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COMPARATIVE GROWTH OF COMPOSITE ENTIRE FUNCTIONS WITH FINITE LOGARITHMIC ORDER

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Abstract. In this article we studied some growth properties of composite entire functions with finite logarithmic order. Also we proved some results on the growth of composite entire functions of finite logarithmic order with respect to their maximum terms. Further we proved some results on the relative growth of one set of composite entire functions with another set of composite entire functions having the same right factor as well as having different left and right factors with respect to logarithmic order.

1. Introduction

The maximum modulus of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is defined as $M_f(r) = \max\{|f(z)| : |z| \leq r\}$ for $r > 0$. It follows immediately that $M_f(r)$ is nondecreasing function of r.

The order $\rho(f)$ and lower order $\lambda(f)$ of the entire function $f(z)$ are

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

respectively.

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Also by Nevanlinna theory [3], the order $\rho(f)$ and lower order $\lambda(f)$ of an entire function $f(z)$ are defined as,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}$$
$$\lambda(f) = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

where $T_f(r)$ is the Nevanlinna's characteristic function.

Definition 1. [1] *An entire function $f(z)$ is said to have finite logarithmic order $\rho_{\log}$ if*

$$\rho_{\log} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\log r}$$

and $f(z)$ is said to have finite lower logarithmic order $\lambda_{\log}$ if

$$\lambda_{\log} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\log r}.$$

One can easily check that $\rho_{\log} < \lambda_{\log} + 1$ and there is a constant c satisfying $0 \leq c < \rho_{\log} - \lambda_{\log}$.

Definition 2. *Also for a transcendental entire function $f(z)$ with order zero, the logarithmic order $\rho_{\log}(f)$ and lower logarithmic order $\lambda_{\log}(f)$ are defined as*

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\log r}, \tag{1}$$

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\log r}. \tag{2}$$

Definition 3. [5] *We have for $0 \leq r < R$,*

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R). \tag{3}$$

Definition 4. *Using above result we can define $\rho_{\log}(f)$ and $\lambda_{\log}(f)$ as*

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log \mu_f(r)}{\log\log r} \tag{4}$$

and

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log \mu_f(r)}{\log\log r}, \tag{5}$$

where $\mu_f(r)$ *is the maximum term of the function* $f(z)$ *on* $|z| = r$ *is defined as* $\mu_f(r) = \max_{n\geq 0} |a_n| r^n$.

Now it is already known [8] that for any two transcendental entire functions $f(z)$ and $g(z)$ with $0 < \lambda(f) \leq \rho(f) < \infty$,

$$\lim_{r\to\infty} \frac{\log\log M_{f\circ g}(r)}{\log\log M_f(r)} = \infty. \tag{6}$$

A.P.Singh and M.S.Baloria [6], proved that for sufficiently large $R = R(r)$,

$$\limsup_{r\to\infty} \frac{\log\log M_{f\circ g}(r)}{\log\log M_f(R)} < \infty. \tag{7}$$

Also they derived that for very large R, say $R = r^A$ along with the condition $\lambda(f) > 0, \lambda(g) > 0$, the result (7) does not hold good for every positive constant A.

However if we consider $R = \exp r^{\rho(f)}$, they showed that the limit in (7) became zero.

In this paper we will develop results made by A.P.Singh and M.S.Baloria [6] with respect to logarithmic order and will obtain the limit as zero. In the composition we shall also deal with the right factor instead of left factor in the denominator of (7) .We will also prove some related results for the maximum term using parallel technique.

Further we will prove some results on relative growth (developed from [7], where both numerator and denominator are of composite type) of entire functions with respect to the logarithmic order.

2. Preliminary Lemmas

In this section we shall present first the following known lemmas.

Lemma 5. [8] *Let* $\lambda(g) < \infty$. *Then for any* $\varepsilon > 0$ *and sufficiently large* r,

$$M_{f\circ g}\left(r^{1+\varepsilon}\right) \geq M_f\left(M_g(r)\right). \tag{8}$$

Lemma 6. [2] *If* $f(z)$ *and* $g(z)$ *are two entire functions with* $g(0) = 0$, *then for* $r > 0$

$$M_{f\circ g}(r) \geq M_f\left(c(\alpha)\, M_g(\alpha r)\right), \tag{9}$$

where α *satisfy* $0 < \alpha < 1$ *and take* $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$.

Further if $g(z)$ is any entire function then with $\alpha = \frac{1}{2}$, for sufficiently large values of r,

$$M_{f \circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right) - |g(0)|\right). \tag{10}$$

Also from the definition it follows immediately that

$$M_{f \circ g}(r) \leq M_f\left(M_g(r)\right) \tag{11}$$

Lemma 7. [5] *Let $f(z)$ and $g(z)$ be entire functions, then for $\alpha > 1$, and $0 < r < R$,*

$$\mu_{f \circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f\left(\frac{\alpha R}{R - r} \mu_g(r)\right).$$

In particular taking $\alpha = 2$ and $R = 2r$,

$$\mu_{f \circ g}(r) \leq 2\mu_f\left(4\mu_g(2r)\right) \tag{12}$$

Lemma 8. [5] *Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Also let $0 < \delta < 1$ then*

$$\mu_{f \circ g}(r) \geq (1 - \delta)\,\mu_f\left(c(\alpha)\mu_g(\alpha\delta r)\right).$$

And if $g(z)$ is any entire function, then with $\alpha = \delta = \frac{1}{2}$, for sufficiently large values of r,

$$\mu_{f \circ g}(r) \geq \frac{1}{2} \mu_f\left(\frac{1}{8} \mu_g\left(\frac{r}{4}\right) - |g(0)|\right). \tag{13}$$

