"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 30 (2020), No. 1, 45 - 56

A NEW TYPE OF CLOSURE-LIKE OPERATOR VIA FUZZY SEMIOPEN SETS

ANJANA BHATTACHARYYA

Abstract. This paper deals with a new type of closure operator in fuzzy topological spaces, called s^c-closure operator, which is an idempotent operator. Then the mutual relationships of this operator with the operators defined in [2, 3, 4, 6, 9, 10] are established. Afterwards, a new type of separation axiom is introduced and studied here. In every space with this axiom assumed fuzzy semiclosure operator and this new operator are identical. In the last section some characterizations of s^c-closure operator have been done via fuzzy net.

1. Introduction

After the introduction by Chang of the concept of fuzzy closure operator [7] several types of fuzzy closure-like operators have been introduced and studied. In this context we have to mention [2, 3, 4, 6, 9, 10, 11]. Here we introduce a different type of fuzzy closure-like operator, called s^c-closure operator which is shown to be an idempotent and isotonic operator. It is shown that, in every fuzzy s^c-regular space, fuzzy semiclosure operator and fuzzy s^c-closure operator coincide.

Keywords and phrases: Fuzzy semiopen set, fuzzy s^c-closure operator, fuzzy regular open set, fuzzy preopen set, fuzzy s^c-regular space, s^c-convergence of a fuzzy net.
(2010) Mathematics Subject Classification: 54A40, 54D99.

2. Preliminaries

Throughout the paper, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [7]. A fuzzy set A is a function from a non-empty set X into a closed interval $I = [0, 1]$, i.e., $A \in I^X$ [13]. The support of a fuzzy set A in X will be denoted by $suppA$ [13] and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [12] with the singleton support $x \in X$ and the value t ($0 < t \leq 1$) at x will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement of a fuzzy set A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$ [13]. For two fuzzy sets A and B in X, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B if $A(x) + B(x) > 1$, for some $x \in X$ [12]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure [7] and fuzzy interior [7] of A in X. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [11], fuzzy β-open [8], fuzzy γ-open [5]) if $A = intclA$ (resp., $A \leq clintA$, $A \leq intclA$, $A \leq clintclA$, $A \leq (intclA) \bigcup (clintA)$). The complement of a fuzzy semiopen (resp., fuzzy preopen, fuzzy β-open, γ-open) set is called a fuzzy semiclosed [1] (resp., fuzzy preclosed [11], fuzzy β-closed [8], γ-closed [5]) set. The smallest fuzzy semiclosed (resp., fuzzy preclosed, fuzzy β-closed, fuzzy γ-closed) set containing a fuzzy set A is called fuzzy semiclosure [1] (resp., fuzzy preclosure [11], fuzzy β-closure [8], γ-closure [5]) of A and is denoted by $sclA$ (resp., $pclA$, βclA, γclA). The collection of all fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy β-open, fuzzy γ-open) sets in an fts X is denoted by $FRO(X)$ (resp., $FSO(X)$, $FPO(X)$, $F\beta O(X)$, $F\gamma O(X)$) and that of fuzzy ssemiclosed (resp., fuzzy preclosed, fuzzy β-closed, fuzzy γ-closed) sets is denoted by $FSC(X)$ (resp., $FPC(X)$, $F\beta C(X)$, $F\gamma C(X)$).

3. **Fuzzy s^c-Closure Operator: Some Properties**

In this section a new type of fuzzy closure operator is introduced and then some properties of this operator are established.

Definition 3.1. A fuzzy point x_t in an fts (X, τ) is called a fuzzy s^c-cluster point of a fuzzy set A in an fts X if $clUqA$ for every

$U \in FSO(X)$ with $x_t q U$.
The union of all fuzzy s^c-cluster points of A is called fuzzy s^c-closure of A, and will be denoted by $[A]_s^c$. A is called fuzzy s^c-closed set if $A = [A]_s^c$ and the complement of a fuzzy s^c-closed set in an fts X is called fuzzy s^c-open set in X.

It is clear from the definition that $A \leq [A]_s^c$, for all fuzzy set A in an fts (X, τ).

We now characterize fuzzy s^c-closure operator of a fuzzy set in an fts X.

