"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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$\widetilde{G}\alpha$-CLOSED SETS IN TERMS OF GRILLS

A. SELVAKUMAR AND S. JAFARI

Abstract. In this paper we introduce the new notions of $\widetilde{g}\alpha(\theta)$-convergence and $\widetilde{g}\alpha(\theta)$-adherence of a grill in a topological space. We prove necessary and sufficient conditions for a grill to be $\widetilde{g}\alpha(\theta)$-adherent to a point, respectively $\widetilde{g}\alpha(\theta)$-convergent to a point, then we provide a characterization of relative $\widetilde{g}\alpha(\theta)$-closedness of a set in terms of grills.

1. Introduction and preliminaries

Recently, R. Devi et al. [2] introduced and studied the concept of $\widetilde{g}\alpha$-closed sets in topological spaces. The notion of grill was introduced in General Topology by G. Choquet [1] in 1947 and since then it has been used as a powerful tool in the study of proximity spaces, closure spaces, theory of compactifications, etc. In 2006, M.N. Mukherjee and B. Roy [4] studied the notion of p-closed sets in topological spaces in terms of grills.

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In this paper, we introduce the notions of $\widetilde{g}\alpha(\theta)$-convergence and $\widetilde{g}\alpha(\theta)$-adherence of a grill in a topological space and study these concepts to some extent in view of further applications.

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) stand for topological spaces with no separation axioms assumed, unless otherwise stated. For a subset $A \subseteq X$, the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1.1. [1] A grill $\mathcal{G}$ on a topological space X is defined to be a collection of non empty subsets of X such that

(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$ and
(ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 1.2. A subset A of a space (X, τ) is called a

1. *semi-open set* [3] if $A \subseteq cl(int(A))$ and a semi-closed set [4] if $int(cl(A)) \subseteq A$ and
2. *α-open set* [5] if $A \subseteq int(cl(int(A)))$ and an α-closed set [6] if $cl(int(cl(A))) \subseteq A$.

Recall that a set in a topological space is called semi-closed (respectively, α-closed) if its complement is semi-open (respectively, α- open). The semi-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. α-closed) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

Definition 1.3. A subset A of a space (X, τ) is said to be a

1. *$\widehat{g}$-closed set* [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ); the complement of every $\widehat{g}$-closed set is called $\widehat{g}$-open,
2. *${}^*g$-closed set* [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\widehat{g}$-open in (X, τ); the complement of every ${}^*g$-closed set is called ${}^*g$-open.
3. *${}^\sharp gs$-closed set* [9] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^*g$-open in (X, τ); the complement of every ${}^\sharp gs$-closed set is called ${}^\sharp gs$-open.
4. *$\widetilde{g}\alpha$-closed set* [2] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^\sharp gs$-open in (X, τ); the complement of every $\widetilde{g}\alpha$-closed set is called $\widetilde{g}\alpha$-open.

The set of all $\widetilde{g}\alpha$-open sets of X will be denoted by $\widetilde{G}\alpha O(X)$ and the set of all those members of $\widetilde{G}\alpha O(X)$, which contain a given point x of X will be designated by $\widetilde{G}\alpha O(x)$.

The intersection of all $\widetilde{g}\alpha$-closed sets in X, which are contained in a given set $A(\subseteq X)$ is called the $\widetilde{g}\alpha$-closure of A, to be denoted by $cl_{\widetilde{g}\alpha}(A)$. It is known that for $x \in X$ and $A \subseteq X$, $x \in \widetilde{g}\alpha\text{-cl}(A)$ if and only if $U \cap A \neq \emptyset$, for all $U \in \widetilde{G}\alpha O(x)$. For any set A in X, $\widetilde{g}\alpha(\theta)$-$cl(A)$, denoted by $\widetilde{g}\alpha(\theta)$-$cl(A)$, is defined as

$$\widetilde{g}\alpha(\theta) - cl(A) = \left\{ x \in X : \widetilde{g}\alpha - cl(U) \cap A \neq \emptyset \text{ for all } U \in \widetilde{G}\alpha O(x) \right\}.$$

2. Grills: $\widetilde{G}\alpha(\theta)$-convergence and $\widetilde{G}\alpha(\theta)$-adherence

Definition 2.1. *A grill* $\mathcal{G}$ on a topological space X is said to

(i) $\widetilde{g}\alpha(\theta)$-*adhere* at $x \in X$ if for each $U \in \widetilde{G}\alpha O(x)$ and each $G \in \mathcal{G}$, $cl_{\widetilde{g}\alpha}(U) \cap G \neq \emptyset$,

(ii) $\widetilde{g}\alpha(\theta)$-*converge* to a point $x \in X$ if for each $U \in \widetilde{G}\alpha O(x)$, there is some $G \in \mathcal{G}$ such that $G \subseteq cl_{\widetilde{g}\alpha}(U)$ (in this case we shall also say that $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent to x).

Remark 2.2. It at once follows that a grill $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent to a point $x \in X$ if and only if $\mathcal{G}$ contains the collection $\left\{ cl_{\widetilde{g}\alpha}(U) : U \in \widetilde{G}\alpha O(x) \right\}$.