Lemma 9. [8] *If $f(z)$ and $g(z)$ are two entire functions with $M_g(r) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then*

$$T_{f \circ g}(r) \leq (1 + \varepsilon)\, T_f\left(M_g(r)\right). \tag{14}$$

In particular if $g(0) = 0$, then for all $r > 0$

$$T_{f \circ g}(r) \leq T_f\left(M_g(r)\right). \tag{15}$$

Lemma 10. [4] *If $f(z)$ and $g(z)$ are two entire functions, then*

$$\begin{aligned} T_{f \circ g}(r) &\geq \frac{1}{3} \log M_f\left\{\frac{1}{8} M_g\left(\frac{r}{4}\right) + o(1)\right\} \\ &\geq \frac{1}{3} \log M_f\left\{\frac{1}{9} M_g\left(\frac{r}{4}\right)\right\}. \end{aligned} \tag{16}$$

3. Main results

Now we will prove our main results here.

Theorem 11. *Let $f(z)$ and $g(z)$ be two entire functions of positive lower logarithmic order and of finite logarithmic order. Then for every constant $A > 0$,*

$$\liminf_{r\to\infty} \frac{\log\log M_{f\circ g}(r)}{\log\log M_f(r^A)} = 0$$

and

$$\liminf_{r\to\infty} \frac{\log\log M_{f\circ g}(r)}{\log\log M_g(r^A)} = 0.$$

Proof. We know by Lemma 6 that for $r \geq r_0$,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{16} M_g\left(\frac{r}{2}\right)\right).$$

That implies ,

$$\log\log M_{f\circ g}(r) \geq \log\log M_f\left(\frac{1}{16} M_g\left(\frac{r}{2}\right)\right). \tag{17}$$

Since $\lambda_{\log}(f) > 0$, for $r \geq r_0$,

$$\liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} = \lambda_{\log}(f)$$

$$\Rightarrow \log\log M_f(r) > (\lambda_{\log}(f) - \varepsilon)\log\log r, \text{ for any } \varepsilon > 0. \tag{18}$$

Combining (17) and (18) ,

$$\begin{aligned} \log\log M_{f\circ g}(r) &> (\lambda_{\log}(f) - \varepsilon)\log\log\left(\frac{1}{16} M_g\left(\frac{r}{2}\right)\right) \\ &= (\lambda_{\log}(f) - \varepsilon)\log\left\{\log\frac{1}{16} + \log M_g\left(\frac{r}{2}\right)\right\} \\ &> (\lambda_{\log}(f) - \varepsilon)\log\log M_g\left(\frac{r}{2}\right). \end{aligned} \tag{19}$$

Again by Definition 2 for $r \geq r_0$,

$$\log\log M_g(r) > (\lambda_{\log}(g) - \varepsilon)\log\log r, \text{ for any } \varepsilon > 0. \tag{20}$$

Using (20) in (19) we have,

$$\log\log M_{f\circ g}(r) > (\lambda_{\log}(f) - \varepsilon)(\lambda_{\log}(g) - \varepsilon)\log\log\left(\frac{r}{2}\right). \tag{21}$$

Also by Definition 2,

$$\log\log M_f(r) < (\rho_{\log}(f)+\varepsilon)\log\log r \text{ , for any } \varepsilon > 0.$$

Now for sufficiently large r $(\geq r_0)$, so that $r^A \geq r_0$, we have from above

$$\begin{aligned} \log\log M_f(r^A) &< (\rho_{\log}(f)+\varepsilon)\log\log r^A \\ &< (\rho_{\log}(f)+\varepsilon)\log r^A \\ &< (\rho_{\log}(f)+\varepsilon)\, r^A. \end{aligned} \tag{22}$$

Now using (21) and (22) for sufficiently large r,

$$\frac{\log\log M_{f\circ g}(r)}{\log\log M_f(r^A)} > \frac{(\lambda_{\log}(f)-\varepsilon)(\lambda_{\log}(g)-\varepsilon)\log\log\left(\frac{r}{2}\right)}{(\rho_{\log}(f)+\varepsilon)\, r^A}.$$

We choose $\varepsilon > 0$ so that $(\lambda_{\log}(f)-\varepsilon) > 0, (\lambda_{\log}(g)-\varepsilon) > 0$ and as $r \to \infty$, $\frac{\log\log\left(\frac{r}{2}\right)}{r^A} \to 0$.

Thus we finally get,

$$\liminf_{r\to\infty}\frac{\log\log M_{f\circ g}(r)}{\log\log M_f(r^A)} = 0.$$

In a similar manner using

$$\log\log M_g(r^A) < (\rho_{\log}(g)+\varepsilon)\, r^A$$

along with (21) we get

$$\liminf_{r\to\infty}\frac{\log\log M_{f\circ g}(r)}{\log\log M_g(r^A)} = 0.$$

□

Remark 12. *Theorem* 11 *need not be true if either* $\lambda_{\log}(g) = 1$ *or* $\lambda_{\log}(f) = 1$. *For this purpose consider* $g(z) = z$ *and* $A = 1$, *then* $\lambda_{\log}(g) = 1$ *and* $\liminf_{r\to\infty}\frac{\log\log M_{f\circ g}(r)}{\log\log M_f(r^A)} = 1$.

Similarly, if we consider $f(z) = z$ *and* $A = 1$, *then* $\lambda_{\log}(f) = 1$ *and* $\liminf_{r\to\infty}\frac{\log\log M_{f\circ g}(r)}{\log\log M_g(r^A)} = 1$.