Theorem 3.2. For any fuzzy set A in an fts (X, τ),
$[A]_s^c = \bigcap\{[U]_s^c : U$ is fuzzy open set in X with $A \leq U\}$.
Proof. Clearly $[A]_s^c \leq \bigcap\{[U]_s^c : U$ is fuzzy open set in X with $A \leq U\}$. Assume that there exists some x_t such that $x_t \in \bigcap\{[U]_s^c : U$ is fuzzy open set in X with $A \leq U\}$, but $x_t \notin [A]_s^c$. Then there exists $V \in FSO(X)$ with $x_t q V$ and $clV \not q A \Rightarrow A \leq 1_X \setminus clV (\in \tau)$. By hypothesis, $x_t \in [1_X \setminus clV]_s^c$. But as $clV \not q (1_X \setminus clV)$, $x_t \notin [1_X \setminus clV]_s^c$, a contradiction.
Note 3.3. It is clear from definition that for any $A \in I^X$, $sclA \leq [A]_s^c$. But the reverse relation is not necessarily true, as follows from the following example.
Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.45, B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $U(a) = 0.5, 0.45 \leq U(b) \leq 0.55$, $V \geq B$. Consider the fuzzy point $b_{0.51}$ and the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.47$. We claim that $b_{0.51} \in [C]_s^c$, but $b_{0.51} \in\in sclC$. Now any fuzzy semiopen set other than 1_X q-coincident with $b_{0.51}$ is of the form U_1, V_1 where $U_1(a) = 0.5, 0.49 < U_1(b) \leq 0.55$ and $V_1 \geq B$. Then $clU_1 = (1_X \setminus A)qC$ and $clV_1 = 1_X qC \Rightarrow b_{0.51} \in [C]_s^c$. But $U_1 \in FSC(X)$ also and so $sclU_1 = U_1 \not q C \Rightarrow b_{0.51} \notin sclC$.

The following theorem shows that on the class of fuzzy open sets fuzzy semiclosure and fuzzy s^c-closure operators coincide.
Theorem 3.5. For a fuzzy open set A in an fts X, $sclA = [A]_s^c$.
Proof. By Note 3.3, it suffices to show that $[A]_s^c \leq sclA$ for every fuzzy open set A in X. Let $x_t \notin sclA$. Then there exists $V \in FSO(X)$, $x_t qV, V \not q A \Rightarrow V \leq 1_X \setminus A$ where $1_X \setminus A$ is fuzzy closed set in X. Therefore, $clV \leq cl(1_X \setminus A) = 1_X \setminus A \Rightarrow clV \not q A \Rightarrow x_t \notin [A]_s^c$. Hence the proof.
Remark 3.6. From Theorem 3.2 and Theorem 3.5, we conclude that

$[A]^c_s$ is fuzzy semiclosed set in X for any $A \in I^X$.

Theorem 3.7. In an fts (X, τ), the following statements are true :
(a) 0_X and 1_X are fuzzy s^c-closed sets in X,
for all fuzzy sets A, B, (b) $A \leq B \Rightarrow [A]^c_s \leq [B]^c_s$,
(c) $[A \bigcup B]^c_s = [A]^c_s \bigcup [B]^c_s$,
(d) $[A \bigcap B]^c_s \leq [A]^c_s \bigcap [B]^c_s$, the equality does not hold, in general, as follows from the next example,
(e) union of any two fuzzy s^c-closed sets in X is also so,
(f) intersection of any two fuzzy s^c-closed sets in X is also so.

Proof. (a) and (b) are obvious.
(c) By (b), we can write, $[A]^c_s \bigcup [B]^c_s \leq [A \bigcup B]^c_s$.
To prove the converse, let $x_t \in [A \bigcup B]^c_s$. Then for any $U \in FSO(X)$ with $x_t qU$, $clUq(A \bigcup B)$. Then there exists $y \in X$ such that $(clU)(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $(clU)(y) + A(y) > 1$ or $(clU)(y) + B(y) > 1 \Rightarrow$ either $clUqA$ or $clUqB \Rightarrow$ either $x_t \in [A]^c_s$ or $x_t \in [B]^c_s \Rightarrow x_t \in [A]^c \bigcup [B]^c_s$.
(d) Follows from (b).
(e) Follows from (c).
(f) From (d), we have $[A \bigcap B]^c_s \leq [A]^c_s \bigcap [B]^c_s$ for any two fuzzy sets $A, B \in X$.
Conversely, let A, B be two fuzzy s^c-closed sets in X. Then $[A]^c_s = A, [B]^c_s = B$. Let $x_t \in [A]^c_s \bigcap [B]^c_s = A \bigcap B \leq [A \bigcap B]^c_s \Rightarrow [A]^c_s \bigcap [B]^c_s \leq [A \bigcap B]^c_s$.

