, \epsilon_n)\} \in \mathcal{I}$. Let $F_n = \{n-1, ..., n^2\}$. Then $B_q(y, \epsilon_n) = [0, y + \epsilon_n)$. So for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup_{n \in \mathbb{N}} \bigcup_{y \in F_n} B_q(y, \epsilon_n)\}$ is countable and hence (X, q) is pre Hurewicz modulo $\mathcal{I}$. We now show that (X, q^s) is not pre Menger modulo $\mathcal{I}$ and hence not pre Hurewicz modulo $\mathcal{I}$. Consider the sequence $\{\frac{1}{2^n} : n \in \mathbb{N}\}$. Then for any choice of sequences $\{F_n : n \in \mathbb{N}\}$ of finite subsets, $X \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{x \in F_n} B_{q^s}(x, \dfrac{1}{2^n}) \notin \mathcal{I}$. Hence (X, q) is not $\mathcal{I}$-Hurewicz bounded.

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