Theorem 13. *Suppose* $f(z)$ *and* $g(z)$ *are two entire functions of finite logarithmic orders* $\rho_{\log}(g)$ *and* $\rho_{\log}(f)$ *respectively with* $\rho_{\log}(g) > \rho_{\log}(f)$, *then*

$$\limsup_{r\to\infty}\frac{\log\log M_{f\circ g}(r)}{\log\log M_f(\exp(R_n)^{\rho_{\log}(f)})} = \infty.$$

Proof. We know by Lemma 6 that for $r \geq r_0$,

$$M_{f \circ g}(r) \leq M_f(M_g(r)).$$

Thus

$$\log \log M_{f \circ g}(r) \leq \log \log M_f(M_g(r)).$$

Also by Definition 2, for all $r \geq r_0$ and for any $\varepsilon > 0$,

$$\log \log M_f(r) < (\rho_{\log}(f) + \varepsilon) \log \log r. \tag{23}$$

Therefore combining the above two,

$$\log \log M_{f \circ g}(r) < (\rho_{\log}(f) + \varepsilon) \log \log M_g(r). \tag{24}$$

Also since $\rho_{\log}(g) < \infty$, we get by Definition 2

$$\limsup_{r \to \infty} \frac{\log \log M_g(r)}{\log \log r} = \rho_{\log}(g).$$

So for any $\varepsilon > 0$,

$$\log \log M_g(r) < (\rho_{\log}(g) + \varepsilon) \log \log r.$$

Hence from (24)we have for a sequence $r = r_n \to \infty$ and for any $\varepsilon > 0$,

$$\begin{aligned} \log \log M_{f \circ g}(r) &< (\rho_{\log}(f) + \varepsilon)(\rho_{\log}(g) + \varepsilon) \log \log r \\ &< (\rho_{\log}(f) + \varepsilon) \log (\log r)^{(\rho_{\log}(g) + \varepsilon)} \\ &< (\rho_{\log}(f) + \varepsilon) (\log r)^{(\rho_{\log}(g) + \varepsilon)} \\ &< (\rho_{\log}(f) + \varepsilon) r^{(\rho_{\log}(g) + \varepsilon)}. \end{aligned} \tag{25}$$

On the otherhand, for a sequence $r = r_n \to \infty$ and given any $\varepsilon > 0$, it follows

$$\log \log M_f(r) > (\rho_{\log}(f) - \varepsilon) \log \log r. \tag{26}$$

Consider $R_n = (\log r_n)^{\frac{1}{\rho_{\log}(f)}}$, we get from (26),

$$\begin{aligned} \log \log M_f(\exp(R_n)^{\rho_{\log}(f)}) &> (\rho_{\log}(f) - \varepsilon) \log \log \exp(R_n)^{\rho_{\log}(f)} \\ &> (\rho_{\log}(f) - \varepsilon) \log (R_n)^{\rho_{\log}(f)}. \end{aligned} \tag{27}$$

Thus for $r = R_n (\geq r_0)$, we get from (25) and (27),

$$\frac{\log \log M_{f \circ g}(r)}{\log \log M_f(\exp(r^{\rho_{\log}(f)}))} < \frac{(\rho_{\log}(f) + \varepsilon) r^{(\rho_{\log}(g) + \varepsilon)}}{(\rho_{\log}(f) - \varepsilon) \log r^{\rho_{\log}(f)}}. \tag{28}$$

Now since $\rho_{\log}(g) > \rho_{\log}(f)$, for any $\varepsilon > 0$,

$$\rho_{\log}(g) + \varepsilon > \rho_{\log}(f)$$

and thus as $r \to \infty$,

$$\frac{r^{\left(\rho_{\log}(g)+\varepsilon\right)}}{\log r^{\rho_{\log}(f)}} \to \infty.$$

Therefore,

$$\limsup_{r\to\infty} \frac{\log\log M_{f\circ g}(r)}{\log\log M_f(\exp\left(r^{\rho_{\log}(f)}\right))} = \infty.$$

$\square$

Example 14. *Consider* $f(z) = z$ *and* $g(z) = z$, *then* $\rho_{\log}(f) = 1, \rho_{\log}(g) = 1$. *Hence by* (28), $\frac{\log\log M_{f\circ g}(r)}{\log\log M_f(\exp\left(r^{\rho_{\log}(f)}\right))} < \frac{r}{\log r} \to \infty$ *as* $r \to \infty$.

Theorem 15. *Consider two entire functions* $f(z)$ *and* $g(z)$ *of finite logarithmic orders* $\rho_{\log}(f)$, $\rho_{\log}(g)$ *and finite lower logarithmic orders* $\lambda_{\log}(g)$, $\lambda_{\log}(f)$ *respectively with* $\rho_{\log}(f) \geq \rho_{\log}(g) > \lambda_{\log}(g) \geq \lambda_{\log}(f) > 0$. *Then*

$$\liminf_{r\to\infty} \frac{\log\log M_{f\circ g}(r)}{\log\log M_g(\exp\left(r^{\rho_{\log}(f)}\right))} = 0.$$

Proof. For all $r \geq r_0$ from (21) we have for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\log\log M_{f\circ g}(r) > (\lambda_{\log}(f) - \varepsilon)(\lambda_{\log}(g) - \varepsilon)\log\log\left(\frac{r}{2}\right)$$

Also by Definition 2, for all $r \geq r_0$ and chosen $\varepsilon > 0$,

$$\log\log M_g(r) < (\rho_{\log}(g) + \varepsilon)\log\log r.$$

We choose r large enough so that $\exp\left(r^{\rho_{\log}(f)}\right) \geq r_0$. Thus we get from above

$$\begin{aligned} \log\log M_g(\exp\left(r^{\rho_{\log}(f)}\right)) &< (\rho_{\log}(g) + \varepsilon)\log\log\exp\left(r^{\rho_{\log}(f)}\right) \\ &= (\rho_{\log}(g) + \varepsilon)\log\left(r^{\rho_{\log}(f)}\right). \end{aligned}$$