Example 3.8. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $U \geq A$, $B \leq V \leq 1_X \setminus B$. Consider two fuzzy sets C and D defined by $C(a) = 0.4, C(b) = 0.1, D(a) = 0.1, D(b) = 0.55$ and the fuzzy point $a_{0.6}$. We claim that $a_{0.6} \in [C]^c_s \bigcap [D]^c_s$, but $a_{0.6} \notin [C \bigcap D]^c_s$. The fuzzy semiopen sets q-coincident with $a_{0.6}$ are of the form U, V_1, V_2 where $0.4 < V_1(a) \leq 0.7, V_1(b) = 0.5, V_2(a) > 0.7, V_2(b) \geq 0.55$. Then $clU = clV_2 = 1_X$ and so $clU = clV_2 qC$ and $clU = clV_2 qD$. Also $clV_1 = 1_X \setminus B$ and so $clV_1 qC$ and $clV_1 qD$. As a result $a_{0.6} \in [C]^c_s$ and $a_{0.6} \in [D]^c_s \Rightarrow a_{0.6} \in [C]^c_s \bigcap [D]^c_s$. Let $E = C \bigcap D$. Then $E(a) = E(b) = 0.1$. Then $clV_1 = (1_X \setminus B) \not{q} E \Rightarrow a_{0.6} \notin [E]^c_s$.

Note 3.9. Since arbitrary intersection (union) of fuzzy closed (open) sets is fuzzy closed (respectively, fuzzy open), so we can conclude that fuzzy s^c-open sets in an fts (X, τ) form a fuzzy topology τ_{s^c} (say) which is coarser than fuzzy topology τ of (X, τ).

Result 3.10. We conclude that $x_t \in [y_{t'}]^c_s$ does not necessarily imply $y_{t'} \in [x_t]^c_s$ where $x_t, y_{t'}$ $(0 < t, t' \leq 1)$ are fuzzy points in X as shown from the following example.

Example 3.11. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $U(a) = 0.5, 0.4 \leq U(b) \leq 0.6, V \geq B$. Now consider two fuzzy points $a_{0.1}$ and $b_{0.61}$. We claim that $a_{0.1} \in [b_{0.61}]^c_s$, but $b_{0.61} \notin [a_{0.1}]^c_s$. Now any fuzzy semiopen set q-coincident with $a_{0.1}$ is of the form V_1 where $V_1(a) > 0.9, V_1(b) \geq 0.5$. So $clV_1 = 1_X q b_{0.61} \Rightarrow a_{0.1} \in [b_{0.61}]^c_s$. Now $A \in FSO(X)$ with $b_{0.61} q A$, but $clA = (1_X \setminus A) \not{q} a_{0.1} \Rightarrow b_{0.61} \notin [a_{0.1}]^c_s$.

The next theorem shows that fuzzy s^c-closure operator is an idempotent operator.

Theorem 3.12. For any fuzzy set A in an fts (X, τ), $[A]^c_s = [[A]^c_s]^c_s$.

Proof. we first show that $A \subseteq [A]^c_s$. Let $x_t \in A$ be arbitrary. If possible, let $x_t \notin [A]^c_s$. Then there exists $U \in FSO(X)$ with $x_t q U$, $clU \not{q} A \Rightarrow A \leq 1_X \setminus clU$. Since $x_t \in A$, $x_t \in 1_X \setminus clU \Rightarrow 1 - (clU)(x) \geq t \Rightarrow x_t \not{q} clU$ which contradicts the fact that $x_t q U$. So $A \subseteq [A]^c_s$. Then by Theorem 3.7(b), $[A]^c_s \subseteq [[A]^c_s]^c_s$....(1).

Conversely, let $x_t \in [[A]^c_s]^c_s$. We have to show that $x_t \in [A]^c_s$. Let $U \in FSO(X)$ with $x_t q U$. By hypothesis, $clU q B$ where $B = [A]^c_s$. Then there exists $y \in X$ such that $(clU)(y) + B(y) > 1$. Let $B(y) = k$. Then $y_k \in B = [A]^c_s$ and $y_k q clU$. Since $U \in FSO(X) \Rightarrow clU \in FSO(X)$, then for $y_k \in [A]^c_s$, we have $cl(clU) = clU q A \Rightarrow x_t \in [A]^c_s \Rightarrow [[A]^c_s]^c_s \subseteq [A]^c_s$ .... (2). Combining (1) and (2), we have $[A]^c_s = [[A]^c_s]^c_s$.