Definition 2.3. *A filter* $\mathcal{F}$ on a topological space X is said to $\widetilde{g}\alpha(\theta)$-*adhere* at $x \in X$ ($\widetilde{g}\alpha(\theta)$-*converge* to $x \in X$) if for each $F \in \mathcal{F}$ and each $U \in \widetilde{G}\alpha O(x)$, $F \cap cl_{\widetilde{g}\alpha}(U) \neq \emptyset$ (respectively, to each $U \in \widetilde{G}\alpha O(x)$, there corresponds $F \in \mathcal{F}$ such that $F \subseteq cl_{\widetilde{g}\alpha}(U)$).

Definition 2.4. [6] If $\mathcal{G}$ is a grill (or a filter) on a space X, then the section of $\mathcal{G}$, denoted by $sec\mathcal{G}$ is given by

$$sec\mathcal{G} = \left\{ A \subseteq X : A \cap G \neq \emptyset, \text{for all } G \in \mathcal{G} \right\}.$$

Lemma 2.5. [6]

(a) *For any grill (filter) G on a space X, $secG$ is a filter(resp. grill) on X.*

(b) *If F and G are respectively a filter and a grill on a space X with $F \subseteq G$, then there is an ultrafilter U on X such that $F \subseteq U \subseteq G$.*

Theorem 2.6. *If a grill G on a topological space X, $\widetilde{g}\alpha(\theta)$-adheres at some point $x \in X$, then G is $\widetilde{g}\alpha(\theta)$-convergent to x.*
Proof. Assume that $\mathcal{G}$ is a grill on X, that $\widetilde{g}\alpha(\theta)$-adheres at $x \in X$. Then for each $U \in \widetilde{G}\alpha O(x)$ and each $G \in \mathcal{G}$, $cl_{\widetilde{g}\alpha}(U) \cap G \neq \emptyset$. Therefore, for each $U \in \widetilde{G}\alpha O(x)$ we have $cl_{\widetilde{g}\alpha}(U) \in sec\mathcal{G}$ and hence $X - cl_{\widetilde{g}\alpha}(U) \notin \mathcal{G}$. Then $cl_{\widetilde{g}\alpha}(U) \in \mathcal{G}$ (as $\mathcal{G}$ is a grill and $X \in \mathcal{G}$), for each $U \in \widetilde{G}\alpha O(x)$. Hence $\mathcal{G}$ must $\widetilde{g}\alpha(\theta)$-converge to x.

The converse need not be true by the following Example.

Example 2.7. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. It is easy to verify that (X, τ) is a topological space such that $\widetilde{G}\alpha O(X) = \tau$. Let $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then the grill $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-convergent but not $\widetilde{g}\alpha(\theta)$-adherent.

Notation 2.8. Let X be a topological space. Then for any $x \in X$, we will use the following notations:

$$\begin{aligned} \mathcal{G}(\widetilde{g}\alpha(\theta), x) &= \Big\{A \subseteq X : x \in \widetilde{g}\alpha(\theta)\text{-cl}(A)\Big\} \\ sec\mathcal{G}(\widetilde{g}\alpha(\theta), x) &= \Big\{A \subseteq X : A \cap G \neq \emptyset, \text{ for all } G \in \mathcal{G}(\widetilde{g}\alpha(\theta), x)\Big\} \end{aligned}$$

In the next two theorems, we characterize the $\widetilde{g}\alpha(\theta)$-adherence and $\widetilde{g}\alpha(\theta)$-convergence of grills in terms of the above notations.

Theorem 2.9. *A grill G on a space X, $\widetilde{g}\alpha(\theta)$-adheres to a point $x \in X$ if and only if $G \subseteq G(\widetilde{g}\alpha(\theta), x)$.*
Proof. Assume that a grill $\mathcal{G}$ on a space X, $\widetilde{g}\alpha(\theta)$-adheres at $x \in X$.

Then for every $G \in \mathcal{G}$ we have $cl_{\widetilde{g}\alpha}(U) \cap G \neq \emptyset$, for all $U \in \widetilde{G}\alpha O(x)$, which implies $x \in \widetilde{g}\alpha(\theta)$-$cl(G)$.

It follows that $G \in \mathcal{G}(\widetilde{g}\alpha(\theta), x)$ for all $G \in \mathcal{G}$, which implies $\mathcal{G} \subseteq \mathcal{G}(\widetilde{g}\alpha(\theta), x)$.

Conversely, assume that $\mathcal{G} \subseteq \mathcal{G}(\widetilde{g}\alpha(\theta), x)$. Then for all $G \in \mathcal{G}$, $x \in \widetilde{g}\alpha(\theta)$-$cl(G)$, so that for all $U \in \widetilde{G}\alpha O(x)$ and for all $G \in \mathcal{G}$, $cl_{\widetilde{g}\alpha}(U) \cap G \neq \emptyset$. It follows that $\mathcal{G}$ is $\widetilde{g}\alpha(\theta)$-adheres at x.