Therefore for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\frac{\log\log M_{f\circ g}(r)}{\log\log M_g(\exp\left(r^{\rho_{\log}(f)}\right))} > \frac{(\lambda_{\log}(f) - \varepsilon)(\lambda_{\log}(g) - \varepsilon)\log\log\left(\frac{r}{2}\right)}{(\rho_{\log}(g) + \varepsilon)\log\left(r^{\rho_{\log}(f)}\right)}.$$

Now since $\rho_{\log}(f) \geq \rho_{\log}(g) > \lambda_{\log}(g) \geq \lambda_{\log}(f) > 0$,for any $\varepsilon > 0$,

$$\frac{(\lambda_{\log}(f) - \varepsilon)}{(\rho_{\log}(g) + \varepsilon)} < 1, \rho_{\log}(f) > \lambda_{\log}(g) - \varepsilon.$$

Hence as $r \to \infty$, the ratio

$$\frac{\log\left(\log\frac{r}{2}\right)^{\left(\lambda_{\log}(g)-\varepsilon\right)}}{\log r^{\rho_{\log}(f)}} \to 0.$$

Therefore

$$\liminf_{r\to\infty}\frac{\log\log M_{f\circ g}(r)}{\log\log M_g(\exp\left(r^{\rho_{\log}(f)}\right))} = 0.$$

□

Theorem 16. *Suppose $f(z)$ and $g(z)$ are two entire functions of positive lower logarithmic orders $\lambda_{\log}(g)$, $\lambda_{\log}(f)$ and of finite logarithmic orders $\rho_{\log}(f)$, $\rho_{\log}(g)$. Then for every constant $A > 0$,*

$$\liminf_{r\to\infty}\frac{\log\log\mu_{f\circ g}(r)}{\log\log\mu_f(r^A)} = 0$$

and

$$\liminf_{r\to\infty}\frac{\log\log\mu_{f\circ g}(r)}{\log\log\mu_g(r^A)} = 0.$$

Proof. By applying Lemma 8, we have for r $(\geq r_0)$ large enough,

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)$$

and thus

$$\begin{aligned}
\log\log\mu_{f\circ g}(r) &\geq \log\log\frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\\
&\geq \log\left[\log\frac{1}{2}+\log\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\right]\\
&\geq \log\log\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\\
&> (\lambda_{\log}(f)-\varepsilon)\log\log\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\ \text{(for any } \varepsilon > 0\text{)}\\
&> (\lambda_{\log}(f)-\varepsilon)\log\left[\log\frac{1}{16}+\log\mu_g\left(\frac{r}{4}\right)\right]\\
&> (\lambda_{\log}(f)-\varepsilon)\log\log\mu_g\left(\frac{r}{4}\right)\\
&> (\lambda_{\log}(f)-\varepsilon)(\lambda_{\log}(g)-\varepsilon)\log\log\left(\frac{r}{4}\right). \qquad (29)
\end{aligned}$$

Also Definition4 for all $r \geq r_0$ and for any $\varepsilon > 0$,

$$\log\log \mu_f(r) < (\rho_{\log}(f) + \varepsilon) \log\log r \ .$$

Now for sufficiently large r, so that $r^A \geq r_0$, we have from above

$$\begin{aligned} \log\log \mu_f(r^A) &< (\rho_{\log}(f) + \varepsilon) \log\log r^A \\ &< (\rho_{\log}(f) + \varepsilon) \log r^A \\ &< (\rho_{\log}(f) + \varepsilon) r^A. \end{aligned} \tag{30}$$

Combining (29) and (30),

$$\frac{\log\log \mu_{f\circ g}(r)}{\log\log \mu_f(r^A)} > \frac{(\lambda_{\log}(f) - \varepsilon)(\lambda_{\log}(g) - \varepsilon) \log\log\left(\frac{r}{4}\right)}{(\rho_{\log}(f) + \varepsilon) r^A}.$$

We choose $\varepsilon > 0$ so that $(\lambda_{\log}(f) - \varepsilon) > 0, (\lambda_{\log}(g) - \varepsilon) > 0$ and as $r \to \infty$, $\frac{\log\log\left(\frac{r}{4}\right)}{r^A} \to 0$.

Thus we finally get,

$$\liminf_{r\to\infty} \frac{\log\log \mu_{f\circ g}(r)}{\log\log \mu_f(r^A)} = 0.$$

Proof of the second part of the theorem is omitted as it is similar. □

Theorem 17. *Consider two entire functions $f(z)$ and $g(z)$ of positive lower logarithmic order $\lambda_{\log}(f)$, $\lambda_{\log}(g)$ and of finite logarithmic orders $\rho_{\log}(f)$, $\rho_{\log}(g)$. Then for every $\xi > 0$,*

$$\liminf_{r\to\infty} \frac{\log\log \mu_{f\circ g}\left(r^{1+\xi}\right)}{\log\log \mu_g(r)} = 0.$$

Proof. By Definition 4, for the entire function $g(z)$, there exists a sequence r_n $(n = 1, 2, 3, ...)$,such that

$$\log\log \mu_g(r_n) > (\rho_{\log}(g) - \varepsilon) \log\log r_n \ , \text{ for any } \varepsilon > 0.$$

Let $R_n = (4r_n)^{\frac{1}{1+\xi}}$, then

$$\log \mu_g\left(\frac{R_n^{1+\xi}}{4}\right) > \left(\log \frac{R_n^{1+\xi}}{4}\right)^{(\rho_{\log}(g)-\varepsilon)} , n = 1, 2, 3, ...$$