4. Mutual Relationships

In this section we first recall some definitions of different types of fuzzy closure-like operators from [2, 3, 4, 6, 9, 10] and then establish the mutual relationships between these closure operators with fuzzy s^c-closure operator.

Definition 4.1. A fuzzy point x_t in an fts (X, τ) is called fuzzy s^*-cluster point [3] (respectively, fuzzy p^*-cluster point [4], fuzzy β^*-cluster point [2], fuzzy γ^*-cluster point [6]) of a fuzzy set A in X if for every $U \in FSO(X)$ (respectively, $U \in FPO(X)$, $U \in F\beta O(X)$, $F\gamma O(X)$) with $x_t q U$, $sclU q A$ (respectively, $pclU q A$, $\beta clU q A$, $\gamma clU q A$).

The union of all fuzzy s^*-cluster (respectively, fuzzy p^*-cluster, fuzzy β^*-cluster, fuzzy γ^*-cluster) points of a fuzzy set A is called fuzzy s^*-closure [3] (respectively, fuzzy p^*-closure [4], fuzzy β^*-closure [2], fuzzy γ^*-closure [6]) of A, denoted by $[A]_s$ (respectively, $[A]_p$, $[A]_\beta$, $[A]_\gamma$).

Definition 4.2 [10]. A fuzzy point x_t in an fts (X,τ) is called a fuzzy θ-cluster point of a fuzzy set A in X if $clUqA$ for every fuzzy open set Uin X with x_tqU.

The union of all fuzzy θ-cluster points of a fuzzy set A in an fts X is called fuzzy θ-closure of A, denoted by $[A]_\theta$.

Definition 4.3 [9]. A fuzzy point x_t in an fts (X,τ) is called a fuzzy δ-cluster point of a fuzzy set A in X if UqA for every fuzzy regular open set Uin X with x_tqU.

The union of all fuzzy δ-cluster points of a fuzzy set A in an fts X is called fuzzy δ-closure of A, denoted by $[A]_\delta$.

Note 4.4. It is clear from discussion that for any fuzzy set A in an fts (X,τ),

(i) $[A]_s \subseteq [A]_s^c \subseteq [A]_\theta$ and $[A]_\gamma \subseteq [A]_s^c$, $[A]_\beta \subseteq [A]_s^c$. But the reverse implications are not true, in general, as follow from the following examples.

(ii) $[A]_s^c$ is an independent concept of $[A]_p, [A]_\delta, clA$ as follow from the following examples.

Example 4.5. It is possible to have $[A]_s^c \not\subseteq [A]_s$ for some $A \in I^X$
Consider Example 3.4. Here $b_{0.51} \in [C]_s^c$. We claim that $b_{0.51} \notin [C]_s$. Here $U_1(a) = 0.5 = U_1(b)$ is a fuzzy semiopen set in X with $b_{0.51}qU_1$. But $sclU_1 = U_1 \not{q} C \Rightarrow b_{0.51} \notin [C]_s$.

Example 4.6. It is possible to have $[A]_s^c \not\subseteq [A]_\beta$, for some $A \in I^X$
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.7$. Then (X,τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U\}$ where $U \geq A$ and $F\beta O(X) = \{0_X, 1_X, V\}$ where $V \not\leq 1_X \setminus A$ and so $F\beta C(X) = \{0_X, 1_X, 1_X \setminus V\}$ where $1_X \setminus V \not\geq A$. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.6$ and the fuzzy point $a_{0.6}$. We claim that $a_{0.6} \notin [B]_\beta$. Indeed, $a_{0.6}qU_1 \in F\beta O(X)$ where $U_1(a) = 0.41, U_1(b) = 0.31$. But $\beta clU_1 = U_1 \not{q} B$. Again any $V_1 \in FSO(X)$ with $a_{0.6}qV_1$ is of the form $V_1(a) > 0.4, V_1(b) \geq 0.7$. Then $clV_1 = 1_XqB \Rightarrow a_{0.6} \in [B]_s^c$.

Example 4.7. It is possible to have $[A]_s^c \not\subseteq [A]_\gamma$, for some $A \in I^X$
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.45, A(b) = 0.4, B(a) = 0.6, B(b) = 0.5$. Then (X,τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus A, V \geq B$. Now consider the fuzzy

set D defined by $D(a) = 0.4, D(b) = 0.6$ and the fuzzy point $a_{0.5}$. Then $a_{0.5}qU_1 \in FSO(X)$ where $0.5 < U_1(a) \leq 0.55, 0.4 \leq U_1(b) \leq 0.6$ and $a_{0.5}qV_1 \in FSO(X)$ where $V_1 \geq B$. Then $clU_1 = (1_X \setminus A)qD$ and $clV_1 = 1_X qD \Rightarrow a_{0.5} \in [D]^c_s$. Now consider the fuzzy set E defined by $E(a) = 0.51, E(b) = 0.4$. Then $E \in F\gamma O(X)$ and $a_{0.5}qE$, but $\gamma clE = E \not{q} D \Rightarrow a_{0.5} \notin [D]_\gamma$.