Theorem 2.10. *A grill G on a topological space X is $\widetilde{g}\alpha(\theta)$-convergent to a point x of X if and only if $secG(\widetilde{g}\alpha(\theta), x) \subseteq G$.*
Proof. Let $\mathcal{G}$ be a grill on X that is $\widetilde{g}\alpha(\theta)$-convergent to a point $x \in X$. Then for each $U \in \widetilde{G}\alpha O(x)$ there exists $G \in \mathcal{G}$ such that $G \subseteq cl_{\widetilde{g}\alpha}(U)$ and hence

$$cl_{\widetilde{g}\alpha}(U) \in \mathcal{G} \text{ for each } U \in \widetilde{G}\alpha O(x)$$

Now, $B \in sec\mathcal{G}(\widetilde{g}\alpha(\theta), x)$ implies $X - B \notin \mathcal{G}(\widetilde{g}\alpha(\theta), x)$, hence $x \notin \widetilde{g}\alpha(\theta)$-cl$(X - B)$, therefore there exists $U \in \widetilde{G}\alpha O(x)$ such that $cl_{\widetilde{g}\alpha}(U) \cap (X - B) = \emptyset$, which implies $cl_{\widetilde{g}\alpha}(U) \subseteq B$ for some $U \in \widetilde{G}\alpha O(x)$. It follows that $B \in \mathcal{G}$, by (1).

We prove the converse by contradiction. Assume that $\mathcal{G}$ does not $\widetilde{g}\alpha(\theta)$-converge to x and $sec\mathcal{G}(\widetilde{g}\alpha(\theta), x) \subseteq \mathcal{G}$. Then for some $U \in \widetilde{G}\alpha O(x)$, $cl_{\widetilde{g}\alpha}(U) \notin \mathcal{G}$ and hence $cl_{\widetilde{g}\alpha}(U) \notin sec\mathcal{G}(\widetilde{g}\alpha(\theta), x)$. Thus for some $A \in \mathcal{G}(\widetilde{g}\alpha(\theta), x)$,

$$A \cap cl_{\widetilde{g}\alpha}(U) = \emptyset$$

But $A \in \mathcal{G}(\widetilde{g}\alpha(\theta), x)$ implies $x \in \widetilde{g}\alpha(\theta)$-cl$(A)$ follows $cl_{\widetilde{g}\alpha}(U) \cap A \neq \emptyset$, contradicting (2).

Definition 2.11. A non empty subset A of a topological space X is called $\widetilde{g}\alpha$-closed relative to X if for every cover $\mathcal{U}$ of A by $\widetilde{g}\alpha$-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\left\{ \text{cl}_{\widetilde{g}\alpha}(U) : U \in \mathcal{U}_0 \right\}$. If X is $\widetilde{g}\alpha$-closed relative to itself, then X is called a $\widetilde{g}\alpha$-closed space.

Theorem 2.12. *A subset A of a topological space X is $\widetilde{g}\alpha$-closed relative to X if and only if every grill G on X with $A \in G$, $\widetilde{g}\alpha(\theta)$-converges to a point in A.*
Proof. Let A be $\widetilde{g}\alpha$-closed relative to X and $\mathcal{G}$ a grill on X satisfying $A \in \mathcal{G}$ such that $\mathcal{G}$ does not $\widetilde{g}\alpha(\theta)$-converges to any $a \in A$. Then to each $A \in A$, there corresponds some $U_a \in \widetilde{G}\alpha O(a)$ such that $cl_{\widetilde{g}\alpha}(U_a) \notin \mathcal{G}$. Now $\{U_a : a \in A\}$ is a cover of A by $\widetilde{g}\alpha$-open sets of X. Then $A \subseteq \cup_{i=1}^{n} cl_{\widetilde{g}\alpha}(U_{a_i}) = U$ (say), for some positive integer n. Since $\mathcal{G}$ is a grill, $U \notin \mathcal{G}$ and hence $A \notin \mathcal{G}$, which is a contradiction.

Conversely, assume that A is not $\widetilde{g}\alpha$-closed relative to X and that every grill $\mathcal{G}$ on X with $A \in \mathcal{G}, \widetilde{g}\alpha(\theta)$-converges to a point in A. Then for some cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of A by $\widetilde{g}\alpha$-open sets of X, $\mathcal{F} = \left\{ A - \cup_{\lambda \in \Lambda_0} cl_{\widetilde{g}\alpha}(U_\lambda) : \Lambda_0 \text{ is a finite subset of } \Lambda \right\}$ is a filterbase on

X. Then the family $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on X (by Lemma 2.5). Then $\mathcal{F}^*$ is a grill on X with $A \in \mathcal{F}^*$. Now for each $x \in A$, there must exist $\beta \in \Lambda$ such that $x \in U_\beta$, as $\mathcal{U}$ is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A - cl_{\widetilde{g}\alpha}(U_\beta))$ is non-empty, therefore, G is not contained in $cl_{\widetilde{g}\alpha}(U_\beta)$, for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot $\widetilde{g}\alpha(\theta)$-converge to any point of A. The contradiction proves the desired result.

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SNS College of Engineering, Coimbatore, Tamilnadu, India.
E-mail : selvam_mphil@yahoo.com

College of Vestsjaelland Syd, Herrestraede 11, 4200 Slagelse, Denmark.