Now from (29) we have for $r \geq r_0$,

$$\begin{aligned} \log \mu_{f\circ g}(r) &\geq \log \left\{ \frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\right\} \\ &\geq \log\frac{1}{2} + \log \mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right) \\ &\geq \frac{1}{2}\log \mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right) \\ &> \frac{1}{2}\left\{\log\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\right\}^{\left(\lambda_{\log}(f)-\varepsilon\right)} \\ &> \frac{1}{2}\left\{\log \mu_g\left(\frac{r}{4}\right) + o(1)\right\}^{\left(\lambda_{\log}(f)-\varepsilon\right)}. \end{aligned}$$

So, if we take $R_n \geq r_0$ then for any $\xi > 0$, $R_n^{1+\xi} \geq r_0$ and from above we get

$$\log \mu_{f\circ g}\left(R_n^{1+\xi}\right) > \frac{1}{2}\left\{\left(\log\frac{R_n^{1+\xi}}{4}\right)^{\left(\rho_{\log}(g)-\varepsilon\right)}\right\}^{\left(\lambda_{\log}(f)-\varepsilon\right)}.$$

Therefore for the sequence $R_n\ (\geq r_0)$,

$$\begin{aligned} \log\log \mu_{f\circ g}\left(R_n^{1+\xi}\right) &> \left(\lambda_{\log}(f)-\varepsilon\right)\log\frac{1}{2}\left\{\left(\log\frac{R_n^{1+\xi}}{4}\right)^{\left(\rho_{\log}(g)-\varepsilon\right)}\right\} \\ &> \left(\lambda_{\log}(f)-\varepsilon\right)\left\{\log\frac{1}{2} + \log\left(\log\frac{R_n^{1+\xi}}{4}\right)^{\left(\rho_{\log}(g)-\varepsilon\right)}\right\} \\ &> \left(\lambda_{\log}(f)-\varepsilon\right)\left(\rho_{\log}(g)-\varepsilon\right)\log\left(\log\frac{R_n^{1+\xi}}{4}\right). \end{aligned} \tag{31}$$

Also for all $r \geq r_0$,

$$\begin{aligned} \log\log \mu_g(r) &< \left(\rho_{\log}(g)+\varepsilon\right)\log\log r \\ &< \left(\rho_{\log}(g)+\varepsilon\right) r. \end{aligned} \tag{32}$$

Combining (31) and (32) we have for the sequence $R_n\ (\geq r_0)$,

$$\frac{\log\log \mu_{f\circ g}\left(R_n^{1+\xi}\right)}{\log\log \mu_g(R_n)} > \frac{\left(\lambda_{\log}(f)-\varepsilon\right)\left(\rho_{\log}(g)-\varepsilon\right)\log\left(\log\frac{R_n^{1+\xi}}{4}\right)}{\left(\rho_{\log}(g)+\varepsilon\right)R_n}.$$

Since for $\varepsilon > 0, \lambda_{\log}(f) - \varepsilon > 0$ and $\frac{\log\left(\log \frac{R_n^{1+\xi}}{4}\right)}{R_n} \to 0$ as $R_n \to \infty$, hence the result is proved. □

Theorem 18. *Let $f(z), g(z)$ and $h(z)$ be entire functions with positive lower logarithmic order $\lambda_{\log}(f), \lambda_{\log}(g), \lambda_{\log}(h)$ and of finite logarithmic order $\rho_{\log}(f), \rho_{\log}(g), \rho_{\log}(h)$. Further let $\rho_{\log}(h) > \lambda_{\log}(g)$. Then*

$$\liminf_{r\to\infty} \frac{\log M_{f\circ g}(r)}{\log M_{f\circ h}(r)} = 0.$$

Proof. By Definition 2,

$$\log\log M_f(r) < (\rho_{\log}(f) + \varepsilon)\log\log r$$

and

$$\log\log M_h(r) < (\rho_{\log}(h) + \varepsilon)\log\log r,$$

for given $\varepsilon > 0$.

From Lemma 6,

$$M_{f\circ h}(r) \leq M_f(M_h(r))$$

i.e;

$$\begin{aligned} \log M_{f\circ h}(r) &\leq \log M_f(M_h(r)) \\ &< \exp\{(\rho_{\log}(f) + \varepsilon)\log\log M_h(r)\} \\ &< \exp\{(\rho_{\log}(f) + \varepsilon)(\rho_{\log}(h) + \varepsilon)\log\log r\} \\ &< (\log r)^{\left(\rho_{\log}(f)+\varepsilon\right)\left(\rho_{\log}(h)+\varepsilon\right)}. \end{aligned} \tag{33}$$

which implies

$$\log M_{f\circ h}(r) < r^{\left(\rho_{\log}(f)+\varepsilon\right)\left(\rho_{\log}(h)+\varepsilon\right)}. \tag{34}$$

Again by Definition 2,

$$\log\log M_f(r) > (\lambda_{\log}(f) - \varepsilon)\log\log r$$

and

$$\log\log M_g(r) > (\lambda_{\log}(g) - \varepsilon)\log\log r,$$

for any given $\varepsilon > 0$.