Example 4.8. It is possible to have $clA \not\supseteq [A]^c_s$, for some $A \in I^X$
Consider Example 4.6 and the fuzzy set S defined by $S(a) = 0.2, S(b) = 0.3$ and the fuzzy point $b_{0.4}$. Then $b_{0.4} \notin clS = 1_X \setminus A$. Here $FSO(X) = \{0_X, 1_X, U\}$ where $U \geq A$. Then any $U \in FSO(X)$, $b_{0.4}qU$ and $clU = 1_X qS \Rightarrow b_{0.4} \in [S]^c_s$.

Example 4.9. It is possible to have $[A]^c_s \subsetneqq [A]_\theta, clA$, for some $A \in I^X$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $U \geq A, 0.3 \leq V(a) \leq 0.7, V(b) = 0.5$. Consider the fuzzy set D defined by $D(a) = 0.2, D(b) = 0.5$ and the fuzzy point $a_{0.41}$. Then $a_{0.41} \in clD = 1_X \setminus B$. Now $1_X \in \tau$ only such that $a_{0.41}q1_X$ and so clearly $a_{0.41} \in [D]_\theta$. Consider the fuzzy set $V_1 \in FSO(X)$ such that $V_1(a) = 0.6, V_1(b) = 0.5$. Then $a_{0.41}qV_1$, but $clV_1 = (1_X \setminus B) \not{q} D \Rightarrow a_{0.41} \notin [D]^c_s$.

Example 4.10. It is possible to have $[A]_p \subsetneqq [A]^c_s$, for some $A \in I^X$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V, W\}$ where $U \leq A$ and $U \not\leq 1_X \setminus B, V > 1_X \setminus A, W \geq B$ and $FSO(X) = \{0_X, 1_X, S, W\}$ where $A \leq S \leq 1_X \setminus A, W \geq B$. Consider the fuzzy set D defined by $D(a) = D(b) = 0.6$ and the fuzzy point $b_{0.71}$. Then the fuzzy set U_1, defined by $U_1(a) = 0.4, U_1(b) = 0.3$, is fuzzy preopen set in X with $b_{0.71}qU_1$. But $pclU_1 = U_1 \not{q} D \Rightarrow b_{0.71} \notin [D]_p$. Now any fuzzy semiopen set other than 0_X in X is S and W, q-coincident with $b_{0.71}$ and clearly $clSqD$ and $clW = cl1_X = 1_X qD \Rightarrow b_{0.71} \in [D]^c_s$.

Example 4.11. It is possible to have $[A]_p, [A]_\delta \not\subseteq [A]^c_s$, for some $A \in I^X$
Let $X = \{a\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, B(a) = 0.7$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V\}$ where $1_X \setminus B \leq U \leq A, V \geq B$ and $FSO(X) = \{0_X, 1_X, V, W\}$ where $V \geq B, A \leq W \leq 1_X \setminus A$. Consider the fuzzy set D defined by $D(a) = 0.4$ and the fuzzy point $a_{0.5}$. Then for any $V \in FPO(X), Vqa_{0.5}$.

Then $pclV = 1_X qD \Rightarrow a_{0.5} \in [D]_p$. Again $FRO(X) = \{0_X, 1_X, A\}$. Then $1_X \in FRO(X)$ only with $a_{0.5}q1_X$ and so clearly $a_{0.5} \in [D]_\delta$. Now $W_1(a) = 0.51$ be such that $W_1 \in FSO(X)$ with $a_{0.5}qW_1$, but $clW_1 = (1_X \setminus A) \not{q} D \Rightarrow a_{0.5} \notin [D]^c_s$.

Example 4.12. It is possible to have $[A]^c_s \not\subseteq [A]_\delta$, for some $A \in I^X$
Consider Example 4.10 and the fuzzy set D defined by $D(a) = 0.3, D(b) = 0.6$ and the fuzzy point $b_{0.71}$. Here $FRO(X) = \{0_X, 1_X, A\}$. So $b_{0.71}qA \in FRO(X)$, but $A \ /qD \Rightarrow b_{0.71} \notin [D]_\delta$. Again the fuzzy set of the form $S \in FSO(X)$ is such that $Sqb_{0.71}$and $clS = (1_X \setminus A)qD$. Also the fuzzy set of the form $W \in FSO(X)$ is such that $b_{0.71}qW$ and $clW = 1_X qD \Rightarrow b_{0.71} \in [D]^c_s$.