Now from Lemma 6,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{16}M_g\left(\frac{r}{2}\right)\right).$$

Now for all $r \geq r_0$ and given any $\varepsilon > 0$, we have

$$\begin{aligned}
\log M_{f \circ g}(r) &\geq \log M_f\left(\frac{1}{16}M_g\left(\frac{r}{2}\right)\right) \\
&> \exp\left\{(\lambda_{\log}(f)-\varepsilon)\log\log\left(\frac{1}{16}M_g\left(\frac{r}{2}\right)\right)\right\} \\
&> \exp\left[(\lambda_{\log}(f)-\varepsilon)\log\left\{\log\frac{1}{16}+\log M_g\left(\frac{r}{2}\right)\right\}\right] \\
&> \exp\left\{(\lambda_{\log}(f)-\varepsilon)\log\log M_g\left(\frac{r}{2}\right)\right\}+o(1) \\
&> \exp\left\{(\lambda_{\log}(f)-\varepsilon)(\lambda_{\log}(g)-\varepsilon)\log\log\frac{r}{2}\right\} \\
&> \left(\log\frac{r}{2}\right)^{\left(\lambda_{\log}(f)-\varepsilon\right)\left(\lambda_{\log}(g)-\varepsilon\right)}, \qquad (35)
\end{aligned}$$

Hence by equation(34) and (35),

(36)

$$\frac{\log M_{f\circ g}(r)}{\log M_{f\circ h}(r)} > \frac{\left(\log\frac{r}{2}\right)^{\left(\lambda_{\log}(f)-\varepsilon\right)\left(\lambda_{\log}(g)-\varepsilon\right)}}{(\log r)^{\left(\rho_{\log}(f)+\varepsilon\right)\left(\rho_{\log}(h)+\varepsilon\right)}} > \frac{\left(\log\frac{r}{2}\right)^{\left(\lambda_{\log}(f)-\varepsilon\right)\left(\lambda_{\log}(g)-\varepsilon\right)}}{r^{\left(\rho_{\log}(f)+\varepsilon\right)\left(\rho_{\log}(h)+\varepsilon\right)}}.$$

Since $\rho_{\log}(h) > \lambda_{\log}(g)$, for any given $\varepsilon > 0$, $\rho_{\log}(h)+\varepsilon > \lambda_{\log}(g)-\varepsilon$. Thus we have

$$\liminf_{r\to\infty}\frac{\log M_{f\circ g}(r)}{\log M_{f\circ h}(r)} = 0.$$

□

Example 19. *In the above Theorem* 18,*consider* $f(z) = g(z) = z$ *and* $h(z) = e^{(\log z)^2}$, *then* $\rho_{\log}(f) = \lambda_{\log}(f) = \lambda_{\log}(g) = 1$ *and* $\rho_{\log}(h) = 2$. *Then* $\rho_{\log}(h) > \lambda_{\log}(g)$ *and therefore from the ratio* (36), *we have* $\frac{\log M_{f\circ g}(r)}{\log M_{f\circ h}(r)} > \frac{\left(\log\frac{r}{2}\right)}{(\log r)^2} \to 0$ *as* $r \to \infty$.

Next two theorems deal with composite entire functions having same right factor.

Theorem 20. *Let* $f(z), g(z)$ *and* $h(z)$ *be entire functions of finite logarithmic orders* $\rho_{\log}(f), \rho_{\log}(g)$ *and* $\rho_{\log}(h)$ *respectively. Suppose* $\lambda_{\log}(h) > 0$ *and* $\rho_{\log}(f) < \rho_{\log}(h)$, *then*

$$\liminf_{r\to\infty}\frac{\log\log M_{f\circ g}\left(\frac{r}{2}\right)}{\log\log M_{h\circ g}(r)} \leq \frac{\rho_{\log}(h)}{\lambda_{\log}(h)}.$$

Proof. For all $r \geq r_0$ and given any $\varepsilon > 0$, from (33) we get

$$\log M_{f \circ g}(r) \leq (\log r)^{\left(\rho_{\log}(f)+\varepsilon\right)\left(\rho_{\log}(g)+\varepsilon\right)}$$

i.e;

$$\log \log M_{f \circ g}(r) \leq \left(\rho_{\log}(f) + \varepsilon\right)\left(\rho_{\log}(g) + \varepsilon\right) \log \log r. \tag{37}$$

which implies

$$\log \log M_{f \circ g}\left(\frac{r}{2}\right) \leq \left(\rho_{\log}(f) + \varepsilon\right)\left(\rho_{\log}(g) + \varepsilon\right) \log \log \frac{r}{2}. \tag{38}$$

Also for all $r \geq r_0$ and given any $\varepsilon > 0$, we have from (35)

$$\log M_{h \circ g}(r) > \left(\log \frac{r}{2}\right)^{\left(\lambda_{\log}(h)-\varepsilon\right)\left(\rho_{\log}(g)-\varepsilon\right)},$$

i.e;

$$\log \log M_{h \circ g}(r) > \left(\lambda_{\log}(h) - \varepsilon\right)\left(\rho_{\log}(g) - \varepsilon\right) \log \log \frac{r}{2}. \tag{39}$$

for a sequence $r = r_n \to \infty$ as $n \to \infty$.

Thus combining (38)and (39) it follows that for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\frac{\log \log M_{f \circ g}\left(\frac{r}{2}\right)}{\log \log M_{h \circ g}(r)} \leq \frac{\left(\rho_{\log}(f) + \varepsilon\right)\left(\rho_{\log}(g) + \varepsilon\right) \log \log \frac{r}{2}}{\left(\lambda_{\log}(h) - \varepsilon\right)\left(\rho_{\log}(g) - \varepsilon\right) \log \log \frac{r}{2}}.$$

Since $\rho_{\log}(f) < \rho_{\log}(h)$, we thus have from above

$$\liminf_{r \to \infty} \frac{\log \log M_{f \circ g}\left(\frac{r}{2}\right)}{\log \log M_{h \circ g}(r)} \leq \frac{\rho_{\log}(h)}{\lambda_{\log}(h)}.$$

□

Theorem 21. *Let $f(z), g(z)$ and $h(z)$ be transcendental entire functions of finite logarithmic orders $\rho_{\log}(f), \rho_{\log}(g)$ and $\rho_{\log}(h)$ respectively. Suppose $\lambda_{\log}(h) > 0$ and, then*

$$\liminf_{r \to \infty} \frac{\log T_{f \circ g}(r)}{\log T_{h \circ g}(r)} \leq \frac{\rho_{\log}(f)}{\lambda_{\log}(h)}.$$