5. Fuzzy s^c-Regular Space

In this section a new type of fuzzy separation axiom is introduced and characterized by fuzzy s^c-closure operator. Also it is shown that in this space fuzzy semiclosure operator and fuzzy s^c-closure operator coincide.

Definition 5.1. An fts (X, τ) is called fuzzy s^c-regular space if for each fuzzy point x_t and each fuzzy semiopen set U in X with x_tqU, then there exists $V \in \tau$ such that $x_tqV \leq clV \leq U$.

Theorem 5.2. For an fts (X, τ), the following statements are equivalent :

(a) X is fuzzy s^c-regular space,
(b) for any $A \in I^X$, $sclA = [A]^c_s$,
(c) for each fuzzy point x_t and each $U \in FSC(X)$ with $x_t \notin U$, there exists $V \in \tau$ such that $x_t \notin clV$ and $U \leq V$,
(d) for each fuzzy point x_t and each $U \in FSC(X)$ with $x_t \notin U$, there exist $V, W \in \tau$ such that x_tqV, $U \leq W$ and $V \not{q} W$,
(e) for any $A \in I^X$ and any $U \in FSC(X)$ with $A \not\leq U$, there exist $V, W \in \tau$ such that $AqV, U \leq W$ and $V \not{q} W$,
(f) for any $A \in I^X$ and any $U \in FSO(X)$ with AqU, there exists $V \in \tau$ such that $AqV \leq clV \leq U$.

Proof. (a) $\Rightarrow$ (b) By Note 3.3, it suffices to show that $[A]^c_s \subseteq sclA$, for any $A \in I^X$. Let $x_t \in [A]^c_s$ be arbitrary and $V \in FSO(X)$ with x_tqV. By (a), there exists $U \in \tau$ such that $x_tqU \leq clU \leq V$. Since $U \in \tau \Rightarrow U \in FSO(X)$, by hypothesis, $clUqA \Rightarrow VqA \Rightarrow x_t \in sclA \Rightarrow [A]^c_s \subseteq sclA$.
(b) $\Rightarrow$ (a) Let x_t be a fuzzy point in X and $U \in FSO(X)$ with x_tqU.

Then $U(x)+t > 1 \Rightarrow x_t \notin 1_X \backslash U (\in FSC(X)) = scl(1_X \backslash U) = [1_X \backslash U]_s^c$ (by (b)). Then there exists $V \in FSO(X)$ with $x_t qV$, $clV \not{q} (1_X \backslash U) \Rightarrow clV \leq U$. Therefore, $x_t qV \leq clV \leq U \Rightarrow X$ is fuzzy s^c-regular space.
(a) $\Rightarrow$ (c) Let x_t be a fuzzy point in X and $U \in FSC(X)$ with $x_t \notin U$. Then $x_t q(1_X \setminus U) \in FSO(X)$. By (a), there exists $V \in \tau$ such that $x_t qV \leq clV \leq 1_X \setminus U$. Therefore, $U \leq 1_X \setminus clV$ $(= W$, say). Then $W \in \tau$. Now $x_t qV = intV \Rightarrow x_t qintV \leq V \leq intclV \Rightarrow x_t q(intclV) \Rightarrow (intclV)(x) + t > 1 \Rightarrow 1 - (intclV)(x) < t \Rightarrow x_t \notin 1_X \setminus intclV = cl(1_X \setminus clV) = clW$.
(c) $\Rightarrow$ (d) Let x_t be a fuzzy point in X and $U \in FSC(X)$ with $x_t \notin U$. By (c), there exists $V \in \tau$ such that $U \leq V$ and $x_t \notin clV \Rightarrow$ there exists $W \in \tau$ such that $x_t qW$, $W \not{q} V$. Now $V \not{q} (1_X \setminus clV)$. Now $x_t q(1_X \setminus clV)$ $(= W$, say). Then $W \in \tau$ and $V \not{q} W$. So we get, $V, W \in \tau$ with $x_t qW$, $U \leq V$ and $V \not{q} W$.
(d) $\Rightarrow$ (e) Let $A \in I^X$ and $U \in FSC(X)$ with $A \not\leq U$. Then there exists $x \in X$ such that $A(x) > U(x)$. Let $A(x) = t$. Then $x_t \notin U$. By (d), there exist $V, W \in \tau$ such that $x_t qV, U \leq W$ and $V \not{q} W$. Again $V(x) + t > 1 \Rightarrow V(x) + A(x) > 1 - t + t = 1 \Rightarrow AqV$.
(e) $\Rightarrow$ (f) Let $A \in I^X$ and $U \in FSO(X)$ with AqU. Then $A \not\leq 1_X \backslash U \in FSC(X)$. By (e), there exist $V, W \in \tau$ such that $A \leq V, 1_X \setminus U \leq W$ and $V \not{q} W \Rightarrow V \leq 1_X \setminus W \in \tau^c \Rightarrow clV \leq cl(1_X \setminus W) = 1_X \setminus W \leq U$. Therefore, $A \leq V \leq clV \leq U$.
(f) $\Rightarrow$ (a) Obvious.

Corollary 5.3. An fts (X, τ) is fuzzy s^c-regular if and only if fuzzy semiclosed set in X is fuzzy s^c-closed set in X.

Proof. Let (X, τ) be fuzzy s^c-regular space and $A \in FSC(X)$. Then by Theorem 5.2 (a)$\Rightarrow$ (b), $A = sclA = [A]_s^c \Rightarrow A$ is fuzzy s^c-closed set in X.
Conversely, let $A = [A]_s^c$ for any $A \in FSC(X)$. Let $B \in I^X$. Then $sclB \in FSC(X)$ and so by hypothesis, $sclB = [sclB]_s^c$. Then $[B]_s^c \leq [sclB]_s^c = sclB$. By Note 3.3, $sclB \leq [B]_s^c$. Combining these two, we get $[B]_s^c = sclB$ for any $B \in I^X$. Then by Theorem 5.2 (b)$\Rightarrow$ (a), X is fuzzy s^c-regular space.

6. Fuzzy s^c-Closure Operator : More characterizations

In this section we first introduce the notions of fuzzy s^c-cluster point and fuzzy s^c-convergence of a fuzzy net and then fuzzy s^c-closure operator of a fuzzy set is characterized in terms of these concepts.

Definition 6.1. A fuzzy point x_t in an fts (X,τ) is called a fuzzy s^c-cluster point of a fuzzy net $\{S_t : n \in (D,\geq)\}$ if for every fuzzy semiopen set U in X with $x_t qU$ and for any $n \in D$, there exists $m \in D$ with $m \geq n$ such that $S_m q clU$.

Definition 6.2. A fuzzy net $\{S_n : n \in (D,\geq)\}$ in an fts (X,τ) is said to s^c-converge to a fuzzy point x_t if for every fuzzy semiopen set U in X, $x_t qU$, there exists $m \in D$ such that $S_n q clU$, for all $n \geq m$ $(n \in D)$. This is denoted by $S_n \overrightarrow{s^c} x_t$.

Theorem 6.3. A fuzzy point x_t is a fuzzy s^c-cluster point of a fuzzy net $\{S_n : n \in (D,\geq)\}$ in an fts (X,τ) iff there exists a fuzzy subset of $\{S_n : n \in (D,\geq)\}$ which s^c-converges to x_t.

Proof. Let x_t be a fuzzy s^c-cluster point of the fuzzy net $\{S_n : n \in (D,\geq)\}$. Let $C(Q_{x_t})$ denote the set of fuzzy closures of all fuzzy semiopen sets of X q-coincident with x_t. Then for any $A \in C(Q_{x_t})$, there exits $n \in D$ such that $S_n qA$. Let E denote the set of all ordered pairs (n, A) such that $n \in D$, $A \in C(Q_{x_t})$ and $S_n qA$. Then $(E,\gg)$ is a directed set, where $(m,A) \gg (n,B)$ $((m,A),(n,B) \in E)$ iff $m \geq n$ in D and $A \leq B$. Then $T : (E,\gg) \to (X,\tau)$ given by $T(m,A) = S_m$ is clearly a fuzzy subnet of $\{S_n : n \in (D,\geq)\}$. We claim that $T \overrightarrow{s^c} x_t$. Let V be any fuzzy semiopen set in X with $x_t qV$. Then there exists $n \in D$ such that $(n, clV) \in E$ and so $S_n q clV$. Now for any $(m,A) \gg (n, clV)$, $T(m,A) = S_m qA \leq clV \Rightarrow T(m,A) q clV$. Consequently, $T \overrightarrow{s^c} x_t$.