Proof. From Definition2 we get,

$$T_f(r) < (\log r)^{\left(\rho_{\log}(f)+\varepsilon\right)}$$

By Lemma 9 and using above result for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\begin{aligned} T_{f\circ g}(r) &\leq T_f(M_g(r)) \\ &\leq (\log M_g(r))^{\left(\rho_{\log}(f)+\varepsilon\right)}, \end{aligned}$$

i.e;

$$\begin{aligned} \log T_{f\circ g}(r) &\leq (\rho_{\log}(f)+\varepsilon)\log\log M_g(r) \\ &\leq (\rho_{\log}(f)+\varepsilon)(\rho_{\log}(g)+\varepsilon)\log\log r. \end{aligned} \tag{40}$$

Again by Lemma 10 for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\begin{aligned} T_{h\circ g}(r) &\geq \frac{1}{3}\log M_h\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right) \\ &\geq \frac{1}{3}\log M_h\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right) \\ &\geq \frac{1}{3}\log\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right)^{\left(\lambda_{\log}(h)-\varepsilon\right)}, \end{aligned} \tag{41}$$

i.e.

$$\begin{aligned} \log T_{h\circ g}(r) &\geq (\lambda_{\log}(h)-\varepsilon)\log\log\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right)+o(1) \\ &\geq (\lambda_{\log}(h)-\varepsilon)\log\left\{c_1\left(\log\frac{r}{4}\right)^{\left(\rho_{\log}(g)-\varepsilon\right)}\right\}+o(1) \\ &\geq (\lambda_{\log}(h)-\varepsilon)\log c_1+(\lambda_{\log}(h)-\varepsilon)(\rho_{\log}(g)-\varepsilon)\log\log\frac{r}{4}+o(1) \\ &\geq (\lambda_{\log}(h)-\varepsilon)(\rho_{\log}(g)-\varepsilon)\log\log\frac{r}{4}+o(1), \end{aligned} \tag{42}$$

where c_1 is a positive constants.

Hence from (40) and (42),we have for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\frac{\log T_{f\circ g}(r)}{\log T_{h\circ g}(r)} \leq \frac{(\rho_{\log}(f)+\varepsilon)(\rho_{\log}(g)+\varepsilon)\log\log r}{(\lambda_{\log}(h)-\varepsilon)(\rho_{\log}(g)-\varepsilon)\log\log\frac{r}{4}}.$$

For a given $\varepsilon > 0$ and for a sequence $r = r_n \to \infty$ from above inequality it follows that

$$\liminf_{r\to\infty}\frac{\log T_{f\circ g}(r)}{\log T_{h\circ g}(r)} \leq \frac{\rho_{\log}(f)}{\lambda_{\log}(h)}.$$

□

Theorem 22. *Let $f(z), g(z), h(z)$ and $k(z)$ be transcendental entire functions of finite logarithmic order $\rho_{\log}(f), \rho_{\log}(g), \rho_{\log}(h)$ and $\rho_{\log}(k)$ respectively and of positive lower logarithmic order $\lambda_{\log}(f), \lambda_{\log}(g), \lambda_{\log}(h)$ and $\lambda_{\log}(k)$ respectively. Further assume $\rho_{\log}(g) = \lambda_{\log}(g)$ and $\lambda_{\log}(k) = \rho_{\log}(k)$, then*

$$\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_{h\circ k}(r)} = \frac{\rho_{\log}(g)}{\rho_{\log}(k)}.$$

Proof. From Definition 2 we get,

$$T_f(r) < (\log r)^{\left(\rho_{\log}(f)+\varepsilon\right)}.$$

By Lemma 9 and using above result,

$$\begin{aligned} T_{f\circ g}(r) &\leq T_f(M_g(r)) \\ &\leq (\log M_g(r))^{\left(\rho_{\log}(f)+\varepsilon\right)}, \end{aligned}$$

i.e;

$$\begin{aligned} \log T_{f\circ g}(r) &\leq (\rho_{\log}(f)+\varepsilon)\log\log M_g(r) \\ &\leq (\rho_{\log}(f)+\varepsilon)(\rho_{\log}(g)+\varepsilon)\log\log r \\ &\leq c(\rho_{\log}(g)+\varepsilon)\log\log r, \qquad (43) \end{aligned}$$

where we take $c > \rho_{\log}(f)$.

Also by Lemma 10,

$$\begin{aligned} T_{h\circ k}(r) &\geq \frac{1}{3}\log M_h\left(\frac{1}{8}M_k\left(\frac{r}{4}\right)+o(1)\right) \\ &\geq \frac{1}{3}\log M_h\left(\frac{1}{9}M_k\left(\frac{r}{4}\right)\right) \\ &\geq \frac{1}{3}\log\left(\frac{1}{9}M_k\left(\frac{r}{4}\right)\right)^{\left(\lambda_{\log}(h)-\varepsilon\right)}, \qquad (44) \end{aligned}$$

i.e;

$$\begin{aligned} \log T_{h\circ k}(r) &\geq (\lambda_{\log}(h)-\varepsilon)\log\log\left(\frac{1}{9}M_k\left(\frac{r}{4}\right)\right)+o(1) \\ &\geq c_1\log\left\{c_2\left(\log\frac{r}{4}\right)^{\left(\rho_{\log}(k)-\varepsilon\right)}\right\}+o(1), \qquad (45) \end{aligned}$$

where c_1, c_2 are positive constants.

Now from (43) and (45) we have

$$\liminf_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_{h\circ k}(r)} \leq \frac{\rho_{\log}(g)}{\rho_{\log}(k)}, \tag{46}$$

Replacing $h \circ k$ by $f \circ g$ in (45) we get,

$$\log T_{f\circ g}(r) \geq d_1 \log \left\{ d_2 \left(\log \frac{r}{4} \right)^{\left(\lambda_{\log}(g)-\varepsilon\right)} \right\} + o(1), \tag{47}$$

and also replacing $f \circ g$ by $h \circ k$ in (43),

$$\log T_{h\circ k}(r) \leq d\left(\rho_{\log}(k) + \varepsilon\right) \log\log r, \tag{48}$$

where we take $d > \rho_{\log}(h)$ and d_1, d_2 are positive constants.

Now from (47) and (48),

$$\liminf_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_{h\circ k}(r)} \geq \frac{\lambda_{\log}(g)}{\rho_{\log}(k)}. \tag{49}$$

Combining (46) and (49) we thus have,

$$\frac{\lambda_{\log}(g)}{\rho_{\log}(k)} \leq \liminf_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_{h\circ k}(r)} \leq \frac{\rho_{\log}(g)}{\rho_{\log}(k)}. \tag{50}$$

In a similar way we can see also,

$$\frac{\rho_{\log}(g)}{\rho_{\log}(k)} \leq \limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_{h\circ k}(r)} \leq \frac{\rho_{\log}(g)}{\lambda_{\log}(k)}. \tag{51}$$

Hence from(50) and (51) the result follows, since it it given that g and k are of regular growth. □

Theorem 23. *Let $g(z), h(z)$ and $k(z)$ be transcendental entire functions of finite logarithmic order $\rho_{\log}(g), \rho_{\log}(h)$ and $\rho_{\log}(k)$ respectively. Let $\lambda_{\log}(h)$ be the lower logarithmic order of $h(z)$ with $\lambda_{\log}(h) > 0$. Suppose $0 < \rho_{\log}(k) < \rho_{\log}(g)$, then*

$$\liminf_{r\to\infty} \frac{T_{h\circ k}(r)}{T_{f\circ g}(r)} = 0$$

holds for every transcendental entire function $f(z)$ of finite logarithmic order $\rho_{\log}(f)$.

Proof. By (43) we have for a sequence $r = r_n \to \infty$ as $n \to \infty$,

$$\begin{aligned} T_{f\circ g}(r) &\leq \exp\{c(\rho_{\log}(g)+\varepsilon)\log\log r\} \\ &< (\log r)^{c\left(\rho_{\log}(g)+\varepsilon\right)} \\ &< r^{c\left(\rho_{\log}(g)+\varepsilon\right)}, \end{aligned} \tag{52}$$

where $c > \rho_{\log}(f)$.

Also by (44), for a sequence $r = r_n \to \infty$,

$$\begin{aligned} T_{h\circ k}(r) &\geq \frac{1}{3}\log\left(\frac{1}{9}M_k\left(\frac{r}{4}\right)\right)^{\left(\lambda_{\log}(h)-\varepsilon\right)} \\ &\geq \frac{1}{3}\exp\left\{c_1\log\log\left(\frac{1}{9}M_k\left(\frac{r}{4}\right)\right)\right\} \\ &\geq \frac{1}{3}\exp\left\{c_1\log\left\{c_2\left(\log\frac{r}{4}\right)^{\left(\rho_{\log}(k)-\varepsilon\right)}\right\}\right\}, \end{aligned} \tag{53}$$

where c_1, c_2 are positive constants.

Combining (53) and (52) we have for a sequence $r = r_n \to \infty$ as $n \to \infty$

$$\frac{T_{h\circ k}(r)}{T_{f\circ g}(r)} > \frac{\frac{1}{3}\exp\left\{c_1\log\left\{c_2\left(\log\frac{r}{4}\right)^{\left(\rho_{\log}(k)-\varepsilon\right)}\right\}\right\}}{r^{c\left(\rho_{\log}(g)+\varepsilon\right)}}. \tag{54}$$

Since it is given $\rho_{\log}(g) > \rho_{\log}(k)$, so for any given $\varepsilon > 0$ we have $\rho_{\log}(g) + \varepsilon > \rho_{\log}(k) - \varepsilon$.

Therefore right hand side of the inequality (54) tends to 0 as $r \to \infty$.

Hence the result follows. □

Theorem 24. *Let $g(z), h(z), k(z)$ be transcendental entire functions of finite logarithmic order $\rho_{\log}(g), \rho_{\log}(h)$ and $\rho_{\log}(k)$ respectively. Let $\lambda_{\log}(h)$ and $\lambda_{\log}(k)$ be the lower logarithmic order of $h(z)$ and $k(z)$ respectively. Also let $\lambda_{\log}(h) > 0$ and $0 < \lambda_{\log}(k) < \rho_{\log}(g)$, then*

$$\lim_{r\to\infty}\frac{T_{h\circ k}(r)}{T_{f\circ g}(r)} = 0$$

holds for every transcendental entire function f of finite logarithmic order.

Proof. From above theorem we have in a same manner,

$$\frac{T_{h\circ k}(r)}{T_{f\circ g}(r)} > \frac{\frac{1}{3}\exp\left\{c_1\left(\lambda_{\log}(k)-\varepsilon\right)\log\log\frac{r}{4}+o(1)\right\}}{r^{c\left(\rho_{\log}(g)+\varepsilon\right)}}. \tag{55}$$

where we take $c > \rho_{\log}(f)$ and c_1 is positive constant.

Since it is given $\rho_{\log}(g) > \lambda_{\log}(k)$, so for any given $\varepsilon > 0$ we have $\rho_{\log}(g) + \varepsilon > \lambda_{\log}(k) - \varepsilon$.

Therefore right hand side of the inequality (55) tends to 0 as $r \to \infty$.

Hence the result follows. □

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