Conversely, let x_t be not a fuzzy s^c-cluster point of the fuzzy net $\{S_n : n \in (D,\geq)\}$. Then there exists $U \in FSO(X)$ with $x_t qU$ and an $n \in D$ such that $S_m \not{q} clU$, for all $m \geq n$. Then clearly no fuzzy subnet of the net $\{S_n : n \in (D,\geq)\}$ can s^c-converge to x_t.

Theorem 6.4. Let A be a fuzzy set in an fts (X,τ). A fuzzy point $x_t \in [A]^c_s$ iff a fuzzy net $\{S_n : n \in (D,\geq)\}$ in A, which s^c-converges to x_t.

Proof. Let $x_t \in [A]^c_s$. Then for any $U \in FSO(X)$ with $x_t qU$, $clUqA$, i.e., there exists $y^U \in suppA$ and a real number s_U with $0 < s_U \leq A(y^U)$ such that the fuzzy point $y^U_{s_U}$ with support y^U and the value s_U belong to A and $y^U_{s_U} q clU$. We choose and fix one such $y^U_{s_U}$ for each U. Let $\mathcal{D}$ denote the set of all fuzzy semiopen set in X q-coincident with x_t. Then $(\mathcal{D},\succeq)$ is a directed set under inclusion relation, i.e., $B, C \in \mathcal{D}$, $B \succeq C$ iff $B \leq C$. Then $\{y^U_{s_U} \in A : y^U_{s_U} q clU, U \in \mathcal{D}\}$ is a fuzzy net in A such that it s^c-converges to x_t. Indeed, for any fuzzy semiopen set U in X with $x_t qU$, if $V \in \mathcal{D}$ and $V \succeq U$ (i.e., $V \leq U$) then $y^V_{s_V} qclV \leq clU \Rightarrow y^V_{s_V} qclU$.

Conversely, let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A such that $S_n \overrightarrow{s^c} x_t$. Then for any $U \in FSO(X)$ with $x_t qU$, there exists $m \in D$ such that $n \geq m \Rightarrow S_n qclU \Rightarrow AqclU$ (since $S_n \in A$). Hence $x_t \in [A]^c_s$.

Remark 6.5. It is clear that an improved version of the converse of the last theorem can be written as "$x_t \in [A]^c_s$ if there exists a fuzzy net in A with x_t as a fuzzy s^c-cluster point".

References

[1] K. K. Azad, **On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity**, J.Math. Anal. Appl. 82 (1981), 14-32.

[2] A. Bhattacharyya, **Fuzzy generalized closed sets in a fuzzy topological space**, J. Fuzzy Math. 25 (2) (2017), 285-301.

[3] A. Bhattacharyya, **s^*-closure operator and s^*-regularity in fuzzy setting**, Int. J. Pure Appl. Math. 96 (2) (2014), 279-288.

[4] A. Bhattacharyya, **p^*-closure operator and p^*-regularity in fuzzy setting**, Math. Morav. 19 (1) (2015), 131-139.

[5] A. Bhattacharyya, **Fuzzy γ-continuous Multifunction**, Int. J. Adv. Res. Sci. Eng. 4 (2) (2015), 195-209.

[6] A. Bhattacharyya, **γ^*-regular space**, J. Fuzzy Math. 28 (2) (2020), 709-719.

[7] C.L. Chang, **Fuzzy topological spaces**, J. Math. Anal. Appl. (24) (1968), 182-190.

[8] M.A. Fath Alla,**On fuzzy topological spaces**, Ph.D. Thesis, Assiut Univ., Sohag, Egypt, (1984).

[9] S. Ganguly and S. Saha,**A note on δ-continuity and δ-connected sets in fuzzy set theory**, Simon Stevin 62 (1988), 127-141.

[10] M.N. Mukherjee and S.P. Sinha, **Fuzzy θ-closure operator on fuzzy topological spaces**, Int. J. Math. Math. Sci. 14 (2) (1991), 309-314.

[11] S. Nanda, **Strongly compact fuzzy topological spaces**, Fuzzy Sets Syst. 42 (1991), 259-262.

[12] P. M. Pu and Y. M. Liu, **Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith Convergence**, J. Math Anal. Appl. 76 (1980), 571-599.

[13] L.A. Zadeh, **Fuzzy Sets**, Inf. Control 8 (1965), 338-353.

Victoria Institution (College),
Department of Mathematics,
78B, A.P.C. Road,
Kolkata-700009, India
e-mail: anjanabhattacharyya@hotmail.com